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CLASSES OF GENERALIZED FUNCTIONS WITH FINITE TYPE REGULARITIES

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Abstract. We introduce and analyze spaces and algebras of generalized functions which correspond to Sobolev, Zygmund and Hölder spaces of functions. The main scope of the paper is the characterization of the limited regularity of distributions embedded into the corresponding space or algebra of generalized functions with finite type regularities.

0. Introduction

In this paper we develop regularity theory in generalized function algebras parallel to the corresponding theory within distribution spaces. We consider subspaces or subalgebras in algebras of generalized functions which correspond to Sobolev spaces $W^{k,p}$, Zygmund spaces C^s_* and Hölder spaces $\mathcal{H}^{k,\rho}$. We refer to [2], [4], [5], [11] and [17] for the theory of generalized function algebras and their use in the study of various classes of equations.

It is known that the elements of algebras of generalized functions are represented by nets $(f_\varepsilon)_\varepsilon$ of smooth functions, with appropriate growth as $\varepsilon \to 0$, that the spaces of Schwartz's distributions are embedded into the corresponding algebras, and that for the space of smooth functions the corresponding algebra of smooth generalized function is $\mathcal{G}^\infty$ (see [17]). Intuitively, these algebras are obtained through regularization of distributions (convolving them with delta nets) and the construction of appropriate algebras of moderate nets and null nets of smooth functions and their quotients, as Colombeau did, [4], [5] with his algebra $\mathcal{G}(\mathbb{R}^d)$, in such a way that distributions are included as well as their natural linear operations (in this way the name Colombeau algebras has appeared).

The main goal of this paper is to find out conditions with respect to the growth order in ε which characterize generalized function spaces and algebras with finite type regularities. Actually, our main task is to seek optimal definitions for such generalized function spaces, since we would like to have backward information on the regularity properties of

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Schwartz distributions embedded into the corresponding space of generalized functions. Sobolev and Zygmund type spaces are very suitable for this purpose. Especially Zygmund type spaces are useful since we can almost literary transfer classical properties of these spaces into the corresponding ones within spaces or algebras of generalized functions.

One can find many articles in the literature where local and microlocal properties of generalized functions in generalized function algebras have been considered. Besides the quoted monographs we refer to the papers [6], [8]– [10], [12]– [14], [16]– [21], [24], where various classes of generalized function spaces and algebras were introduced and studied. The motivation of this article came partly from the papers [12] and [14], where Zygmund type algebra of generalized functions were studied and used in the qualitative analysis of certain hyperbolic problems. We shall give another definition which intrinsically characterizes such spaces of generalized functions and allows us to connect them with Hölder type spaces. Moreover, our results concerning Tauberian theorems for regularizing transforms, [22], [26]- [29] (see also [7]), enable us to consider regularity properties of generalized functions.

The paper is organized as follows. We recall in Subsection 1.1 the notion of the valuation v for Colombeau generalized functions and L^p generalized functions (calling $-v$ the calibration function) and in Section 2 explain the main definitions of Sobolev type spaces and algebras of generalized functions. Then, in Section 3, we investigate subspaces and subalgebras which correspond to Sobolev function spaces and study the properties of distributions for which we know an appropriate weak limit growth, called association, after embedding them into the corresponding generalized function space or algebra. In Section 4 we analyze subspaces of the generalized function algebra $\mathcal{G}(\mathbb{R}^d)$ which correspond to Zygmund and Hölder spaces of functions and give the characterization of distributions belonging, after embeddings, to these generalized function spaces. We recall in Section 5 the regularizing transform, generalized boundedness and by the use of our Tauberian results ([22]) give a new proof for the equality $\mathcal{G}^\infty \cap \mathcal{D}' = C^\infty$ which provides a new perspective in the analysis of regularity properties of distributions and functions.

1. Definitions

In the sequel, we shall consider open subsets $\Omega \subset \mathbb{R}^d$ whose boundary $\partial\Omega$ satisfies the strong local Lipschitz condition, implying the existence of a total extension operator for Ω (see Chapter 4 in [1]) which gives a continuous extension of Sobolev functions out of Ω. This implies, for every $p \in [1,\infty]$,

$$\mathcal{D}_{L^p}(\Omega) = \bigcap_{m\in\mathbb{N}_0} W^{m,p}(\Omega) \subset \left\{\phi \in C^\infty(\Omega) : ||\phi^{(\alpha)}||_{L^\infty(\Omega)} < \infty,\ \forall\alpha \in \mathbb{N}_0^d\right\}, \tag{1.1}$$

where $W^{m,p}$ denotes the Sobolev space, $p \in [1,\infty]$, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Let $\mathcal{E}(\Omega)$ be the space of smooth functions in Ω. We consider the families of local Sobolev seminorms $||\phi||_{W^{m,p}(\omega)} = \sup\{||\phi^{(\alpha)}||_{L^p(\omega)};\ |\alpha| \leq m\}$, where $m \in \mathbb{N}_0$, $p \in [1,\infty]$, and ω runs over all open subsets of Ω with compact closure ($\omega \subset\subset \Omega$); in case ω is replaced by Ω, we obtain a family of norms $||\phi||_{W^{m,p}(\Omega)}, m \in \mathbb{N}_0$.

The spaces of moderate nets and negligible nets $\mathcal{E}_{L^p_{loc},M}(\Omega)$ and $\mathcal{N}_{L^p_{loc}}(\Omega)$, resp., $\mathcal{E}_{L^p,M}(\Omega)$ and $\mathcal{N}_{L^p}(\Omega)$, consist of nets $(f_\varepsilon)_{\varepsilon\in(0,1)} = (f_\varepsilon)_\varepsilon \in \mathcal{E}(\Omega)^{(0,1)}$ with the properties

$$(\forall m \in \mathbb{N}_0)(\forall\omega \subset\subset \Omega)(\exists a \in \mathbb{R})(||f_\varepsilon||_{W^{m,p}(\omega)} = O(\varepsilon^a)) \tag{1.2}$$

$$\text{and}\ \ (\forall m \in \mathbb{N}_0)(\forall\omega \subset\subset \Omega)(\forall b \in \mathbb{R})(||f_\varepsilon||_{W^{m,p}(\omega)} = O(\varepsilon^b)),$$

resp.,

$$(\forall m \in \mathbb{N}_0)(\exists a \in \mathbb{R})(||f_\varepsilon||_{W^{m,p}(\Omega)} = O(\varepsilon^a)) \tag{1.3}$$

$$\text{and}\ \ (\forall m \in \mathbb{N}_0)(\forall b \in \mathbb{R})(||f_\varepsilon||_{W^{m,p}(\Omega)} = O(\varepsilon^b))$$

(big O and small o are the Landau symbol). These spaces are actually algebras because of (1.1). Note that, by Sobolev lemma,

$$\mathcal{E}_{L^\infty_{loc},M}(\Omega) = \mathcal{E}_{L^p_{loc},M}(\Omega) = \mathcal{E}_M(\Omega),\ \ \mathcal{N}_{L^\infty_{loc}}(\Omega) = \mathcal{N}_{L^p_{loc}}(\Omega) = \mathcal{N}(\Omega),\ p \geq 1.$$

We obtain the Colombeau algebra of generalized functions as a quotient: $\mathcal{G}(\Omega) = \mathcal{G}_{L^\infty_{loc}}(\Omega) = \mathcal{G}_{L^p_{loc}}(\Omega),\ p \geq 1$. Furthermore, we define the L^p type generalized function algebra $\mathcal{G}_{L^p}(\Omega) = \mathcal{E}_{L^p,M}(\Omega)/\mathcal{N}_{L^p}(\Omega),\ p \geq 1$. Let us note that $\mathcal{E}_{L^p,M}(\Omega) \subset \mathcal{E}_M(\Omega)$ is differential subalgebra and $\mathcal{N}_{L^p}(\Omega) \subset \mathcal{N}(\Omega)$. Thus, there is a canonical differential algebra mapping $\mathcal{G}_{L^p}(\Omega) \to \mathcal{G}(\Omega)$; however, in general, this mapping is not injective. So, $\mathcal{G}_{L^p}(\Omega)$ cannot be seen as a differential subalgebra of $\mathcal{G}(\Omega)$, $p \in [1,\infty]$, we exhibit an explicit counterexample below when $\Omega = \mathbb{R}^d$. Observe that the same holds for the so called tempered generalized function algebra (see Chapter 4 in [5]), which will not be considered in this paper.

Example 1. We show that the canonical mapping $\mathcal{G}_{L^p}(\mathbb{R}^d) \to \mathcal{G}(\mathbb{R}^d)$ is not injective, $p \in [1,\infty]$. Indeed, this would be the case if we are able to find a net $(f_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^d)\cap\mathcal{E}_{L^p,M}(\mathbb{R}^d)$ which does not belong to $\mathcal{N}_{L^p}(\mathbb{R}^d)$. Actually, the same counterexample will work for all $p \in [1,\infty]$. Let $\rho \in \mathcal{D}(\mathbb{R}^d)$ be non-trivial and supported by the ball with center at the origin and radius 1/2. Consider the net of smooth functions

$$f_\varepsilon(x) = \sum_{n=0}^{\infty} \frac{\chi_{[(n+1)^{-1},1]}(\varepsilon)}{(n+1)^2}\rho(x - 2ne_1),$$

where $\chi_{[(n+1)^{-1},1]}$ is the characteristic function of the interval $[1/(n+1),1]$ and $e_1=(1,0,\dots,0)$. Then, clearly $(f_\varepsilon)_\varepsilon\in\mathcal{N}(\mathbb{R}^d)$ because on compact sets it identically vanishes for small enough ε. On the other hand, for each $p\in[1,\infty]$ and $m\in\mathbb{N}_0$,

$$||f_\varepsilon||_{W^{m,p}}=||\rho||_{W^{m,p}}\sum_{\frac{1}{\varepsilon}-1\le n}^{\infty}\frac{1}{(n+1)^2}=\varepsilon\,||\rho||_{W^{m,p}}+O(\varepsilon^2)\text{ as }\varepsilon\to 0.$$

Thus, the net satisfies all the requirements.

