

Circular planar nearrings: geometrical and combinatorial aspects

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Abstract Let (N, Φ) be a circular Ferrero pair. We define the disk with center b and radius a , $\mathcal{D}(a; b)$, as

$$\mathcal{D}(a; b) = \{x \in \Phi(r) + c \mid r \neq 0, b \in \Phi(r) + c, |(\Phi(r) + c) \cap (\Phi(a) + b)| = 1\}.$$

We prove that in the field-generated case there are many analogies with the Euclidean geometry. Moreover, if $\mathcal{B}^{\mathcal{D}}$ is the set of all disks, then, in some interesting cases, we show that the incidence structure $(N, \mathcal{B}^{\mathcal{D}}, \in)$ is actually a balanced incomplete block design.

Key words Planar nearring – BIBD – Circular – Disk – Field

1 Introduction

Almost at the same time, but independently, Clay [1,2] and Ferrero [9] studied a new kind of algebraic structures, namely planar nearrings. It turns out that this class of nearrings is closely related with incidence geometry [5], combinatorics [4], coding theory [10] and experimental designs [12]. An example in [1] proves that the circles of the usual Euclidean plane can be described in terms of the structural parts of a planar nearring whose additive group is $(\mathbb{C}, +)$, leading the author to define the concept of circularity for planar nearrings. Moreover it is possible to construct double planar nearrings with a circular component as the following example shows.

Example 1.1 On the complex number field $(\mathbb{C}, +, \cdot)$, for $a, b \in \mathbb{C}$, define two operations $*$ and \circ :

$$a * b = |a| \cdot b, \\ a \circ b = \begin{cases} 0 & \text{if } a = 0, \\ \frac{a}{|a|} \cdot b & \text{if } a \neq 0. \end{cases}$$

Then $(\mathbb{C}, +, *)$ and $(\mathbb{C}, +, \circ)$ are planar nearrings and each of $*$ and \circ is left distributive over the other, i.e., for any $a, b, c \in \mathbb{C}$ we have

$$\begin{aligned} a * (b \circ c) &= (a * b) \circ (a * c), \\ a \circ (b * c) &= (a \circ b) * (a \circ c). \end{aligned}$$

Hence $(\mathbb{C}, +, *, \circ)$ is a double planar nearring with a circular component.

As circularity proves to be interesting in application (see for example [3]), in [6] and [7], Clay proposes a “tentative” of definition of interior point of a circle (when a double planar nearring has a circular component). Let $(N, +, *, \circ)$ be a double planar nearring and suppose $(N, +, \circ)$ is circular with circles $N^* \circ a + b$. Then $c \in N$ is an interior point for the circle $N^* \circ a + b$ if:

1. $c \notin N^* \circ a + b$;
2. every ray $N \star a + b$ from c intersects the circle $N^* \circ a + b$.

Applied to Example 1.1 this definition works exactly as expected, namely it gives the interior points of a circle in the Euclidean plane. In [8] the author proves some properties of the interior points, but no special combinatorial structure seems to arise from this approach. Moreover, the definition is not very practical, not only because it requires a great amount of calculation, but especially because it is not peculiar to the circular nearring. In fact the definition requires that a circular nearring has a planar “partner”, and in general there exist many possible partners each giving a different set of interior points for a given circle, as we prove in the Example 2.8.

Our aim is to find a definition of interior point for a circle inherent to the structure of circular planar nearring without considering a planar partner, and to study its related geometry.

2 Preliminaries and notations

2.1 Circular planar nearrings

Definition 2.1 A (left) nearring is an algebraic structure $(N, +, \cdot)$ on a nonempty set N with two inner operations, $+$ and \cdot , such that:

1. $(N, +)$ is a group;
2. (N, \cdot) is a semigroup;
3. the left distributive law holds, i.e.

$$\forall x, y, z \in N, \quad x \cdot (y + z) = (x \cdot y) + (x \cdot z).$$

Moreover, if $(N \setminus \{0\}, \cdot)$ is a group, then $(N, +, \cdot)$ is a (left) nearfield.

We remind now some definitions and simple properties of nearrings, for details see [6].

