Korovkin type theorem and iterates of certain positive linear opeartors

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Abstract

In this paper we prove Korovkin type theorem for iterates of general positive linear operators $T : C[0,1] \to C[0,1]$ and derive quantitative estimates in terms of modulus of smoothness. In particular, we show that under some natural conditions the iterates $T^m : C[0,1] \to C[0,1]$ converges strongly to a fixed point of the original operator T. The results can be applied to several well-known operators; we present here the q-MKZ operators, the q-Stancu operators, the genuine q-Bernstein–Durrmeyer operators and the Cesaro operators.

Keywords. Iterates of operators, degree of approximation, K-functionals, modulus of smoothness, Bernstein operators, genuine Bernstein–Durrmeyer operators, Stancu operators, Korovkin type theorem, Cesaro operators, Meyer-Kőnig and Zeller operators.

1 Introduction

These iterated Bernstein operators were investigated in the 60's and 70's by P. C. Sikkema [27], R. P. Kelisky & T. J. Rivlin [16], S. Karlin & Z. Ziegler [15], J. Nagel [20], M. R. da Silva [6] and Gonska [9], [10]. Some of this research was later generalized by Altomare et al. (see, for example, [1], [2], [3]). Altomare suggested to use in this context an approach described by Dickmeis and Nessel [8]. This was done recently by Rasa in [24] and [25]. Another new paper related to the subject of this article was written by S. Ostrovska [23] on iterates of q-Bernstein polynomials.

The methods employed to study the convergence of iterates of some operators occurring in Approximation Theory include Matrix Theory methods, like stochastic matrices [22], [7], [28], Korovkin-type theorems [15], quantitative results about the approximation of functions by positive linear operators [11], [12], fixed point theorems [4], [13], [26], or methods from the theory of C_0-semigroups, like Trotter's approximation theorem [15], [18]. However, these techniques fail to work for the Meyer-Kőnig and Zeller (MKZ) or the May operators. Very recently, I. Gavrea and M. Ivan [37] proved that the iterates of the MKZ operates converges strongly to $P(f;x) = (1-x) f(0) + x f(1)$. Once such convergence have been obtained, the following natural question is to ask for rates of convergence. In Section 3, as a consequence of our results, we obtain the quantitaive estimates for the iterates of the q-MKZ $(0 < q \leq 1)$ operators, which is completely new.

On the other hand, because of its powerful applications, Korovkin's result has been extended in many directions. There is an extensive literature on Korovkin-type theorems, which may have had a summit already about twenty five years ago. In particular, there exist abstract results that cover many naturally arising concrete cases. The contributions up to about 1994 are excellently documented in the book of Altomare and Campiti [3]. More recent results obtained in [33], [34] cover also approximation of q-type operators.

In this paper we establish quantitative Korovkin type theorem for the iterates of certain positive linear operators $T : C[0,1] \to C[0,1]$. As a consequence of our results, we obtain the quantitaive estimates for the iterates of almost all classical and new positive linear operators: the q-MKZ operators, the q-Stancu operators, the genuine q-Bernstein–Durrmeyer operators in the case $0 < q \leq 1$ and the Cesaro operators. It is worth mentioning that for $q = 1$ these operators become classical MKZ, Stancu and genuine Bernstein-Durrmeyer operators.

2 Main results

The following notations will be used throughout this paper. The classical Petree's K-functional and the second modulus of smoothness of a function f are defined respectively by

$$K_2(f,t) := \inf_{g \in C^2[0,1]} \{\|f-g\| + t\|g''\|\}$$

and

$$\omega_2(f,t) := \sup_{0<h\leq t} \sup_{0\leq x\leq 1-2h} |f(x+2h) - 2f(x+h) + f(x)|.$$

It is known there exists a constant $C > 0$ such that

$$K_2(f,t) \leq C\omega_2(f,\sqrt{t}). \tag{1}$$

Let $e_i : [0,1] \to R$ be the monomial functions $e_i(x) = x^i$, $i = 0, 1, 2$.

Now we formulate the main results of the paper. First result shows that under the conditions (2) the iterates of $T : C[0,1] \to C[0,1]$ converges to some linear positive operator $T^\infty : C[0,1] \to C[0,1]$.