If we would like to have a version of $\mathcal{G}_{L^p}(\Omega)$ as a differential subalgebra of $\mathcal{G}(\Omega)$ then we could do this through the following definition:

$$\tilde{\mathcal{G}}_{L^p}(\Omega)=\{f\in\mathcal{G}(\Omega);\ \exists(f_\varepsilon)_\varepsilon\in\mathcal{E}_{L^p,M}(\Omega),\ f=[(f_\varepsilon)_\varepsilon]\}.$$

We shall consider subspaces and subalgebras of $\tilde{\mathcal{G}}_{L^p}(\Omega)$ in Sections 2 and 3.

If the elements of the nets $(f_\varepsilon)_\varepsilon\in\mathcal{E}_M(\Omega)$ are constant functions in Ω (i.e., seminorms reduce to the absolute value), then one obtains the corresponding algebras $\mathcal{E}_0$ and $\mathcal{N}_0$; $\mathcal{N}_0$ is an ideal in $\mathcal{E}_0$ and, as their quotient, one obtains the Colombeau algebra of generalized complex numbers $\bar{\mathbb{C}}=\mathcal{E}_0/\mathcal{N}_0$ (or $\bar{\mathbb{R}}$). It is a ring, not a field.

The embedding of the Schwartz distribution space $\mathcal{E}'(\Omega)$ into $\mathcal{G}(\Omega)$ is realized through the sheaf homomorphism $\mathcal{E}'(\Omega)\ni T\mapsto\iota(T)=[(T*\phi_\varepsilon|\Omega)_\varepsilon]\in\mathcal{G}(\Omega)$, where the fixed net of mollifiers $(\phi_\varepsilon)_\varepsilon$ is defined by $\phi_\varepsilon=\varepsilon^{-d}\phi(\cdot/\varepsilon),\ \varepsilon<1$, and $\phi\in\mathcal{S}(\mathbb{R}^d)$ satisfies

$$\int\phi(t)dt=1,\ \int t^m\phi(t)dt=0,\ m\in\mathbb{N}_0^d,\ |m|>0.$$

($t^m=t_1^{m_1}...t_d^{m_d}$ and $|m|=m_1+...+m_d$). This sheaf homomorphism, extended over $\mathcal{D}'$, gives the embedding of $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega)$, $\Omega\subset\mathbb{R}^d$. We also use the notation ι for the mapping from $\mathcal{E}'(\Omega)$ to $\mathcal{E}_M(\Omega)$, $\iota(T)=(T*\phi_\varepsilon|\Omega)_\varepsilon$. Throughout this article, ϕ will always be fixed and satisfy the above condition over its moments.

The generalized algebra of "smooth generalized functions" $\mathcal{G}^\infty(\Omega)$ is defined in [17] as the quotient of the algebras $\mathcal{E}_M^\infty(\Omega)$ and $\mathcal{N}(\Omega)$, where $\mathcal{E}_M^\infty(\Omega)$ consists of those nets $(f_\varepsilon)_\varepsilon\in\mathcal{E}(\Omega)^{(0,1)}$ with the property

$$(\forall\omega\subset\subset\Omega)(\exists a\in\mathbb{R})(\forall\alpha\in\mathbb{N})(\sup_{|\alpha|\le m}||f_\varepsilon^{(\alpha)}(x)||_{L^\infty(\omega)}=O(\varepsilon^a)).$$

Observe that $\mathcal{G}^\infty$ is a subsheaf of $\mathcal{G}$; it has a similar role as C^∞ in $\mathcal{D}'$.

We will use a continuous Littlewood-Paley decomposition of the unity (see [15], Section 8.4, for instance). Let $\theta\in\mathcal{D}(\mathbb{R}^d)$ be a real valued radial (independent on rotations) function with support contained in the unit ball such that $\theta(y)=1$ if $|y|\le 1/2$. Set $\zeta(y)=-\frac{d}{d\varepsilon}\theta(\varepsilon y)|_{\varepsilon=1}=-y\cdot\nabla\theta(y)$. The support of ζ is contained in the set $1/2\le|y|\le 1$. An easy calculation shows that $1=\theta(y)+\int_0^1\eta^{-1}\zeta(\eta y)d\eta$. Set $\varphi=\mathcal{F}^{-1}(\theta)$

and $\psi = \mathcal{F}^{-1}(\zeta)$, then φ is a mollifier and ψ is a wavelet (all the moments of ψ are equal to zero). In addition, one has $\theta(\varepsilon D)u = u * \varphi_\varepsilon$ and $\zeta(\varepsilon D)u = u * \psi_\varepsilon$, thus ([15], Section 8.6): for any $u \in \mathcal{S}'(\mathbb{R}^d)$,

$$u = u * \varphi + \int_0^1 u * \psi_\eta \frac{d\eta}{\eta} = u * \varphi_\varepsilon + \int_0^\varepsilon u * \psi_\eta \frac{d\eta}{\eta},\ 0 < \varepsilon \leq 1, \quad (1.4)$$

$$u * \varphi_\varepsilon = u * \varphi + \int_\varepsilon^1 u * \psi_\eta \frac{d\eta}{\eta},\ 0 < \varepsilon \leq 1. \quad (1.5)$$

1.1. **Growth function.** Let $g \in \mathcal{G}(\Omega)$ and $\omega \subset\subset \Omega$. We will define a growth function $c_{g,\omega}$ motivated by the results of [24] and [25]. First, we repeat the definition of the usual valuation. Let $\omega \subset\subset \Omega$, $m \in \mathbb{N}_0$, $(g_\varepsilon)_\varepsilon \in \mathcal{E}_M(\Omega)$ and $A^m_{g_\varepsilon,\omega} = \{a \in \mathbb{R}; \varepsilon^a ||g_\varepsilon||_{W^{m,\infty}(\omega)} = O(1)\}$. Then the valuation of $(g_\varepsilon)_\varepsilon$ is defined by $v_{\omega,m}((g_\varepsilon)_\varepsilon) = \sup\{-a,\ a \in A^m_{g_\varepsilon,\omega}\}$. In order to simplify the notation we introduce the notion of calibration, that is, $c_{\omega,m}((g_\varepsilon)_\varepsilon)$ given by:

$$c_{\omega,m}((g_\varepsilon)_\varepsilon) = -v_{\omega,m}((g_\varepsilon)_\varepsilon) = \inf\{a,\ a \in A^m_{g_\varepsilon,\omega}\}.$$

If $c_{\omega,m}(g_\varepsilon) = s \in A^m_{g_\varepsilon,\omega}$, then we say that the calibration is reached at s. The valuations (and calibrations) of different representatives of $g \in \mathcal{G}(\Omega)$ are obviously the same, thus we obtain a family of semi-ultrametrics on $\mathcal{E}_M(\Omega)$, resp., on $\mathcal{G}(\Omega)$, defined by

$$P_{\omega,m}((g_\varepsilon)_\varepsilon) = P_{\omega,m}(g) = e^{c_{\omega,m}((g_\varepsilon)_\varepsilon)}.$$

This family defines the so called sharp topology in $\mathcal{E}_M(\Omega)$, resp., in $\mathcal{G}(\Omega)$. Now, for every $g = [(g_\varepsilon)_\varepsilon]$ and $\omega \subset\subset \Omega$, we define the growth function

$$c_{g,\omega} : \mathbb{N}_0 \to [-\infty, \infty),\ c_{g,\omega}(m) = c_{\omega,m}(g).$$

Clearly, it is an increasing function for any ω and $g \in \mathcal{G}^\infty(\Omega)$. By a theorem of Vernaeve, [25], one has $c_{g,\omega}(j+1) \leq (c_{g,\omega}(j) + c_{g,\omega}(j+2))/2$, for every $\omega \subset\subset \Omega$ and $j \in \mathbb{N}_0$. Therefore, $c_{g,\omega}$ is a convex function. This implies that there are only two possibilities for the growth function on an open connected set $\omega \subset\subset \Omega$: either it is precisely a constant function, or it is constant up to some $m_0 \in \mathbb{N}$ and then becomes a strictly increasing function.

It is proved in [25] that $\mathcal{G}^\infty(\Omega)$ is closed in the sense of the sharp topology in $\mathcal{G}(\Omega)$. We will not consider topological questions in this paper and we thus refer to [20], and [25] for more information concerning the sharp topology.

The growth functions for $\mathcal{E}_{L^p,M}(\Omega)$ (and the corresponding algebras) are defined by

$$c_{g_\varepsilon,L^p}(m) = c_{L^p,m}(g_\varepsilon),\ m \in \mathbb{N}_0,$$

where $c_{L^p,m}(g_\varepsilon) = \inf\{a; a \in A^m_{g_\varepsilon,\Omega,p}\}$ and $A^m_{g_\varepsilon,\Omega,p} = \{a \in \mathbb{R}; \varepsilon^a ||g_\varepsilon||_{W^{m,p}(\Omega)} = O(1)\}$. Once again, if $c_{L^p,m}(g_\varepsilon) = s \in A^m_{g_\varepsilon,\Omega,p}$, then we say that the calibration is reached at s.