Let $(N, +, \cdot)$ be a nearring, then for any $x, y \in N$, $x \cdot (-y) = -(x \cdot y)$ and

$x \cdot 0 = 0$; moreover if for any $x \in N$ we have $x \cdot 0 = 0 \cdot x = 0$, N is said to be *0-symmetric*.

We say that $a, b \in N$ are *equivalent multipliers* if and only if for all $n \in N$, $a \cdot n = b \cdot n$. It is easy to see that to be equivalent multipliers is an equivalence relation and we denote it by \equiv_m . We have the following fundamental definition.

Definition 2.2 *A nearring $(N, +, \cdot)$ is said to be planar if:*

1. $|N / \equiv_m| \geq 3$;
2. $\forall a, b, c \in N$, with $a \not\equiv_m b$, the equation

$$a \cdot x = b \cdot x + c$$

has a unique solution in N .

For planar nearrings we consider the set $A = \{n \in N \mid n \equiv_m 0\}$, called *annihilator* of N , and we write

$$N^0 = N \setminus \{0\} \quad \text{and} \quad N^* = N \setminus A.$$

It is well known that planar nearrings are 0-symmetric; moreover planar nearrings with identity are also planar nearfield. Vice-versa a finite nearfield with at least three elements is planar and 0-symmetric and its additive group is the additive group of a field.

Definition 2.3 *Let $(N, +)$ be a group. A subgroup of automorphisms Φ , $\{1\} \neq \Phi < \text{Aut}N$, is said to be regular if for any $\varphi \in \Phi \setminus \{1\}$, φ is a fixed point free (f.p.f.) automorphism, i.e., $\varphi(x) = x \Leftrightarrow x = 0$. Moreover, if for any $\varphi \in \Phi \setminus \{1\}$, $-\varphi + 1$ is surjective, the pair (N, Φ) is called a Ferrero pair. Then an orbit of Φ is the set*

$$\Phi(a) = \{\varphi(a) \mid \varphi \in \Phi\},$$

for some $a \in N$ ($\Phi(0) = \{0\}$ is the trivial orbit).

The concept of Ferrero pair is central in this framework since every planar nearring can be constructed from such pair by the so called *Ferrero Planar Nearing Factory* (for details, see [6],§4.1). Moreover we remember that, even if non-isomorphic nearrings can be generated by the same pair, the knowledge of the generating pair suffices in the study of the geometrical properties of the nearring.

Theorem 2.4 (Ferrero, Clay) *Let (N, Φ) be a finite Ferrero pair. Then*

1. $\forall a \in N^0$, $|\Phi(a)| = |\Phi|$;
2. $\forall a \in N$, $\forall b \in \Phi(a)$, $\Phi(a) = \Phi(b)$;
3. $\{\Phi(a) \mid a \in N\}$ is a partition of N ;
4. $|\Phi|$ divides $|N| - 1$.

Definition 2.5 *A planar nearring is said to be field-generated if it is generated from a Ferrero pair (N, Φ) , where $(N, +, \cdot)$ is a field and Φ is isomorphic to a subgroup of (N^0, \cdot) .*

It follows immediately that a field-generated planar nearring has a very simple structure, in fact every non trivial orbit is isomorphic to a multiplicative subgroup of a field.

Now, on a planar nearring $(N, +, \cdot)$ we consider the incidence structure (N, \mathcal{B}^*, \in) where $\mathcal{B}^* \subset \mathcal{P}(N)$ is defined by

$$\mathcal{B}^* = \{N^* \cdot a + b \mid a, b \in N, a \neq 0\}.$$

As usual in this contest, we call the elements of N points and those of \mathcal{B}^* blocks. It is immediate to observe that the structure (N, \mathcal{B}^*, \in) depends only on the pair (N, Φ) in the sense that if $(N, +, *)$ and $(N, +, \circ)$ are two planar nearrings constructed from the same Ferrero pair, then they yield identical (N, \mathcal{B}^*, \in) ; this allows the following definition.