Theorem 1 *Suppose that $T : C[0,1] \to C[0,1]$ is a positive linear operator such that*

$$\begin{gathered} T(e_0) = e_0, \quad T(e_1; x) \leq x, \\ \lim_{n\to\infty} \|T^m(e_1) - T^\infty(e_1)\| = \lim_{n\to\infty} \|T^m(e_2) - T^\infty(e_2)\| = 0. \end{gathered} \tag{2}$$

Then there exists a linear positive operator $T^\infty : C[0,1] \to C[0,1]$ such that the following pointwise estimate

$$|(T^m - T^\infty)(f;x)| \leq k\omega_2\left(f, \sqrt{\lambda_n(x)}\right) + k\|f\|\delta_n(x) \tag{3}$$

holds true for $x \in [0,1]$ and $f \in C[0,1]$, where k is an absolute constant and

$$\begin{aligned} \lambda_n(x) &= \max\{|(T^m - T^\infty)(e_1;x)|, |(T^\infty - T^m)(e_2;x)|\}, \\ \delta_n(x) &= |(T^m - T^\infty)(e_1;x)|. \end{aligned}$$

Proof. For every nonincreasing convex $g \in C^2[0,1]$, we have

$$\begin{aligned} g(t) &\geq g(x) + g'(x)(t-x), \\ T(g;x) &\geq g(x) + g'(x)(T(e_1;x) - x) \geq g(x). \end{aligned} \tag{4}$$

It follows that

$$g(x) \leq T^m(g;x) \leq T^{m+1}(g;x).$$

In other words the sequence $\{T^m(g;x)\}$ is nondecreasing for any nonincreasing convex $g \in C^2[0,1]$ and $x \in [0,1]$.

Let $x \in [0,1]$ be fixed and $g \in C^2[0,1]$ be arbitrary. Introduce the following auxiliary functions

$$g_\pm(t) = \frac{1}{2}\|g''\|(1-t)^2 + \|g'\|(1-t) \pm g(t).$$

It is clear that

$$g'_\pm(t) = -\|g''\|(1-t) - \|g'\| \pm g(t) \leq 0, \quad g''_\pm(t) = \|g''\| \pm g(t) \geq 0.$$

Therefore the functions $g_{\pm}(t)$ are nonincreasing convex for both choices of the sign. Since $(T^{m+p} - T^m)(g_{\pm}; x)$ is positive we have

$$0 \le \left(T^{m+p} - T^m\right)(g_{\pm}; x) = \frac{1}{2}\|g''\|\left(T^{m+p} - T^m\right)\left((e_0 - e_1)^2; x\right)$$
$$+ \|g'\|\left(T^{m+p} - T^m\right)(e_0 - e_1; x) \pm \left(T^{m+p} - T^m\right)(g; x).$$

It follows that

$$\left|\left(T^{m+p} - T^m\right)(g; x)\right| \le \frac{1}{2}\|g''\|\left|\left(T^{m+p} - T^m\right)(e_2; x)\right| + (\|g''\| + \|g'\|)\left|\left(T^m - T^{m+p}\right)(e_1; x)\right|. \tag{5}$$

So $\{T^m(f; x)\}$ is a Cauchy sequence in $C[0,1]$ and there is a linear positive operator $T^{\infty}(f)$ such that

$$\lim_{m\to\infty}\|T^m(f) - T^{\infty}(f)\| = 0$$

for any $f \in C[0,1]$. Taking the limit as $p \to \infty$ in (5) and using the well known inequality

$$\|g'\| \le C_1(\|g\| + \|g''\|)$$

we have

$$|(T^{\infty} - T^m)(g; x)| \le \frac{1}{2}\|g''\||(T^{\infty} - T^m)(e_2; x)| + (\|g''\| + \|g'\|)|(T^m - T^{\infty})(e_1; x)|$$
$$\le \left(\frac{3}{2} + C_1\right)\lambda_n(x)\|g''\| + C_1\delta_n(x)\|g\|. \tag{6}$$