Example 2. Let $f \in W^{k,p}(\mathbb{R}) \setminus W^{k+1,p}(\mathbb{R})$, $p \geq 1$, $k \in \mathbb{N}_0$ and let ϕ be a mollifier. Then $g = [(f(\varepsilon x) * f(x) * \phi_\varepsilon)_\varepsilon$ satisfies $c_{g,L^p}(m) = c_{g,L^p}(0) = 0, m \leq 3k$ and $c_{g,L^p}(3k+i) = i$.

2. Sobolev type spaces and algebras of generalized functions

2.1. $L^\infty_{loc} - L^p-$ and $(k,s)-$ generalized functions.

In this paper we are interested in nets $(f_\varepsilon)_\varepsilon \in \mathcal{E}_M(\Omega)$ such that for given $k \in \mathbb{N}_0$ there exists $\tilde{s} \in \mathbb{R}$ such that

$$(\forall \omega \subset\subset \Omega)(\forall i \in \mathbb{N}_0^d, |i| \leq k)(\varepsilon^{\tilde{s}} \|\partial^i f_\varepsilon\|_{L^\infty(\omega)} = O(1), \varepsilon \to 0),$$

$$(\exists \omega_1 \subset\subset \Omega)(\exists i \in \mathbb{N}_0^d, |i| = k+1)(\varepsilon^{\tilde{s}} \|\partial^i f_\varepsilon\|_{L^\infty(\omega)} \neq O(1), \varepsilon \to 0).$$

Such nets will be the representatives of, roughly speaking, $W^{k,\infty}_{loc}-$generalized functions. Similarly, for $p \in [1, \infty]$, we can consider $W^{k,p}-$generalized functions. The precise definitions are as follows.

Definition 1. Let Ω be an open set in $\mathbb{R}^d$ and $(f_\varepsilon)_\varepsilon \in \mathcal{E}_M(\Omega)$, resp., $(f_\varepsilon)_\varepsilon \in \mathcal{E}_{L^p,M}(\Omega)$. Let $s \in \mathbb{R} \cup \{-\infty\}$ and $k \in \mathbb{N}_0$. Then we say that $(f_\varepsilon)_\varepsilon$ is of class (k,s) if

$$(\forall \omega \subset\subset \Omega)((c_{g,\omega}(i) = s, 0 \leq i \leq k) \wedge (\exists \omega_1 \subset\subset \Omega)(c_{g,\omega_1}(k+1) > s) \tag{2.1}$$

$$\text{resp., } ((c_{g,L^p}(i) = s, 0 \leq i \leq k) \wedge (c_{g,L^p}(k+1) > s) \tag{2.2}$$

and there are no other $s_1 < s$ and $k_1 \in \mathbb{N}_0$ so that (2.1), resp., (2.2), holds with (k_1, s_1).

It is said that $(f_\varepsilon)_\varepsilon$ is of class (∞, s) if $s \in \mathbb{R}$ is the infimum of all $\tilde{s}$ for which

$$(\forall \omega \subset\subset \Omega)(\forall i \in \mathbb{N}_0^d)(\varepsilon^{\tilde{s}} \|\partial^i f_\varepsilon\|_{L^\infty(\omega)} = O(1),\ \varepsilon \to 0),$$

$$\text{resp., } (\forall i \in \mathbb{N}_0^d)(\varepsilon^{\tilde{s}} \|\partial^i f_\varepsilon\|_{L^p(\Omega)} = O(1),\ \varepsilon \to 0).$$

It is said that $(f_\varepsilon)_\varepsilon$ is of class $(\infty, -\infty)$ if

$$(\forall \omega \subset\subset \Omega)(\forall k \in \mathbb{N}_0^d)(\forall s \in \mathbb{R})(\varepsilon^{s} \|\partial^k f_\varepsilon\|_{L^\infty(\omega)} = O(1),\ \varepsilon \to 0),$$

$$\text{resp., } (\forall k \in \mathbb{N}_0^d)(\forall s \in \mathbb{R})(\varepsilon^{s} \|\partial^k f_\varepsilon\|_{L^p(\Omega)} = O(1),\ \varepsilon \to 0).$$

Let $s \in [-\infty, \infty),\ k \in \mathbb{N}_0 \cup \{\infty\}$. It is said that $(f_\varepsilon)_\varepsilon \in \widetilde{\mathcal{E}}_M^{(k,s)}$, resp., $\widetilde{\mathcal{E}}^{(k,s)}_{L^p(\Omega),M}$ if it is of class (k,s). By definition, $\widetilde{\mathcal{E}}_M^{(k,\infty)} = \emptyset$, resp., $\widetilde{\mathcal{E}}^{(k,\infty)}_{L^p(\Omega),M} = \emptyset, k \in \mathbb{N}_0 \cup \{\infty\}$.

The following two propositions are direct consequences of these definitions; we therefore omit their proofs.

Proposition 1. *a)* $(f_\varepsilon)_\varepsilon \in \widetilde{\mathcal{E}}_M^{(k,s)}(\Omega)$ *if and only if* $(\varepsilon^s f_\varepsilon)_\varepsilon \in \widetilde{\mathcal{E}}_M^{(k,0)}(\Omega)$.

b) Let $\Omega_1 \subset \Omega$. *If* $(f_\varepsilon)_\varepsilon \in \widetilde{\mathcal{E}}_M^{(k,s)}(\Omega)$, *then there exist* $k_1 \geq k$ *and* $s_1 \leq s$ *such that* $(f_{\varepsilon|\Omega_1})_\varepsilon \in \widetilde{\mathcal{E}}_M^{(k_1,s_1)}(\Omega_1)$.

c) $\widetilde{\mathcal{E}}_M^{(k,s)}(\Omega) \neq \widetilde{\mathcal{E}}_M^{(k_1,s)}(\Omega)$ *if* $k \neq k_1$ *and* $\widetilde{\mathcal{E}}_M^{(k,s)}(\Omega) \neq \widetilde{\mathcal{E}}_M^{(k,s_1)}(\Omega)$ *if* $s \neq s_1$.

d) Let $(f_\varepsilon)_\varepsilon \in \widetilde{\mathcal{E}}_M^{(k_1,s_1)}(\Omega), (g_\varepsilon)_\varepsilon \in \widetilde{\mathcal{E}}_M^{(k_2,s_2)}(\Omega)$. *Then* $(f_\varepsilon + g_\varepsilon)_\varepsilon \in \widetilde{\mathcal{E}}_M^{(r,p)}(\Omega)$, *for some* $r \geq \min\{k_1,k_2\}$ *and* $p \leq \max\{s_1,s_2\}$, *as well as* $(f_\varepsilon g_\varepsilon)_\varepsilon \in \widetilde{\mathcal{E}}_M^{(r,p)}(\Omega)$ *for some* $r \geq \min\{k_1,k_2\}$ *and* $p \leq s_1+s_2$.

e) $\widetilde{\mathcal{E}}_M^{(\infty,-\infty)}(\Omega) = \mathcal{N}(\Omega)$ *(so, it is an algebra).*

f) If for every $\kappa \in C_0^\infty(\Omega)$ *there exist* $k \geq k_0$ *and* $s \leq s_0$ *such that* $\kappa f_\varepsilon \in \widetilde{\mathcal{E}}_M^{(k,s)}(\Omega)$, *then there exist* $\tilde{k} \geq k_0$ *and* $\tilde{s} \leq s_0$ *such that* $f_\varepsilon \in \widetilde{\mathcal{E}}_M^{(\tilde{k},\tilde{s})}(\Omega)$.

Proposition 2. *The corresponding assertions a), b), c), d) and e) of Proposition 1 hold if one considers the spaces* $\widetilde{\mathcal{E}}_{L^p,M}^{(k,s)}(\Omega)$.

Definition 2. Let $k \in \mathbb{N}_0 \cup \{\infty\}, s \in \mathbb{R} \cup \{\infty\}$. Set

$$\mathcal{E}_M^{k,s}(\Omega) := \bigcup_{h\in[-\infty,s],i\in\{k,k+1,\ldots\}\cup\{\infty\}} \widetilde{\mathcal{E}}_M^{i,h}(\Omega),$$

$$\mathcal{G}^{k,s}(\Omega) := \mathcal{E}_M^{k,s}(\Omega)\Big/\mathcal{N}(\Omega),$$

$$\mathcal{E}_{L^p,M}^{k,s}(\Omega) := \bigcup_{h\in[-\infty,s],i\in\{k,k+1,\ldots\}\cup\{\infty\}} \widetilde{\mathcal{E}}_{L^p,M}^{i,h}(\Omega),$$

$$\mathcal{G}_{L^p}^{k,s}(\Omega) := \{u \in \mathcal{G}(\Omega); \exists (f_\varepsilon)_\varepsilon \in \mathcal{E}_{L^p,M}^{k,s}(\Omega),\ u = [(f_\varepsilon)_\varepsilon]\}.$$

Propositions 2 and 1 imply the next result.

