Mutations and instanton knot homology

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1 Statements and proofs

If K is a knot or link in $\mathbb{R}^3$ and $S \subset \mathbb{R}^3$ is a smoothly embedded 2-sphere meeting K transversely in 4-points, then one can form a new knot or link K' as follows. One cuts $\mathbb{R}^3$ along S and then glues the two pieces again using a diffeomorphism $\sigma : S \to S$ that permutes the 4 distinguished points $S \cap K$ and whose class in the mapping class group of $(S, S \cap K)$ is one of the three central involutions. This operation is known as (Conway) mutation, and is introduced in [3]. Many invariants of K (starting with the number of components) are left unchanged by mutation.

The purpose of this paper is to record the fact that the instanton homology group $I^\sharp(K)$, defined in [6], can be added to the list of invariants that are invariant under Conway mutation, as long as we restrict our attention to knots, rather than links. As we will explain, the proof is essentially the observation that the earlier work of the third author [9] concerning instanton homology of closed 3-manifolds can be applied directly to the question.

To make the connection with [9], we state our result in terms of the closely-related operation of genus-2 mutation introduced in [8]. This is the operation of cutting a 3-manifold Y along an embedded genus-2 surface Σ and re-glueing using a diffeomorphism $\tau : \Sigma \to \Sigma$ belonging to the class of the central involution in the mapping class group of Σ. As observed in [8] and [10], a Conway mutation of a knot can always be realized either as a genus-2 mutation along a surface Σ in the knot complement, or as a composite of two such genus-2 mutations. In [6], an instanton homology group is introduced for triples (Y, K, ω), where Y is a closed, oriented 3-

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manifold, $K \subset Y$ is a link, and $\omega \subset Y$ is a one-manifold with boundary, with $\omega \cap K = \partial\omega$, meeting K normally at its endpoints. We require also that $[\omega, \partial\omega]$ defines a class in $H_1(Y, K)$/torsion that is not divisible by 2. This invariant is denoted $I^\omega(Y, K)$. The main result is then:

Theorem 1.1. *Let (Y, K, ω) be as above, and let $\Sigma \subset Y$ be a separating, embedded genus-2 surface that is disjoint from both K and ω. Let Y' be obtained from Y by mutation along Σ and let K' and ω' be the resulting link and 1-manifold. Then we have*

$$I^\omega(Y, K) \cong I^{\omega'}(Y', K')$$

as abelian groups with affine $\mathbb{Z}/4$ gradings.

The main theorem from [9] is essentially the same statement, but is formulated for the instanton homology of a homology 3-sphere Y, in the sense of [5], and does not involve K or ω. For the proof of the theorem, it is already observed in [9] that the argument of that paper is readily adaptable to situations more general than a homology 3-sphere, though at the time that [9] was written, the version of instanton homology which involves also a knot or link $K \subset Y$ had not been developed. In the present context it is straightforward to see that the argument carries over *mutatis mutandis.*

Returning to Conway mutation, we have:

Corollary 1.2. *If K_1 and K_2 are classical knots in $\mathbb{R}^3$ that are Conway mutants, then $I^\sharp(K_1) \cong I^\sharp(K_2)$ as $\mathbb{Z}/4$-graded abelian groups.*

Proof of the corollary. As observed in [8] and mentioned above, it is sufficient to examine the case that K_2 is obtained from K_1 by a genus-2 mutation along some surface Σ in $\mathbb{R}^3 \setminus K_1$. The definition of $I^\sharp(K_1)$ from [6] is in terms of the more general construction $I^\omega(Y, K)$. Specifically, from $K_1 \subset \mathbb{R}^3$ one forms a new link $K_1^\sharp \subset S^3$ by adding to K_1 a Hopf link in a ball at infinity in S^3; and one takes ω to be an arc in that ball, joining the two components of the Hopf link. One then defines

$$I^\sharp(K_1) = I^\omega(S^3, K_1^\sharp).$$

The genus-2 surface Σ in $\mathbb{R}^3 \setminus K_1$ becomes a surface Σ in S^3 which is disjoint from $K_1^\sharp$ and ω. An application of the theorem above, we have

$$I^\omega(S^3, K_1^\sharp) \cong I^\omega(S^3, K_2^\sharp),$$

which is what we need. Unlike the case of a general triple, the affine $\mathbb{Z}/4$ grading has a canonical lift to an absolute $\mathbb{Z}/4$ grading on $I^\omega(S^3, K_1^\sharp)$. As in [9], the isomorphism is derived from an explicit cobordism with orbifold singularities, and it is straightforward to deduce from the definitions in [6] that the map induced by the cobordism has degree 0. So the absolute $\mathbb{Z}/4$ gradings coincide. □

2 Further remarks

In writing this note, the authors were motivated, in part, by the open question of whether Khovanov homology of knots is invariant under Conway mutation. (See [4] for a survey of related questions.) It is shown in [6] that there is a spectral sequence relating the Khovanov homology of K to $I^\sharp(K)$, but we are not able to make any use of this as an approach to the conjecture.

The instanton homology group $I^\sharp(K)$ carries extra structure (gradings and filtrations), some of which arise via the isomorphism between $I^\sharp(K) \otimes \mathbb{Q}$ and the sutured instanton homology of the knot complement [6]. From this extra structure one can extract (amongst other things) the genus of the knot. Knot genus is not invariant under Conway mutation, however. This is a reflection of the fact that the theorem does not imply that this extra structure is mutation-invariant. The situation can be compared to that of the Heegaard knot homology groups $\widehat{HFK}(K)$. It is known that, as a bigraded object, the Heegaard knot homology is sensitive to Conway mutations [7], but in the known examples (up to knots of 12 crossings [1, 2]), the *rank* of $\widehat{HFK}(K)$ is Conway-mutation-invariant.

To return to Khovanov homology, although the question is open for Conway mutation, it is known that $Kh(K)$ (as a bigraded group) is not invariant under the more general genus-2 mutation [4]. The main theorem of this paper raises the question of whether the *rank* of Khovanov homology is genus-2-mutation-invariant.

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