Taking into account that $\|T^m\| = 1$ from (6) and the inequality $\|g\| \le \|f\| + \|f - g\|$ it follows that

$$|T^{\infty}(f; x) - T^m(f; x)| \le |(T^{\infty} - T^m)(f - g; x)| + |T^{\infty}(g; x) - T^m(g; x)|$$
$$\le 2\|f - g\| + \left(\frac{3}{2} + C_1\right)\lambda_n(x)\|g''\| + C_1\delta_n(x)\|g\|$$
$$\le C_2(\|f - g\| + \lambda_n(x)\|g''\|) + C_4\delta_n(x)\|f\|$$

Taking on the right side the infimum over all $g \in C^2[0,1]$ we obtain

$$|T^{\infty}(f; x) - T^m(f; x)| \le C_2K_2\left(f; \frac{1}{4}|(T^{\infty} - T^m)(e_2; x)|\right) + C_4\delta_n(x)\|f\|.$$

Now using (1) we obtain (3). ■

Remark 2 *It is clear that for any $f \in C[0,1]$ the the image $T^{\infty}(f)$ is a fixed point of the original operator $T : C[0,1] \to C[0,1]$. It gives some information on the nature of the limit operator T^{∞}.*

If $T : C[0,1] \to C[0,1]$ preserves the affine functions as a consequence of the above theorem we have the following result.

Theorem 3 *Suppose that $T : C[0,1] \to C[0,1]$ is a positive linear operator such that*

$$T(e_0) = e_0, \quad T(e_1; x) = x, \quad \lim_{n\to\infty}\|T^m(e_2) - T^{\infty}(e_2)\| = 0. \tag{7}$$

Then there exists a linear positive operator $T^{\infty} : C[0,1] \to C[0,1]$ such that the following pointwise estimate

$$|(T^m - T^{\infty})(f; x)| \le k\omega_2\left(f, \frac{1}{2}\sqrt{|(T^{\infty} - T^m)(e_2; x)|}\right) \tag{8}$$

holds true for $x \in [0,1]$ and $f \in C[0,1]$, where k is an absolute constant.

Next result shows that under the conditions (9) the limit T^{∞} of the iterates T^m is exactly the operator $P(f) := (e_0 - e_1)f(0) + e_1f(1)$ with the quantitative estimate (10).

Theorem 4 *Let $T : C[0,1] \to C[0,1]$ be a positive linear operator such that*

$$T(e_0) = e_0, \quad T(e_1; x) = x, \; T(e_2; x) \le ax^2 + bx, a, b \in R \setminus \{0\}, \; a + b = 1. \tag{9}$$

Then the pointwise approximation

$$|T^m(f; x) - P(f; x)| \le k\omega_2\left(f; \sqrt{a^m x(1-x)}\right) \tag{10}$$

holds true for all $x \in [0,1]$ and $f \in C[0,1]$.

Proof. By the assumptions we have

$$x^2 \le T(e_2; x) \le ax^2 + bx \le (a+b)x = x = T(e_1; x).$$

On the other hand by the induction we have

$$x^2 \le T^m(e_2; x) \le a^m x^2 + b\left(1 + a + ... + a^{m-1}\right)x = a^m x^2 + (1 - a^m)x.$$

So

$$0 \le x - T^m(e_2; x) \le a^m x(1-x),$$
$$\lim_{m\to\infty} \|T^m(e_2) - e_1\| = 0,$$

and the operator T^∞ of Theorem 3 satisfies

$$T^\infty(e_0) = e_0, \quad T^\infty(e_1) = e_1, \quad T^\infty(e_2) = e_1$$

and

$$(T^\infty - T^m)(e_2; x) \le a^m x(1-x).$$

It remains to show that $T^\infty(f) = P(f)$ for all $f \in C[0,1]$. It is clear that it is enough to show this equality in $C^2[0,1]$. Let $g \in C^2[0,1]$. Define the following auxiliary functions.

$$G(x) := g(x) - P(g; x), \quad l := \frac{1}{2}\|G''\| = \frac{1}{2}\|g''\|, \quad g_\pm(x) := -lx^2 + lx \pm G(x).$$