Definition 2.6 *A planar nearring $(N, +, \cdot)$, or the pair (N, Φ) from which is generated, or the incidence structure (N, \mathcal{B}^*, \in) yielded, is said circular if every three distinct points of N belong to at most one block of \mathcal{B}^* , and if every two distinct points belong to at least two distinct blocks. In this case the block $N^* \cdot a + b = \Phi(a) + b$ is called circle with center b and radius a (see [6], §5.1).*

Definition 2.7 *An algebraic structure $(N, +, *, \circ)$ is a (left) double planar nearring if each of $(N, +, *)$ and $(N, +, \circ)$ is a (left) planar nearring, and each of $*$ and \circ is left distributive over the other.*

Example 2.8 Let consider the pairs (\mathbb{Z}_{61}, Φ) , $(\mathbb{Z}_{61}, \Gamma)$ and $(\mathbb{Z}_{61}, \Sigma)$, where Φ , Γ and Σ are isomorphic to the multiplicative subgroups of $(\mathbb{Z}_{61}^0, \cdot)$ generated by 11, 9 and 13, respectively. Each pair is circular and moreover if $(\mathbb{Z}_{61}, +, *_{\Phi})$, $(\mathbb{Z}_{61}, +, *_{\Gamma})$ and $(\mathbb{Z}_{61}, +, *_{\Sigma})$ are the nearrings they yield, then it is easy to prove that $(\mathbb{Z}_{61}, +, *_{\Phi}, *_{\Gamma})$ and $(\mathbb{Z}_{61}, +, *_{\Phi}, *_{\Sigma})$ are double planar nearrings. If we try to construct the set of interior points for the circle $\Phi(1)$ using the definition proposed in [6], for $(\mathbb{Z}_{61}, +, *_{\Phi}, *_{\Gamma})$ we obtain the union of the orbits on 0, 4, 5, 7, 9, 10, 13, 19, 20, while for $(\mathbb{Z}_{61}, +, *_{\Phi}, *_{\Sigma})$ we obtain the orbits on 0, 3, 4, 5, 7, 8, 13, 14, 15, 19, 20, 25.

2.2 BIBDs

Definition 2.9 *Let (X, \mathcal{B}, \in) be an incidence structure with $|X| = v$ and $|\mathcal{B}| = b$. If there exist two integers k and r so that for any $B \in \mathcal{B}$, $|B| = k$, and every $x \in X$ belongs to exactly r distinct blocks $B_1, \dots, B_r \in \mathcal{B}$, then (X, \mathcal{B}, \in) is a tactical configuration with parameters v, b, k, r . Moreover if an integer λ exists so that every pair of points belongs to exactly λ distinct blocks, then (X, \mathcal{B}, \in) is a balanced incomplete block design (BIBD) with parameters v, b, k, r, λ . If $a, b \in X$, we use the notation $\llbracket a, b \rrbracket$ to denote the number of blocks which the pair $\{a, b\}$ belongs to.*

Proposition 2.10 ([6],§5) *Let (X, \mathcal{B}, \in) be a BIBD with parameters v, b, k, r, λ . Then $vr = bk$ and $\lambda(v-1) = r(k-1)$.*

If $(N, +, \cdot)$ is a finite planar nearring, then (N, \mathcal{B}^, \in) is a BIBD with parameters $v = |N|$, $k = |N^*/\equiv_m|$, $b = v(v-1)/k$, $r = v-1$, $\lambda = k-1$. Moreover if the BIBD is circular, then $k \leq (3 + \sqrt{4v-7})/2$ and this limit is effective.*

2.3 The family of circles E_c^r

Definition 2.11 *Let (N, Φ) be a circular pair, then for every $r, c \in N^0$ define the family of circles E_c^r by*

$$E_c^r = \{\Phi(r) + b \mid b \in \Phi(c)\}.$$

Obviously $|E_c^r| = |\Phi(c)|$ (remember that $E_c^r \subset \mathcal{P}(N)$).

We give now without proof some well known properties of the family of circles, for details see ([11],§4) and ([6],§6).