Proposition 3. *(i)* $\mathcal{G}^{\infty,\infty}(\Omega)$, *resp.,* $\mathcal{G}_{L^p}^{\infty,\infty}(\Omega)$ *and* $\mathcal{G}^{k,0}(\Omega), k \in \mathbb{N}_0 \cup \{\infty\}$, *resp.,* $\mathcal{G}_{L^p}^{k,0}(\Omega), k \in \mathbb{N}_0 \cup \{\infty\}$, *are algebras.*

(ii) $\mathcal{G}^{k,s}(\Omega)$, *resp.,* $\mathcal{G}_{L^p}^{k,s}(\Omega)$, $k \in \mathbb{N}_0 \cup \{\infty\}, s \neq 0, -\infty$, *are vector spaces.*

(iii) Let $P(D) = \sum_{|\alpha|\leq m} a_\alpha D^\alpha$ *be a differential operator of order* m *with constant coefficients* ($D^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1}\ldots\partial_{x_d}^{\alpha_d}}$) *so that* $m \leq k \in \mathbb{N}$. *Then* $P(D): \mathcal{G}^{k,s}(\Omega) \to \mathcal{G}^{k-m,s}(\Omega)$, *resp.,* $P(D): \mathcal{G}_{L^p}^{k,s}(\Omega) \to \mathcal{G}_{L^p}^{k-m,s}(\Omega)$.

(iv)

$$\mathcal{G}^{k_1,s_1}(\Omega) \subset \mathcal{G}^{k_2,s_2}(\Omega),\ \textit{resp.},\ \mathcal{G}_{L^p}^{k_1,s_1}(\Omega) \subset \mathcal{G}_{L^p}^{k_2,s_2}(\Omega)$$

if and only if $k_1 \geq k_2, s_1 \leq s_2$.

Proof. Parts (i), (ii), and (iv) follow immediately from Proposition 2, Proposition 1 and the definitions. Let us prove (iii). It is enough to consider $\partial_{x_1}^{\alpha_1}$ in the case of $\mathcal{G}^{k,s}(\Omega)$. Since $k \geq 1$, we have that $\varepsilon^{\tilde{s}}|f_\varepsilon^{(m+(1,\ldots,0))}(x)| \leq \infty$, $|m| \leq k-1$, where $\tilde{s}$ cannot be larger than s, the assertion follows. □

If $(f_\varepsilon)_\varepsilon \in \mathcal{E}^{k,0}_{L^p,M}(\Omega)$, then, by Sobolev lemma [1], $(f_\varepsilon)_\varepsilon \in \mathcal{E}^{m,0}_M(\Omega)$ for $m < k - n/p$ (we use here the assumption over $\partial\Omega$, see Section 1). Therefore, $\mathcal{G}^{k,0}_{L^p}(\Omega) \subset \mathcal{G}^{m,0}(\Omega)$.

We compare $\mathcal{E}^{\infty,\infty}_M(\Omega)$ with $\mathcal{E}^{\infty}_M(\Omega)$. Clearly, we have the inclusion $\mathcal{E}^{\infty,\infty}_M(\Omega) \subset \mathcal{E}^{\infty}_M(\Omega)$. Conversely, if $(f_\varepsilon)_\varepsilon \in \mathcal{E}^{\infty}_M(\Omega)$, then $(f_{\varepsilon|\Omega'})_\varepsilon \in \mathcal{E}^{\infty,\infty}_M(\Omega')$ for every bounded open subset $\Omega' \subset\subset \Omega$.

Example 3. 1. Let $f \in C^\infty(\mathbb{R})$. Then $\log(1/\varepsilon) f(\,\cdot \log(1/\varepsilon)) \in \widetilde{\mathcal{E}}^{\infty,0}_M(\mathbb{R})$ and $\iota(f) \in \mathcal{G}^{\infty,0}(\Omega)$.

2. We say that $f \in \mathcal{E}'(\Omega)$ is strictly of order $k \in \mathbb{N}$ if it is of the form $f = \partial^\alpha F$, where $F \in L^\infty_{loc}(\Omega)$, $|\alpha| = k$ and there is no other L^∞_{loc}-function F_1 and a multi-index α_1 such that $f = \partial^{\alpha_1} F_1$ and $|\alpha_1| < k$. Then $\iota(f) = (f * \phi_{\varepsilon|\Omega})_\varepsilon \in \widetilde{\mathcal{E}}^{0,k-\sigma}_M(\Omega)$ with $0 \le \sigma < 1$, and hence $\iota(f) \in \mathcal{G}^{0,k}(\Omega)$.

3. For the delta distribution we have $\iota(\delta^{(m)}) \in \widetilde{\mathcal{E}}^{0,m+1}_M(\mathbb{R}), m \in \mathbb{N}_0$.

4. The embedding of a locally bounded function f, which does not have locally bounded partial derivatives, belongs to $\widetilde{\mathcal{E}}^{0,0}_{L^\infty_{loc},M}(\Omega)$. Since any $f \in \mathcal{E}'(\Omega)$ is of the form $f = \sum_{i=0}^{l} f_i^{(i)}$, where f_i are compactly supported locally bounded functions, it follows that $\iota(f)$ is of class (k,s), for some $k \in \mathbb{N}_0 \cup \{\infty\}$ and $s \in [-\infty, \infty)$.

Example 4. The assumption on $\partial\Omega$ (see the beginning of Section 1) implies that any element of a Sobolev space over Ω can be extended out of Ω to $\tilde{u}$ so that $\tilde{u}$ belongs to the corresponding Sobolev space over $\mathbb{R}^d$ in such a way the extension operator is continuous. So $\tilde{u} * \phi_{\varepsilon_{|\Omega}}, \varepsilon \in (0,1)$, is a representative of $\iota(u)$ in the corresponding algebra of generalized functions.

Let $u \in W^{-m,p}(\Omega), p \ge 1$. Then by the structural theorem (see [1]) there exist functions $v_\alpha \in L^p(\Omega)$, $|\alpha| \le m$, such that $u = \sum_{|\alpha| \le m} \partial^\alpha v_\alpha$ in the distributional sense. Let $(\phi_\varepsilon)_\varepsilon$ be a mollifier. Then a representative of $\iota(u) \in \mathcal{G}_{L^p}(\Omega)$ is given by $u_\varepsilon = \sum_{|\alpha| \le m} \partial^\alpha \tilde{v}_\alpha * \phi_\varepsilon, \varepsilon < 1$, so that

$$\|u_\varepsilon\|_{L^p} = \| \sum_{|\alpha| \le m} \partial^\alpha \tilde{v}_\alpha * \phi_\varepsilon \|_{L^p}$$

$$\le \| \sum_{|\alpha| \le m} \varepsilon^{-|\alpha|} \tilde{v}_\alpha * \partial^\alpha \phi_\varepsilon \|_{L^p} \le C \sum_{|\alpha| \le m} \varepsilon^{-|\alpha|} \|v_\alpha\|_{L^p} \|(\partial^\alpha \phi)_\varepsilon\|_{L^1} \le C C_u C_\phi \varepsilon^{-m},$$

where $C_u = \sum_{|\alpha| \le m} \|v_\alpha\|_{L^p}$, $C_\phi = \max_{|\alpha| \le m} \|\partial^\alpha \phi\|_{L^1}$, and the constant C comes from the extension operator.

Thus, there exists $r \le m$, such that $(u_\varepsilon)_\varepsilon \in \widetilde{\mathcal{E}}^{0,r}_{L^p,M}$ and so $\iota(u) \in \mathcal{G}^{0,r}_{L^p}(\Omega)$.

3. Characterizations through associations

Recall, generalized functions $f, g \in \mathcal{G}(\Omega)$ are associated, $f \sim g$, or their representatives are associated, $(f_\varepsilon)_\varepsilon \sim (g_\varepsilon)_\varepsilon$, if $f_\varepsilon - g_\varepsilon \to 0$ in $\mathcal{D}'(\Omega)$. Observe that if $T \in \mathcal{D}'(\Omega)$, then $f \sim \iota(T)$ if and only if $\lim_{\varepsilon\to 0^+} f_\varepsilon = T$ in $\mathcal{D}'(\Omega)$; we write in this case $(f_\varepsilon)_\varepsilon \sim T$.

Let $r \in \mathbb{R}$; the Zygmund space $C_*^r(\mathbb{R}^d) = C_*^r$ is defined by

$$C_*^r := \{u \in \mathcal{S}' : \ |u|_*^r := ||\varphi * u||_{L^\infty} + \sup_{\varepsilon<1}(\varepsilon^{-r}||\psi_\varepsilon * u||_{L^\infty} < \infty\}, \quad (3.1)$$

where φ and ψ form a continuous Littlewood-Paley decomposition (see (1.4) and (1.5) in Section 1). It can be shown that the definition is independent on the choice of the Littlewood-Paley decomposition (see Section 8.6 in [15]). We refer to [15] for further properties of Zygmund spaces.

Theorem 1. *a) Let $k \in \mathbb{N}_0 \cup \{\infty\}$. Then $\iota(W^{k,\infty}(\Omega)) \subset \mathcal{G}^{k,0}_{L^\infty}(\Omega) \subset \mathcal{G}^{k,0}(\Omega)$.*

b) Let $T \in \mathcal{E}'(\Omega)$. If $\iota(T) \in \mathcal{G}^{k,0}_{L^\infty}(\Omega)$ for some $k \in \mathbb{N}$ and the calibration, $c_{\Omega,k}(\iota(T)) = 0$, is reached at zero, then $T \in W^{k,\infty}(\Omega)$, namely, $T^{(\alpha)} \in L^\infty(\Omega)$, $|\alpha| \le k$.

c) Let $T \in \mathcal{S}'(\mathbb{R}^d)$ such that its Fourier transformation is a bounded function. Then $\iota(T) \in \mathcal{G}^{0,d}(\mathbb{R}^d)$.