It is clear that $g_\pm$ is concave and nonnegative, since

$$g''_\pm(x) = -\|G''\| \pm G''(x) \le 0, \quad G(0) = G(1) = 0.$$

It follows that

$$-l\left(x - x^2\right) \le G(x) \le l\left(x - x^2\right), \quad 0 \le x \le 1.$$

Applying the positive operator T^∞ we get

$$-l(T^\infty(e_1; x) - T^\infty(e_2; x)) \le T^\infty(G; x) = T^\infty(g; x) - P(g; x) \le l(T^\infty(e_1; x) - T^\infty(e_2; x)),$$

for all $0 \le x \le 1$, and consequently

$$T^\infty(g) = P(g)$$

for all $g \in C^2[0,1]$, which completes the proof. ■

Remark 5 *It is worth mentioning that the conditions $T(e_0) = e_0, \quad T(e_1) = e_1, \; T(e_2; x) = ax^2 + bx$ are satisfied by the many classical positive linear operators defined on $C[0,1]$. The condition $T(e_2; x) \le ax^2 + bx, , a, b \in R \setminus \{0\}, \; a + b = 1$, covers the q-MKZ operators.*

The last result shows that under the conditions (11) the limit T^∞ of the iterates T^m is exactly the operator $f(0)$ with the quantitative estimate (12).

Theorem 6 *Let $T : C[0,1] \to C[0,1]$ be a positive linear operator such that*

$$T(e_0) = e_0, \quad T(e_1; x) \le x, \quad \lim_{m\to\infty} \|T^m(e_1)\| = \lim_{m\to\infty} \|T^m(e_2)\| = 0. \tag{11}$$

Then the pointwise approximation

$$|T^m(f;x) - f(0)| \le k\omega_2\left(f, \sqrt{\lambda_m(x)}\right) + k\|f\|\,\delta_m(x) \tag{12}$$

holds true for $x \in [0,1]$ and $f \in C[0,1]$, where k is an absolute constant and

$$\lambda_m(x) = \max\{T^m(e_1;x), T^m(e_2;x)\},$$
$$\delta_m(x) = T^m(e_1;x).$$

Proof. The operator T^∞ of Theorem 3 satisfies

$$T^\infty(e_0) = e_0, \quad T^\infty(e_1) = 0, \quad T^\infty(e_2) = 0.$$

It remains to show that $T^\infty(f) = f(0)$ for all $f \in C[0,1]$. It is clear that it is enough to show this equality in $C^2[0,1]$. Let $g \in C^2[0,1]$. Define the following auxiliary functions.

$$G(x) := g(x) - g(0), \quad l_2 := \frac{1}{2}\|G''\| = \frac{1}{2}\|g''\|, \quad l_1 := l_2 + |g(1) - g(0)|,$$
$$g_\pm(x) := -l_2x^2 + l_1x \pm G(x).$$

It is clear that $g_\pm$ is concave and nonnegative, since

$$g''_\pm(x) = -\|G''\| \pm G''(x) \le 0, \quad g_\pm(0) = 0, \quad g_\pm(1) = -l_2 + l_1 \pm G(1) = |g(1) - g(0)| \pm G(1) \ge 0.$$

It follows that

$$-l_1x + l_2x^2 \le G(x) \le l_1x - l_2x^2, \quad 0 \le x \le 1.$$

Applying the linear positive operator T^∞ we get

$$-l_1T^\infty(e_1;x) + l_2T^\infty(e_2;x) \le T^\infty(G;x) = T^\infty(g;x) - g(0) \le l_1T^\infty(e_1;x) - l_2T^\infty(e_2;x), \quad 0 \le x \le 1,$$

and consequently

$$T^\infty(g;x) - g(0) = 0.$$

for all $g \in C^2[0,1]$, which completes the proof. ■

Many of the linear methods of approximation are given by a sequence of discrete linear positive operators designed as follows.

$$\Lambda_n(f;x) := \sum_{k=0}^{n} f(x_{n,k})\, a_{n,k}(x), \quad f \in C[0,1], \tag{13}$$

where every function $a_{n,k} \in C[0,1]$ is non-negative, $0 = x_{n,0} < ... < x_{n,n} = 1$ forms a mesh of nodes. Assume that the following identities

$$\sum_{k=0}^{n} a_{n,k}(x) = 1, \quad \sum_{k=0}^{n} x_{n,k}a_{n,k}(x) = x, \quad 0 \le x \le 1,$$
$$a_{n,k}(x) \ge 0, \quad a_{n,0}(0) = a_{n,1}(1) = 1, \tag{14}$$

are fulfilled. It is clear that

$$\Lambda_n(f;0) = f(0), \quad \Lambda_n(f;1) = f(1).$$

Iterates of the discrete operators Λ_n was studied by O. Agratini and I. A.Rus [38] via contraction principle. In the next theorem we give uniform estimation for the iterates of Λ_n

Theorem 7 *Let Λ_n be defined by (13) and satisfies (14). Assume that $u_n := \min\{a_{n,0}(x) + a_{n,n}(x) : 0 \le x \le 1\} > 0$. Then the pointwise approximation*

$$|\Lambda_n^m(f;x) - P(f;x)| \le k\omega_2\left(f; \sqrt{|e_1(x) - \Lambda_n^m(e_2;x)|}\right)$$

holds true for all $x \in [0,1]$ and $f \in C[0,1]$. Furthermore, we have the following uniform estimation

$$\|\Lambda_n^m(f) - P(f)\| \le k\omega_2\left(f; \sqrt{(1-u_n)^m \|\Lambda_n(e_2) - e_1\|}\right).$$

Proof. For each $0 \le x \le 1$ we can write

$$\begin{aligned}
\left|\Lambda_n^{m+1}(e_2;x) - x\right| &= \left|\sum_{k=0}^{n}\left(\Lambda_n^m(e_2;x_{n,k}) - x_{n,k}\right)a_{n,k}(x)\right| \le \sum_{k=0}^{n}\left|\Lambda_n^m(e_2;x_{n,k}) - x_{n,k}\right| a_{n,k}(x) \\
&\le \sum_{k=1}^{n-1}\left|\Lambda_n^m(e_2;x_{n,k}) - x_{n,k}\right| a_{n,k}(x) \le |1 - a_{n,0}(x) - a_{n,n}(x)| \, \|\Lambda_n^m(e_2) - e_1\| \\
&\le (1-u_n)\|\Lambda_n^m(e_2) - e_1\|.
\end{aligned}$$

By the induction we have

$$\left\|\Lambda_n^{m+1}(e_2) - e_1\right\| \le (1-u_n)^m \|\Lambda_n(e_2) - e_1\|.$$

If $u_n := \min\{a_{n,0}(x) + a_{n,n}(x) : 0 \le x \le 1\} > 0$ then

$$\lim_{m\to\infty}\left\|\Lambda_n^{m+1}(e_2) - e_1\right\| = 0.$$

■

Remark 8 *Actually, Λ_n is a wide class of discrete operators that include Bernstein-Sheffer (see P. Sablonniere [39]) , Stancu operators and Cheney-Sharma operators*

3 Applications

In this section we employ the standard notations of q-calculus. q-integer and q-factorial are defined by

$$[n]_q := \begin{cases} \dfrac{1-q^n}{1-q} & \text{if} \quad q \in R^+\backslash\{1\}, \\ n & \text{if} \quad q = 1 \end{cases} \qquad \text{for } n \in N \quad \text{and} \quad [0]_q = 0,$$

$$[n]_q! := [1]_q [2]_q \dots [n]_q \qquad \text{for } n \in N \quad \text{and} \quad [0]_q! = 1.$$

For integers $0 \le k \le n$ q-binomial is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! \, [n-k]_q!}.$$

In this section we apply the main result of the paper to discuss the limit of the iterates of a special class of operators.