Proof. a) This follows easily from the definition.

b) We have that for every $|\alpha| \le k$, $T^{(\alpha)} * \phi_\varepsilon$, $\varepsilon < 1$, is a bounded net of smooth functions and thus it has a weakly* convergent subsequence in $L^\infty(\Omega)$, and it also converges to $T^{(\alpha)}$ in $\mathcal{D}'(\Omega)$. Therefore, $T^{(\alpha)} \in L^\infty(\Omega)$.

c) We have that

$$|T * \phi_\varepsilon(x)| \le |\mathcal{F}^{-1}(\hat{T}\hat{\phi}_\varepsilon)(x)| \le \int_{\mathbb{R}^d} |\hat{T}(\xi)\hat{\phi}(\varepsilon\xi)| dx \le M/\varepsilon^d, \ \varepsilon < 1,$$

hence c) follows. □

The next three theorems are the main results of this section.

Theorem 2. *Let $T \in \mathcal{E}'(\Omega)$, $k \in \mathbb{N}_0$ and $(f_\varepsilon)_\varepsilon \in \mathcal{E}^{k,0}_M(\Omega)$ such that*

$$\operatorname{supp} T, \operatorname{supp} f_\varepsilon \subset \omega \subset\subset \Omega, \ \varepsilon < 1. \quad (3.2)$$

(1) Suppose that the calibration of $(f_\varepsilon)_\varepsilon$ ($c_{\omega,k}((f_\varepsilon)_\varepsilon) = 0$) is reached at zero. Assume further that $(f_\varepsilon)_\varepsilon \sim T$, i.e.,

$$(\forall \rho \in \mathcal{D}(\Omega))(\langle T - f_\varepsilon, \rho\rangle = o(1), \ \varepsilon \to 0^+). \quad (3.3)$$

Then

$$(\exists C > 0)(\forall \xi \in \mathbb{R}^d)(|\hat{T}(\xi)| \le C(1 + |\xi|^2)^{-k/2}). \quad (3.4)$$

In particular, $\iota(T) \in \mathcal{G}^{k,d}(\Omega)$ and $T \in C_*^{k-d}$.

Remark 1. The conclusion holds, in particular, if $(f_\varepsilon)_\varepsilon \in \mathcal{E}^{k,s}_M(\Omega)$, $s < 0$.

(2) Suppose that the calibration of $(f_\varepsilon)_\varepsilon$ ($c_{\omega,k}((f_\varepsilon)_\varepsilon)=0$) is not reached at zero. Assume that

$$(\exists b>0)(\forall\rho\in\mathcal{D}(\Omega))(|\langle T-f_\varepsilon,\rho\rangle|=O(\varepsilon^b),\varepsilon\in(0,1)).$$

Then

$$(\forall\eta>0)(\exists C>0)(\forall\xi\in\mathbb{R}^d)(|\hat{T}(\xi)|\le C(1+|\xi|^2)^{-k/2+\eta}). \qquad (3.5)$$

In particular, $T\in C_*^{k-d-\eta}(\mathbb{R}^d)$ for every $\eta\in(0,1)$.

Proof. (1) By the assumption (3.2) over the supports, we obtain that

$$(\forall\rho\in\mathcal{E}(\Omega))(\langle T-f_\varepsilon,\rho\rangle=o(1),\ \varepsilon\to0^+).$$

We consider for each fixed $\xi\in\mathbb{R}^d$ the test function $\rho_\xi(\cdot)=e^{i\xi\cdot}$, then, pointwise,

$$\hat{T}(\xi)=\langle T,\rho_\xi\rangle=\lim_{\varepsilon\to0^+}\langle f_\varepsilon,\rho_\xi\rangle=\lim_{\varepsilon\to0^+}\hat{f}_\varepsilon(\xi).$$

Now, the assumption on the calibration of $(f_\varepsilon)_\varepsilon$ implies the existence of $C>0$ such that

$$|\hat{f}_\varepsilon(\xi)|\le C(1+|\xi|)^{-k},\ \xi\in\mathbb{R}^d,\ \varepsilon<1.$$

Fixing $\xi\in\mathbb{R}^d$ and taking $\varepsilon\to0^+$, the inequality (3.4) follows.

By part c) of the previous theorem we have that $T^{(\alpha)}*\phi_\varepsilon\in\mathcal{E}_M^{0,d}$ for all $|\alpha|\le k$ and thus $\iota(T)\in\mathcal{G}^{k,d}(\Omega)$.

The estimate (3.4) implies that

$$||T*\varphi||_{L^\infty(\mathbb{R}^d)}\le\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\hat{T}(\xi)||\hat{\varphi}(\xi)|d\xi<\infty$$

and, for $\varepsilon\in(0,1)$,

$$\varepsilon^{d-k}||T*\psi_\varepsilon||_{L^\infty(\mathbb{R}^d)}\le\frac{C}{(2\pi)^d}\varepsilon^d\int_{\mathbb{R}^d}\frac{|\hat{\psi}(\varepsilon\xi)|}{|\varepsilon\xi|^k}d\xi=\frac{C}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{|\hat{\psi}(\xi)|}{|\xi|^k}d\xi<\infty.$$

It follows that $T\in C_*^{k-d}$.

(2) The assumption over $(f_\varepsilon)_\varepsilon$ implies that:

$$(\forall a>0)(||(1+|\cdot|)^k\hat{f}_\varepsilon||_{L^\infty(\mathbb{R}^d)}=O(\varepsilon^{-a})). \qquad (3.6)$$

The support condition (3.4) and the equivalence between weak and strong boundedness in $\mathcal{E}'(\Omega)$ yield

$$(\exists r\in\mathbb{N})(\exists C>0)(\forall\rho\in\mathcal{E}(\Omega))(\forall\varepsilon\in(0,1))(\varepsilon^{-b}|\langle T-f_\varepsilon,\rho\rangle|\le C\|\psi\|_r),$$

where $\|\rho\|_r=\sup_{x\in\Omega,|p|\le r}|\rho^{(p)}(x)|$. Let $\xi\in\mathbb{R}^d$ and $\rho_\xi(\cdot)=e^{i\xi\cdot}$. We have

$$\|\rho_\xi\|_r\le C_1(1+|\xi|)^r,\ \xi\in\mathbb{R}^d.$$

Therefore, with $C_2=CC_1$,

$$|\hat{T}(\xi)|\le C_2\varepsilon^b(1+|\xi|)^r+|\hat{f}_\varepsilon(\xi)|,\ \xi\in\mathbb{R}^d,\ 0<\varepsilon<1.$$

By (3.6), given any $a>0$, there exists $M=M_a>0$ such that

$$|\hat{T}(\xi)|\le C_2\varepsilon^b(1+|\xi|)^r+M\varepsilon^{-a}(1+|\xi|)^{-k},\ \xi\in\mathbb{R}^d,\ 0<\varepsilon<1.$$

Let $\varepsilon = (1+|\xi|)^{\frac{-k-r}{b}}$. Then

$$|\hat{T}(\xi)| \leq C_2(1+|\xi|)^{-k} + M(1+|\xi|)^{-k}(1+|\xi|)^{\frac{ak+ar}{b}},\ \xi \in \mathbb{R}^d. \quad (3.7)$$

Choosing small enough $a > 0$ so that $\eta = (ak+ar)/b \in (0,1)$, the estimate (3.5) follows after renaming the constant. The fact that $T \in C_*^{k-d-\eta}$ for arbitrary $\eta \in (0,1)$ can be established as in part (1). □

Theorem 3. *Let $T \in \mathcal{E}'(\Omega)$, $k \in \mathbb{N}_0$, and $(f_\varepsilon)_\varepsilon \in \mathcal{E}_M^{k,s_1}(\Omega)$, for some $s_1 > 0$, be such that (3.2) holds. Assume that*

$$(\exists r \in \mathbb{N})(\forall a > 0)(\forall \rho \in \mathcal{E}^r(\Omega))(|\langle T - f_\varepsilon, \rho\rangle| = O(\varepsilon^a)). \quad (3.8)$$

Then

$$(\forall \eta > 0)(\exists C > 0)(|\hat{T}(\xi)| \leq C(1+|\xi|^2)^{-k/2+\eta},\ \xi \in \mathbb{R}^d)$$

In particular, $T \in C_*^{k-d-\eta}(\mathbb{R}^d)$ for every $\eta \in (0,1)$.

Proof. As in part (2) of the previous theorem, we have

$$\varepsilon^{-a}|\hat{T}(\xi) - \hat{f}_\varepsilon(\xi)| \leq C(1+|\xi|)^r,\ \xi \in \mathbb{R}^d,\ \varepsilon < 1, \text{ i.e.,}$$

$$|\hat{T}(\xi)| \leq C\varepsilon^a(1+|\xi|)^r + |\hat{f}_\varepsilon(\xi)|,\ \xi \in \mathbb{R}^d,\ \varepsilon < 1,$$

for some constant $C = C_a$. By the assumption on the calibration of $(f_\varepsilon)_\varepsilon$, one gets that for some $s > s_1$ there exists another constant $C = C_{s,a} > 0$ such that

$$|\hat{T}(\xi)| \leq C\varepsilon^a(1+|\xi|)^r + C\varepsilon^{-s}(1+|\xi|)^{-k},\ \xi \in \mathbb{R}^d,\ \varepsilon < 1.$$

Let $\varepsilon = (1+|\xi|)^{\frac{-k-r}{a}}$. Then

$$|\hat{T}(\xi)| \leq C(1+|\xi|)^{-k} + C(1+|\xi|)^{-k}(1+|\xi|)^{\frac{sk+sr}{a}},\ \xi \in \mathbb{R}^d.$$

Thus, taking large enough $a > 0$ so that $(sk+sr)/a = \eta \in (0,1)$ and proceeding as in the proof of Theorem 2, one establishes all the assertions of Theorem 3. □

Recall [1], the space $W_0^{k,p}(\Omega) \subset W^{k,p}(\Omega)$ denotes the closure of $\mathcal{D}(\Omega)$ in the corresponding Sobolev norm.

Theorem 4. *a) Let $T \in \mathcal{E}'(\Omega), k \in \mathbb{N}_0, 1 < p \leq \infty$. Assume that $\iota(T) \in \mathcal{E}_{L^p,M}^{k,0}(\Omega)$ so that the calibration of $(T * \phi_\varepsilon)_\varepsilon$ ($c_{L^p,k}(T * \phi_\varepsilon) = 0$) is reached at 0. Then $T \in W^{k,p}(\Omega)$.*

b) Let $1 < p < \infty$, $T \in \mathcal{D}'(\Omega)$ and $(f_\varepsilon)_\varepsilon \in \mathcal{E}_{L^p,M}^{k,0}(\Omega)$ so that the calibration $c_{L^p,k}((f_\varepsilon)_\varepsilon) = 0$ is reached at 0 and $f_\varepsilon \in W_0^{k,p}(\Omega)$ for each $0 < \varepsilon < 1$. If $(f_\varepsilon)_\varepsilon \sim T$, i.e., (2) holds, then $T \in W_0^{k,p}(\Omega)$ ($\subset W^{k,p}(\Omega)$).

Proof. a) Since for some $C > 0$, $||T^{(\alpha)} * \phi_\varepsilon||_{L^p(\Omega)} \leq C,\ |\alpha| \leq k$, it follows that $(T * \phi_\varepsilon)_\varepsilon$ is weakly compact in $W^{k,p}(\Omega)$ if $p \in (1,\infty)$ and weakly* compact in $W^{k,\infty}(\Omega)$ if $p = \infty$. This implies that the limit T belongs to $W^{k,p}(\Omega)$.

b) Let $p' = p/(p-1)$. Since $(W_0^{k,p}(\Omega))' = W^{-k,p'}(\Omega)$, it follows that for every $\rho \in \mathcal{D}(\Omega)$,

$$|\langle T, \rho\rangle| \leq |\langle f_\varepsilon, \rho\rangle| + o(1) \leq C||\rho||_{W^{-k,p'}(\Omega)} + c(\varepsilon)$$

and since $c(\varepsilon) \to 0$, we obtain $(\forall \rho \in \mathcal{D}(\Omega))(|\langle T, \rho\rangle| \leq C||\rho||_{W^{-k,p'}(\Omega)})$ which implies, because of reflexivity of $W_0^{k,p}(\Omega)$, that $T \in W_0^{k,p}(\Omega)$, as required. □

4. Zygmund-type spaces and algebras

If r is of the form $r = k + \rho, k \in \mathbb{N}_0, \rho \in (0,1)$, then $C_*^r(\mathbb{R}^d)$ equals Hölder's space $\mathcal{H}^{k,\rho}(\mathbb{R}^d)$ ([15], Ch. 8) with the equivalent norm

$$||f||_{\mathcal{H}^{k,\rho}(\mathbb{R}^d)} = ||f||_{W^{k,\infty}(\mathbb{R}^d)} + \sup_{|\alpha|=k, x\neq y, x,y\in\mathbb{R}^d} \frac{|f^{(\alpha)}(x) - f^{(\alpha)}(y)|}{|x-y|^\rho} \tag{4.1}$$

Let $r \in \mathbb{R}$, Hörmann ([12]) defined $\tilde{\mathcal{G}}_*^r(\mathbb{R}^d)$ as the space of generalized functions $u \in \mathcal{G}(\mathbb{R}^d)$ with a representative $(u_\varepsilon)_\varepsilon$ such that for $\alpha \in \mathbb{N}_0^d$,

$$\|u_\varepsilon^{(\alpha)}\|_{L^\infty(\mathbb{R}^d)} = \begin{cases} O(1), & 0 \leq |\alpha| < r, \\ O(\log(1/\varepsilon)), & |\alpha| = r \in \mathbb{N}_0 \\ O(\varepsilon^{r-|\alpha|}), & |\alpha| > r. \end{cases} \quad \text{as } \varepsilon \to 0, \tag{4.2}$$

Remark 2. Clearly, if $g = [(g_\varepsilon)_\varepsilon] \in \tilde{\mathcal{G}}_*^r(\mathbb{R}^d)$, then $[(g_\varepsilon * \phi_\varepsilon)_\varepsilon] \in \tilde{\mathcal{G}}_*^r(\mathbb{R}^d)$ but the opposite does not hold, in general. However, $g = [(g_\varepsilon)_\varepsilon]$ and $g = [(g_\varepsilon * \phi_\varepsilon)_\varepsilon]$ are equal in the sense of generalized distributions, which means that

$$\langle g_\varepsilon * \phi_\varepsilon - g_\varepsilon, \theta\rangle = o(\varepsilon^p) \text{ for every } p \text{ and every } \theta \in \mathcal{D}(\mathbb{R}^d).$$

Originally in [12], the "tilde" did not appear in the notation but since we will introduce a new definition, which is intrinsically related to the classical definition of Zygmund spaces (including Hölder spaces), we leave the notation $\mathcal{G}_*^r(\mathbb{R}^d)$ for our space.

Definition 3. The space $\mathcal{G}_*^r(\mathbb{R}^d) = \mathcal{G}_*^{r,0}(\mathbb{R}^d)$, called the Zygmund space of generalized functions of 0−growth order, consists of $u \in \mathcal{G}(\mathbb{R}^d)$ with a representative $(u_\varepsilon)_\varepsilon$ such that

$$||u_\varepsilon||_*^r = ||u_\varepsilon * \varphi||_{L^\infty(\mathbb{R}^d)} + \sup_{\eta<1} \eta^{-r}||u_\varepsilon * \psi_\eta||_{L^\infty(\mathbb{R}^d)} = O(1), \tag{4.3}$$

while $\mathcal{G}_*^{r,s}(\mathbb{R}^d)$, the Zygmund space of generalized functions of s-growth order, consists of $u \in \mathcal{G}(\mathbb{R}^d)$ with a representative $(u_\varepsilon)_\varepsilon$ such that $(\varepsilon^s u_\varepsilon)_\varepsilon$ represents an element of $\mathcal{G}_*^r(\Omega)$.

The main properties of these spaces are summarized in the next theorem. In particular, we show the embedding of the ordinary Zygmund spaces of functions and characterize those distributions which, after embedding, belong to our generalized version of Zygmund classes.

Theorem 5. *Let $r, r_1, r_2, s \in \mathbb{R}$ and ϕ be a radial mollifier from $\mathcal{S}(\mathbb{R}^d)$ with all non-zero moments vanishing. Then*

(i) $i(\mathcal{C}^r_*(\mathbb{R}^d)) \subset \mathcal{G}^{r,0}_*(\mathbb{R}^d)$. If $u \in \mathcal{S}'(\mathbb{R}^d)$ and $i(u) \in \mathcal{G}^{r,0}_*(\mathbb{R}^d)$, then there is $\theta \in C^\infty(\mathbb{R}^d)$ such that $u - \theta \in \mathcal{C}^r_*(\mathbb{R}^d)$.

(ii) $\mathcal{G}^{r_1,s}_*(\mathbb{R}^d) \subset \mathcal{G}^{r,s}_*(\mathbb{R}^d)$ if $r_1 \geq r$; $P(D)\mathcal{G}^{r,s}_*(\mathbb{R}^d) \subset \mathcal{G}^{r-m,s}_*(\mathbb{R}^d)$, where m is the order of the differential operator P with constant coefficients.

(iii) Let $u_i \in \mathcal{G}^{r_i,s}_*(\mathbb{R}^d), i = 1,2$ and $r_1 + r_2 > 0$. Then $u_1u_2 \in \mathcal{G}^{p,2s}_*(\mathbb{R}^d)$, where $p = \min\{r_1, r_2\}$. In particular, $\mathcal{G}^{r,s}_*(\mathbb{R}^d)$ is an algebra if $s = 0$ and $r > 0$.

Proof. (i) Let $u \in \mathcal{C}^r_*(\mathbb{R}^d)$ and $\iota(u) = (u * \phi_\varepsilon)_\varepsilon$. By the assumption, we have

$$||u_\varepsilon * \varphi||_{L^\infty(\mathbb{R}^d)} \leq C(1 * |\phi_\varepsilon|) \leq C_1,\ \varepsilon < 1.$$

On the other hand,

$$\sup_{\eta<1} \eta^{-r}||u_\varepsilon * \psi_\eta||_{L^\infty(\mathbb{R}^d)} \leq ||\phi_\varepsilon||_{L^1(\mathbb{R}^d)} \sup_{\eta<1} \eta^{-r}||u * \psi_\eta||_{L^\infty(\mathbb{R}^d)} \leq C_2,\ 0 < \varepsilon < 1.$$

Let us now prove the second part of (i). Set $\varepsilon = \eta < 1$. We have that $\eta^{-d}(\phi * \psi)(x/\eta) = \phi_\eta * \psi_\eta(x)$. Thus, one obtains that $\phi * \psi$ is a radial wavelet which gives the implication

$$\sup_{\varepsilon<1} \varepsilon^{-r}|u * (\phi_\varepsilon * \psi_\varepsilon)| < \infty, \implies u - \theta \in C^r_*(\mathbb{R}^d) \text{ for some } \theta \in C^\infty(\mathbb{R}^d).$$

(ii) The first part is a consequence of the definition. By the use of Bernstein inequality ([15], Lemma 8.6.2) and the arguments of Proposition 8.6.6 in [15], we obtain the second part.