3.1 Iterates of the q-MKZ operators $(0 < q \le 1)$

q-MKZ operators $M_{n,q} : C[0,1] \to C[0,1]$, $n \in \mathbb{N}$, are defined by

$$M_{n,q}(f;x) = \begin{cases} (1-x)_q^{n+1} \displaystyle\sum_{k=0}^{\infty} f\left(\frac{[k]_q}{[n+k]_q}\right) \begin{bmatrix} n+k \\ k \end{bmatrix}_q x^k, & 0 \in x < 1, \\ f(1), & x = 1. \end{cases}$$

MKZ operators were introduced by Meyer-König and Zeller, q-MKZ operators by T.Trif. It is worth mentioning that the second moment $M_{n,q}(e_2)$ of the q-MKZ operators cannot be expressed as a finite combination of elementary functions since this moment turns out to be a generalized hypergeometric function. This was a major barrier in calculating the limit of the iterates of the q-MKZ operators. Very recently, I. Gavrea and M. Ivan [37] proved that the iterates of the MKZ operates converges strongly to $P(f;x) = (1-x) f(0) + xf(1)$. In what follows we solve this problem for q-MKZ operators using Korovkin type theorem for the iterates of the positive linear operators. Furthermore, we give the quantitative estimate for the iterates of the q-MKZ operators, which is completely new.

Lemma 9 *$M_{n,q}(e_2;x)$ satisfies the condition*

$$M_{n,q}(e_2;x) \leq \left(1 - \frac{1}{[n+1]_q}\right) x^2 + \frac{1}{[n+1]_q} x.$$

Proof. It is easy to see that

$$\begin{aligned}
M_{n,q}(e_2;x) - x^2 &= x(1-x)_q^{n+1} \sum_{k=0}^{\infty} \left(\frac{[k+1]_q}{[n+k+1]_q} - \frac{[k]_q}{[n+k]_q} \right) \begin{bmatrix} n+k \\ k \end{bmatrix}_q x^k \\
&= x \sum_{k=0}^{\infty} \left(\frac{[k+1]_q}{[n+k+1]_q} - \frac{[k]_q}{[n+k]_q} \right) m_{n,k}(q;x) \\
&= x \sum_{k=0}^{\infty} \frac{[k+1]_q [n+k]_q - [k]_q [n+k+1]_q}{[n+k+1]_q [n+k]_q} m_{n,k}(q;x) \\
&= x \sum_{k=0}^{\infty} \frac{q^k [n+k]_q - [k]_q q^{n+k}}{[n+k+1]_q [n+k]_q} m_{n,k}(q;x) \\
&= x \sum_{k=0}^{\infty} \frac{q^k \left([n]_q + q^n [k]_q\right) - [k]_q q^{n+k}}{[n+k+1]_q [n+k]_q} m_{n,k}(q;x) \\
&= x \sum_{k=0}^{\infty} \frac{q^k [n]_q}{[n+k+1]_q [n+k]_q} m_{n,k}(q;x) \\
&= x \sum_{k=0}^{\infty} \frac{q^k [n]_q}{[n+k+1]_q [n+k]_q} m_{n,k}(q;x) \\
&= x(1-x)_q^{n+1} \sum_{k=0}^{\infty} \frac{q^k}{[n+k+1]_q} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q x^k
\end{aligned}$$

It follows that

$$\begin{aligned}
M_{n,q}(e_2;x) &= x^2 + x(1-x)_q^{n+1} \sum_{k=0}^{\infty} \frac{(qx)^k}{[n+k+1]_q} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \\
&= x^2 + \frac{x(1-x)}{[n+1]_q} (1-qx)_q^n \sum_{k=0}^{\infty} \frac{[n+1]_q}{[n+k+1]_q} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q (qx)^k \\
&\leq x^2 + \frac{x(1-x)}{[n+1]_q} = \left(1 - \frac{1}{[n+1]_q}\right) x^2 + \frac{1}{[n+1]_q} x \leq x.
\end{aligned}$$

■

Thus it is obvious that $M_{n,q}$ satisfies the requirements of Theorem 4. We arrive at the following theorem.

Theorem 10 *Let $0 < q \leq 1$. Let $M_{n,q}$ be a sequence of q-MKZ operators. Then the pointwise approximation*

$$\left| M_{n,q}^m(f;x) - P(f;x) \right| \leq k\omega_2 \left(f; \sqrt{\left(1 - \frac{1}{[n+1]_q}\right)^m x(1-x)} \right)$$

holds true for all $x \in [0,1]$ *and* $f \in C[0,1]$.

3.2 Iterates for the Cesaro operators

Define the Cesaro operator $C : C[0,1] \to C[0,1]$

$$C(f;x) = \begin{cases} f(0), & x = 0, \\ \frac{1}{x}\int_0^x f(s)\,ds, & 0 < x < 1, \\ \int_0^1 f(s)\,ds, & x = 1. \end{cases}$$

Simple calculations show that

$$C(e_0;x) = 1, \quad C(e_1;x) = \frac{x}{2}, \quad C(e_2;x) = \frac{x^2}{3},$$
$$C^m(e_0;x) = 1, \quad C^m(e_1;x) = \frac{x}{2^m}, \quad C^m(e_2;x) = \frac{x^2}{3^m}.$$

Hence an application of Theorem 6 yields the following statement.

Theorem 11 *Let* $C : C[0,1] \to C[0,1]$ *be a Cesaro operator. Then the pointwise approximation*

$$|T^m(f;x) - f(0)| \le k\omega_2\left(f, \sqrt{\lambda_m(x)}\right) + k\,\|f\|\,\delta_m(x)$$

holds true for $x \in [0,1]$ *and* $f \in C[0,1]$*, where* k *is an absolute constant and*

$$\lambda_m(x) = \max\left\{\frac{x}{2^m}, \frac{x^2}{3^m}\right\}, \quad \delta_m(x) = \frac{x}{2^m}.$$

3.3 Iterates of the genuine q-Bernstein-Durrmeyer operators $(0 < q \le 1)$

We consider now the genuine q-Bernstein–Durrmeyer operators

$$U_{n,q}(f;x) := f(0)\,p_{n,0}(q;x) + f(1)\,p_{n,n}(q;x) + [n-1]_q \sum_{k=1}^{n-1} q^{1-k} p_{n,k}(q;x) \int_0^1 p_{n-2,k-1}(q;qt)\,f(t)\,d_qt, \quad (15)$$

studied in [35]. Classical genuine Bernstein–Durrmeyer operators appeared first in papers W. Chen [29] and T.N.T. Goodman and A. Sharma [30].

In case of the genuine q-Bernstein–Durrmeyer operators we have

$$U_{n,q}(e_0;x) = 1, \quad U_{n,q}(e_1;x) = x,$$
$$U_{n,q}(e_2;x) = \left(1 - \frac{[2]_q}{[n+1]_q}\right)x^2 + \frac{[2]_q}{[n+1]_q}x,$$
$$U_{n,q}^m(e_2;x) = \left(1 - \frac{[2]_q}{[n+1]_q}\right)^m x^2 + \left[1 - \left(1 - \frac{[2]_q}{[n+1]_q}\right)^m\right]x.$$

Hence an application of Theorem 4 yields the following statement.

Theorem 12 *Let* $0 < q \le 1$*. Let* $U_{n,q}$ *be a sequence of genuine* q*-Bernstein–Durrmeyer operators. Then the pointwise approximation*

$$\left|U_{n,q}^m(f;x) - P(f;x)\right| \le k\omega_2\left(f; \sqrt{\left(1 - \frac{[2]_q}{[n+1]_q}\right)^m x(1-x)}\right)$$

holds true for all $x \in [0,1]$ *and* $f \in C[0,1]$.

3.4 Iterates of the q-Stancu operators $(0 < q \leq 1)$

We apply now the above results for iterates of the q-Stancu operators

$$S_{n,q}^{\langle \alpha,\beta,\gamma\rangle} : C[0,1] \ni f \to \sum_{k=0}^{n} f\left(\frac{[k]_q + [\beta]_q}{[n]_q + [\gamma]_q}\right) p_{n,k}(q;\alpha;\cdot) \in P_n,$$

$$p_{n,k}(q;\alpha;x) = \left[\begin{array}{c} n \\ k \end{array}\right]_q \frac{\prod_{i=0}^{k-1}\left(x + \alpha [i]_q\right) \prod_{s=0}^{n-k-1}\left(x - q^s x + \alpha [s]_q\right)}{\prod_{i=0}^{n-1}\left(1 + \alpha [i]_q\right)}.$$