(iii) This part is a consequence of [15], Proposition 8.6.8. Actually, we have, by this proposition, that there exists $\varepsilon_0 \in (0,1)$ and $K = K(r_1, r_2)$, which does not depend on ε, such that

$$||\varepsilon^{2s}u_{1,\varepsilon}u_{2,\varepsilon}||^r_* \leq K||\varepsilon^s u_{1,\varepsilon}||^{r_1}_*||\varepsilon^s u_{2,\varepsilon}||^{r_2}_*,\ \varepsilon \leq \varepsilon_0.$$

□

Remark 3. As in the case of multiplication of continuous functions, we have that $[((u_1u_2) * \phi_\varepsilon)_\varepsilon] \neq [(u_1 * \phi_\varepsilon)_\varepsilon][(u_2 * \phi_\varepsilon)_\varepsilon]$ but these products are associated.

Several properties are listed in the next corollary.

Corollary 1. *(i) Let $r \in \mathbb{R}$, $T \in \mathcal{E}'(\mathbb{R}^d)$ and $\iota(T) \in \mathcal{G}^{r,0}_*(\mathbb{R}^d)$. Then $T \in \mathcal{C}^r_*(\mathbb{R}^d)$. (ii) Let $r \in \mathbb{R}$ and $u \in \mathcal{G}^{r,0}_*(\mathbb{R}^d)$. If $r \in \mathbb{R}_+ \setminus \mathbb{N}$, then $u \in \mathcal{G}^{[r],0}_{L^\infty}(\mathbb{R}^d)$. If $r \in \mathbb{N}$, then $u \in \mathcal{G}^{r-1,0}_{L^\infty}(\mathbb{R}^d)$. (iii) $\mathcal{G}^{r,0}_{L^p}(\mathbb{R}^d) \subset \mathcal{G}^{r-d/p,0}_*(\mathbb{R}^d)$.*

Proof. (i) is a direct consequence of Theorem 5, while (ii) follows directly from the equivalence between the norms in (3.1) and (4.1) for $0 < \rho < 1$. Part (iii) is implied by Theorems 7.33 and 7.37 from [1]. □

The following remark makes some partial comparisons between our definition and Hörmann's definition, [12]. We also formulate an open question.

Remark 4. Let $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}^r_*(\mathbb{R}^d)$ and let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ be the function from the Littlewood-Paley decomposition (see (1.4)). We show that $[(u_\varepsilon * \varphi_\varepsilon)_\varepsilon] \in \tilde{\mathcal{G}}^r_*(\mathbb{R}^d)$. For this, we will make use of Lemma 8.6.5 of [15], which asserts that given κ with $\hat{\kappa} \in \mathcal{D}(\mathbb{R}^d)$, there exist constants $K_{r,\alpha}$, $\alpha \in \mathbb{N}_0$, such that for all $v \in C^r_*(\mathbb{R}^d)$ and $0 < \eta \leq 1$, there holds

$$\|(v * \kappa_\eta)^{(\alpha)}\|_{L^\infty(\mathbb{R}^d)} \leq \begin{cases} K_{r,\alpha}||v||^r_*, & 0 \leq |\alpha| < r, \\ K_{r,\alpha}||v||^r_*(1 + \log(1/\eta)), & |\alpha| = r \in \mathbb{N}_0, \\ K_{r,\alpha}||v||^r_*(\eta^{r-|\alpha|}), & |\alpha| > r, \end{cases} \tag{4.4}$$

where, as usual, $\kappa_\eta(x) = \eta^{-d}\kappa(x/\eta)$. Thus, if we employ (4.4) with $v = u_\varepsilon$, $\kappa = \varphi$, and $\eta = \varepsilon$, together with the the fact that $||u_\varepsilon||^r_*$ is uniformly bounded with respect to ε, we obtain at once $[(u_\varepsilon * \varphi_\varepsilon)_\varepsilon] \in \tilde{\mathcal{G}}^r_*(\mathbb{R}^d)$, as claimed. At this point we should mention that the precise relation between the spaces $\mathcal{G}^r_*(\mathbb{R}^d)$ and $\tilde{\mathcal{G}}^r_*(\mathbb{R}^d)$ is still unknown; therefore, we can formulate an *open question: Find the precise inclusion relation between these two spaces.*

Remark 5. As seen from the given assertions, our Zygmund generalized function spaces are suitable for the analysis of pseudodifferential operators. We leave such investigations for further work.

4.1. **Hölder-type spaces and algebras of generalized functions.** Recall that $\mathcal{H}^{k,1}(\mathbb{R}^d) \subsetneq C^{k+1}_*(\mathbb{R}^d), k \in \mathbb{N}_0$.

Definition 4. Let $k \in \mathbb{N}_0, s \in [-\infty, \infty), \rho \in (0, 1]$ and let $u \in \mathcal{G}(\mathbb{R}^d)$. Then it is said that $u \in \mathcal{G}^{k,\rho,0}(\mathbb{R}^d)$ if it has a representative $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(\mathbb{R}^d)$ such that

$$||u_\varepsilon||_{\mathcal{H}^{k,\rho}} = ||u_\varepsilon||_{W^{k,\infty}} + \sup_{x,y\in\mathbb{R}^d, x\neq y, |\alpha|=k} |u^{(\alpha)}_\varepsilon(x) - u^{(\alpha)}_\varepsilon(y)||x - y|^{-\rho} = O(1) \tag{4.5}$$

It is said that $u \in \mathcal{G}^{k,\rho,s}(\mathbb{R}^d)$ if it has a representative $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(\mathbb{R}^d)$ such that $[(\varepsilon^s u_\varepsilon)_\varepsilon] \in \mathcal{G}^{k,\rho,0}(\mathbb{R}^d)$.

Proposition 4. *If* $r = k + \rho, \rho \in (0, 1)$, *then* $\mathcal{G}^{r,s}_*(\mathbb{R}^d) = \mathcal{G}^{k,\rho,s}(\mathbb{R}^d)$.

Proof. There exists $C > 0$ such that for every $\varepsilon < 1$,

$$C^{-1}||\varepsilon^s u_\varepsilon||^r_* \leq ||\varepsilon^s u_\varepsilon||_{\mathcal{H}^{k,\rho}} \leq C||\varepsilon^s u_\varepsilon||^r_*,$$

as follows from the equivalence between the norms (3.1) and (4.1) ([15]). This implies the assertion. □

Because of that, we will consider below only the cases $\mathcal{G}^{k,1,s}(\mathbb{R}^d), k \in \mathbb{N}_0$.

Theorem 6. *(i)* $\iota(\mathcal{H}^{k,1}(\mathbb{R}^d)) \subset \mathcal{G}^{k,1,0}(\mathbb{R}^d)$.

(ii) Let $k \in \mathbb{N}_0$, $u \in \mathcal{S}'(\mathbb{R}^d)$ *and* $\iota(u) \in \mathcal{G}^{k,1,0}(\mathbb{R}^d)$. *Then* $u \in \mathcal{H}^{k,1}(\mathbb{R}^d)$.

(iii)

$$\mathcal{G}^{k,1,s}(\mathbb{R}^d) \subsetneq \mathcal{G}_*^{k+1,s}(\mathbb{R}^d), k \in \mathbb{N}_0.$$

(iv) Let $P(D) = \sum_{|\alpha| \le m} a_\alpha D^\alpha$ *be a differential operator of order* m *with constant coefficients and* $m \le k \in \mathbb{N}$. *Then* $P(D) : \mathcal{G}^{k,\rho,s}(\mathbb{R}^d) \to \mathcal{G}^{k-m,\rho,s}(\mathbb{R}^d)$.

(v) $\mathcal{G}^{k_1,1,s}(\mathbb{R}^d) \subset \mathcal{G}^{k,1,s}(\mathbb{R}^d)$ *if* $k_1 \ge k$.

(vi) Concerning the multiplication, we have

$$\mathcal{G}^{k_1,1,s}(\mathbb{R}^d) \cdot \mathcal{G}^{k_2,1,s}(\mathbb{R}^d) \subset \mathcal{G}^{p,1,2s}(\mathbb{R}^d)$$

so that $p = \min\{k_1, k_2\}$. *In particular,* $\mathcal{G}^{k_1,1,s}(\mathbb{R}^d)$ *is an algebra if and only if* $s = 0$.

Proof. The proofs of assertions (i), (iv) and (v) are clear. We will prove (ii), (iii) and (vi).

(ii) By the assumption $\{f_\varepsilon^{(\alpha)}, \varepsilon < 1\}$ is a bounded and equicontinuous net of functions on any compact set in $\mathbb{R}^d$, for every $|\alpha| \le k$. Thus, by Arzelà-Ascoli theorem, it has a convergent subsequence for every $|\alpha| \le k$ and, by diagonalization, there exists a sequence $(f_{\varepsilon_n})_n$ and $f \in C^k(\mathbb{R}^d)$ such that $f_{\varepsilon_n}^{(\alpha)} \to f^{(\alpha)}, n \to \infty, |\alpha| \le k$, uniformly on any compact set $K \subset \mathbb{R}^d$. Now let $|\alpha| = k$. For every $x, y \in \mathbb{R}^d,\ x \ne y$,

$$\frac{|f^{(\alpha)}(x) - f^{(\alpha)}(y)|}{|x-y|^\rho} = \lim_{n\to\infty} \frac{|f_{\varepsilon_n}^{(\alpha)}(x) - f_{\varepsilon_n}^{(\alpha)}(y)|}{|x-y|^\rho} \le C,$$

since

$$\sup_{x,y\in\mathbb{R}^d,\, x\ne y} \lim_{n\to\infty} \frac{|f_{\varepsilon_n}^{(\alpha)}(x) - f_{\varepsilon_n}^{(\alpha)}(y)|}{|x-y|^\rho} \le \sup_{x,y\in\mathbb{R}^d,\, x\ne y} \sup_{n\in\mathbb{N}} \frac{|f_{\varepsilon_n}^{(\alpha)}(x) - f_{\varepsilon_n}^{(\alpha)}(y)|}{|x-y|^\rho}$$

$$\le \sup_{x,y\in\mathbb{R}^d,\, x\ne y,\varepsilon<1} \frac{|f_\varepsilon^{(\alpha)}(x) - f_\varepsilon^{(\alpha)}(y)|}{|x-y|^\rho} \le C,$$

and the assertion follows. A similar argument shows that $||f||_{W^{k,\infty}(\mathbb{R}^d)} < \infty$.

(iii) This follows from the fact $\mathcal{H}^{k,1}(\mathbb{R}^d) \ne C_*^{k+1}(\mathbb{R}^d)$ and parts (i) and (ii).

(vi) By the Leibnitz formula, the claim is reduced to the proof of

$$\sup_{x,y\in\mathbb{R}^d, x\ne y} \frac{|f_\varepsilon^{(\alpha)}(x)g_\varepsilon^{(\beta)}(x) - f_\varepsilon^{(\alpha)}(y)g_\varepsilon^{(\beta)}(x)|}{|x-y|^\rho} \le C,\ |\alpha+\beta| = p.$$

We just have to add and subtract $f_\varepsilon^{(\alpha)}(x)g_\varepsilon^{(\beta)}(y)$ in the numerator and then, by the use of the boundedness of $||f_\varepsilon^{(\alpha)}||_{L^\infty(\mathbb{R}^d)}$ and of $||g_\varepsilon^{(\beta)}||_{L^\infty(\mathbb{R}^d)}$, the result follows. □

5. Regularity properties of generalized functions

We recall some definitions concerning regularizing transforms and quasiasymptotics of distributions. Let ϕ be a mollifier as in Section 2. We set

$$F_\phi(f)(x,y) = \langle f(x+yt), \phi(t)\rangle = f * \check{\phi}_y(x),\ (x,y) \in \mathbb{H}^{d+1} := \mathbb{R}^d \times \mathbb{R}_+,$$

and call it the regularizing transform with kernel ϕ (the $\phi-$transform, in ([22]). Recall ([7], [27], [28]) that f is said to be quasiasymptotically bounded at the origin, resp., at ∞, with respect to ε^α if for each $\rho \in \mathcal{S}(\mathbb{R}^d)$

$$\frac{1}{\varepsilon^\alpha}\langle f(\varepsilon t), \rho(t)\rangle = O(1),\ \varepsilon \to 0^+,$$
$$\text{resp., } \frac{1}{\varepsilon^\alpha}\langle f(\varepsilon t), \rho(t)\rangle = O(1),\ \varepsilon \to \infty. \tag{5.1}$$

Remark 6. The quasiasymptotic bounds and behaviors (see [7], [26], or [27], for example) can also be defined by comparison with regularly varying functions which have the form $r^\alpha L(r), r > 0$, where L is a slowly varying function [3]. We assume in this paper that $L \equiv 1$ in order to simplify the notation and the exposition.

Our aim is to give another proof of the following well known theorem in Colombeau theory:

Theorem 7. $\mathcal{D}'(\mathbb{R}^d) \cap \mathcal{G}^\infty(\mathbb{R}^d) = C^\infty(\mathbb{R}^d)$.

Since Theorem 7 is a local statement, this equality is equivalent to the one $\mathcal{E}'(\mathbb{R}^d) \cap \mathcal{G}^\infty(\mathbb{R}^d) = C^\infty_c(\mathbb{R}^d)$, which means that if $f \in \mathcal{E}'(\mathbb{R}^d)$ and $\iota(f) = (f * \phi_\varepsilon)_\varepsilon \in \mathcal{E}^\infty_M(\mathbb{R}^d)$, then $f \in C^\infty(\mathbb{R}^d)$.

The proof follows from our analysis developed in [22], [?] which enables us to analyze properties of distributions f by knowing their growth order properties with respect to ε after regularizing, $f_\varepsilon = f * \theta_\varepsilon$, where θ is not necessarily a mollifier. We already demonstrated this in the previous subsection where we have used a Tauberian theorem from [22] for the mollifier transform. We now use Tauberian theorems for wavelet transforms of vector-valued distributions ([22,29]). Let E be a Banach space, $\mathbf{g} \in \mathcal{S}'(\mathbb{R}^d, E) := L_b(\mathcal{S}(\mathbb{R}^d), E)$ (the space of continuous linear mappings), and $\psi \in \mathcal{S}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \psi(t)dt = 0$. The wavelet transform is defined as the E-valued function $\mathcal{W}_\psi \mathbf{g} \in C^\infty(\mathbb{H}^{d+1}, E)$ given by

$$\mathcal{W}_\psi \mathbf{g}(x,y) = \frac{1}{y^d}\langle \mathbf{g}(x+yt), \overline{\psi}(t)\rangle := \mathbf{g} * \check{\overline{\psi}}_y(x) \in E,\ (x,y) \in \mathbb{H}^{d+1}.$$

Proof of Theorem 7. We may assume $f \in \mathcal{E}'(\mathbb{R}^d)$. Let $\Omega \subset\subset \mathbb{R}^d$ and $\beta \in \mathbb{N}^d$. We will show that $f^{(\beta)} \in C(\Omega)$. Find $\Omega \cup B(0,1) \subset V \subset\subset \mathbb{R}^d$. Since $[(f * \phi_\varepsilon)_\varepsilon] \in \mathcal{G}^\infty(\mathbb{R}^d)$, there is $s > 0$ such that for every $\alpha \in \mathbb{N}^d$,

$$\sup_{x \in \overline{V}} |(f * \phi_\varepsilon)^{(\beta+\alpha)}(x)| = O(\varepsilon^{-s}),\ \text{ i.e., } \sup_{x \in \overline{V}} |(f^{(\beta)} * \phi^{(\alpha)}_\varepsilon)(x)| = O(\varepsilon^{|\alpha|-s}).$$

Let $k \in \mathbb{N}$ be so that $p = 2k - s > 0$, we may assume $p \notin \mathbb{N}$. Then, with $\psi = \Delta^k \check{\bar{\phi}}$, there exists $M > 0$ such that $|\mathcal{W}_\psi f^{(\beta)}(x, \varepsilon)| \leq M\varepsilon^p$, $x \in \overline{V}$, $0 < \varepsilon < 1$. Define now $\mathbf{g} \in \mathcal{S}'(\mathbb{R}^d, C(\overline{\Omega}))$ as $\langle \mathbf{g}, \rho \rangle := (f^{(\beta)} * \check{\rho})_{|\overline{\Omega}} \in C(\overline{\Omega})$, $\rho \in \mathcal{S}(\mathbb{R}^d)$. Thus, $\mathcal{W}_\psi \mathbf{g}(x, y) \in C(\overline{\Omega})$ is the function $\mathcal{W}_\psi \mathbf{g}(x, y)(\xi) = \mathcal{W}_\psi f^{(\beta)}(x + \xi, y)$, and hence

$$\limsup_{\varepsilon \to 0} \sup_{|x|^2 + y^2 = 1} \varepsilon^{-p} ||\mathcal{W}_\psi \mathbf{g}(\varepsilon x, \varepsilon y)||_{C(\overline{\Omega})} \leq \sup_{0 < \varepsilon, y \leq 1} \sup_{t \in \overline{V}} \varepsilon^{-p} |\mathcal{W}_\psi f^{(\beta)}(t, \varepsilon y)| \leq M.$$

Applying the Tauberian theorem for the wavelet transform of Banach space-valued distributions (see Subsection 6.3 in [22]), we conclude the existence of an $C(\overline{\Omega})$-valued polynomial $\mathbf{P}(t) = \sum_{|\alpha| < p} v_\alpha t^\alpha$, $v_\alpha \in C(\overline{\Omega})$, such that for each $\rho \in \mathcal{S}(\mathbb{R}^p)$

$$\sup_{\xi \in C(\overline{\Omega})} |(f^{(\beta)} * \check{\rho}_\varepsilon)(\xi) - \sum_{|\alpha| < p} v_\alpha(\xi) \varepsilon^\alpha \int_{\mathbb{R}^d} t^\alpha \rho(t) dt| = ||\langle \mathbf{g} - \mathbf{P}, \rho_\varepsilon \rangle||_{C(\overline{\Omega})} = O(\varepsilon^p);$$

in particular, if we choose the mollifier $\rho = \check{\phi}$, we obtain that $\lim_{\varepsilon \to 0} (f^{(\beta)} * \phi_\varepsilon)(\xi) = v_0(\xi)$ uniformly for $\xi \in \overline{\Omega}$. This shows that $f^{(\beta)}_{|\Omega} = v_0$ and actually $f^{(\beta)}$ is continuous on Ω. Since β and Ω are arbitrary, the proof is complete.

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