q-Stancu operators $S_{n,q}^{\langle \alpha,0,0\rangle}$ were introduced and studied in [31]. Let $0 < q \leq 1$ and $\alpha \geq 0$. Then we have

$$S_{n,q}^{\langle \alpha,0,0\rangle}(1;x) = 1, \quad S_{n,q}^{\langle \alpha,0,0\rangle}(t;x) = x,$$

$$\left(S_{n,q}^{\langle \alpha,0,0\rangle}\right)^m (t^2;x) = \left(\left(1 - \frac{1}{[n]_q}\right)\frac{1}{1+\alpha}\right)^m x^2 + \left[1 - \left(\left(1 - \frac{1}{[n]_q}\right)\frac{1}{1+\alpha}\right)^m\right] x.$$

It is worthwhile to mention that already in 1978 G. Mastroianni & M. R. Occorsio [32] have introduced and investigated the iterates of $S_n^{\langle \alpha,0,0\rangle}$ $(q = 1)$ by extending a procedure used by R. P. Kelisky & T. J. Rivlin for the Bernstein operators.

q-Stancu operators $S_{n,q}^{\langle \alpha,\beta,\gamma\rangle}$ were introduced and studied in [36]. Let $\alpha = \beta = 0,\ \gamma > 0$. Then it is easy to show

$$S_{n,q}^{\langle 0,0,\gamma\rangle}(e_0;x) = 1, \quad S_{n,q}^{\langle 0,0,\gamma\rangle}(e_1;x) = \frac{[n]_q}{[n]_q + [\gamma]_q} x,$$

$$S_{n,q}^{\langle 0,0,\gamma\rangle}(e_2;x) = \frac{[n]_q^2 - [n]_q}{\left([n]_q + [\gamma]_q\right)^2} x^2 + \frac{[n]_q}{\left([n]_q + [\gamma]_q\right)^2} x,$$

$$\left(S_{n,q}^{\langle 0,0,\gamma\rangle}\right)^m (e_1;x) = \left(\frac{[n]_q}{[n]_q + [\gamma]_q}\right)^m x,$$

$$\left(S_{n,q}^{\langle 0,0,\gamma\rangle}\right)^m (e_2;x) = \left(\frac{[n]_q^2 - [n]_q}{\left([n]_q + [\gamma]_q\right)^2}\right)^m x^2 + \frac{[n]_q^m}{\left([n]_q + [\gamma]_q\right)^{m+1}} \sum_{i=0}^{m-1}\left(\frac{[n]_q - 1}{[n]_q + [\gamma]_q}\right)^i x.$$

We arrive at the following theorem.

Theorem 13 *Let $0 < q \leq 1$. Let $S_{n,q}^{\langle \alpha,\beta,\gamma\rangle}$ be a sequence of q-Stancu operators.*

1. *If $\alpha \geq 0,\ \beta = \gamma = 0$ then the pointwise approximation*

$$\left|\left(S_{n,q}^{\langle \alpha,0,0\rangle}\right)^m (f;x) - P(f;x)\right| \leq k\omega_2\left(f; \sqrt{\left(\left(1 - \frac{1}{[n]_q}\right)\frac{1}{1+\alpha}\right)^m x(1-x)}\right)$$

holds true for all $x \in [0,1]$ and $f \in C[0,1]$.

2. *If $\alpha = \beta = 0,\ \gamma > 0$ then*

$$\left|\left(S_{n,q}^{\langle 0,0,\gamma\rangle}\right)^m (f;x) - f(0)\right| \leq k\omega_2\left(f, \sqrt{\lambda_m(x)}\right) + k\|f\|\,\delta_m(x)$$

holds true for $x \in [0,1]$ *and* $f \in C[0,1]$, *where* k *is an absolute constant and*

$$\lambda_m(x) = \max\left\{ \left(S_{n,q}^{\langle 0,0,\gamma\rangle}\right)^m (e_1;x), \left(S_{n,q}^{\langle 0,0,\gamma\rangle}\right)^m (e_2;x) \right\},$$
$$\delta_m(x) = \left(S_{n,q}^{\langle 0,0,\gamma\rangle}\right)^m (e_1;x).$$

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