Lemma 2.12 *Let $r, r', c, c' \in N^0$. Then $E_c^r = E_{c'}^{r'}$ if and only if $\Phi(r) = \Phi(r')$ and $\Phi(c) = \Phi(c')$.*

Lemma 2.13 *Let $A = \Phi(r) + a$ and $B = \Phi(r) + b$ be in E_c^r . Then there exists $\varphi \in \Phi$ such that $\varphi(A) = B$ and $\varphi(a) = b$. In particular if $\varphi \in \Phi$ exists such that $\varphi(a) = b$, then $\varphi(\Phi(r) + a) = \Phi(r) + b$ and so for any $\varphi \in \Phi$, we have $\varphi(E_c^r) = E_c^r$.*

Theorem 2.14 *Let $A \in E_c^r$ intersect in only one point (i.e. it's tangent) exactly m circles of E_c^r , and intersect exactly n circles of E_c^r , each in two points. Then every $B \in E_c^r$ has this property. Moreover, if $A, B \in E_c^r$ and $r' \in N^0$, then $|A \cap \Phi(r')| = |B \cap \Phi(r')|$.*

3 Disks

Definition 3.1 *Let (N, Φ) be a circular Ferrero pair and $\Phi(a) + b \in \mathcal{B}^*$, then we define $\mathcal{D}(a; b)$, the disk of center b and radius a , as*

$$\mathcal{D}(a; b) = \{x \in \Phi(r) + c \mid r \neq 0, b \in \Phi(r) + c, |(\Phi(r) + c) \cap (\Phi(a) + b)| = 1\}.$$

Now it is obvious to define the interior part of the circle $\Phi(a) + b$, $\mathcal{I}(\Phi(a) + b)$, as

$$\mathcal{I}(\Phi(a) + b) = \mathcal{D}(a; b) \setminus (\Phi(a) + b).$$

This definition is coherent with what we know by Euclidean geometry, in particular the idea is to reconstruct the interior of a circle joining together all the circles tangent to it and containing its center, and this forces these circles to be somehow internal at the given one. Moreover the definition is also inherent to the fixed nearring, in the sense that it does not require another

structure to be used, and this is of course an improvement towards Clay's definition ([6], Def.(7.115)). Now we want to study the geometrical structure of the disks and in particular to give, at least in the more interesting cases, a very fast way to construct them and to check the membership of a point.

3.1 Geometrical considerations

First we prove that under some enough general conditions, the definition of interior part is stable under translations and dilatations, and then we investigate the cases in which this interior part is surely nonempty.

To simplify the notations, we write for short $\sqcup E_c^r$ to mean the set of points belonging to the circles of E_c^r , i.e.

$$\sqcup E_c^r = \{x \in \Phi(r) + b \mid \Phi(r) + b \in E_c^r\}.$$

Lemma 3.2 *Let (N, Φ) be a Ferrero pair and $a, b \in N$ with $a \neq 0$, then*

$$\mathcal{D}(a; b) = \mathcal{D}(a; 0) + b$$

and

$$\mathcal{I}(\Phi(a) + b) = \mathcal{I}(\Phi(a)) + b,$$

that is, the definitions of disk and of interior part are homogeneous respect to translation.

Proof It is enough to observe, remembering that we always consider $r \neq 0$, that

$$\begin{aligned} \mathcal{D}(a; 0) + b &= \\ &= \{x \in \Phi(r) + c \mid 0 \in \Phi(r) + c, |(\Phi(r) + c) \cap (\Phi(a))| = 1\} + b = \\ &= \{x \in \Phi(r) + \bar{c} - b \mid 0 \in \Phi(r) + \bar{c} - b, |(\Phi(r) + \bar{c} - b) \cap (\Phi(a))| = 1\} + b = \\ &= \{x + b \in \Phi(r) + \bar{c} \mid b \in \Phi(r) + \bar{c}, |(\Phi(r) + \bar{c}) \cap (\Phi(a) + b)| = 1\} + b = \\ &= \{x \in \Phi(r) + \bar{c} \mid b \in \Phi(r) + \bar{c}, |(\Phi(r) + \bar{c}) \cap (\Phi(a) + b)| = 1\} - b + b = \\ &= \mathcal{D}(a; b). \end{aligned}$$

For the interior part, we have

$$\begin{aligned} \mathcal{I}(\Phi(a) + b) &= \mathcal{D}(a; b) \setminus (\Phi(a) + b) = (\mathcal{D}(a; 0) + b) \setminus (\Phi(a) + b) = \\ &= (\mathcal{D}(a; 0) \setminus \Phi(a)) + b = \mathcal{I}(\Phi(a)) + b. \end{aligned}$$

Lemma 3.3 *Let (N, Φ) be a field-generated Ferrero pair and let $(N, +, \cdot)$ be the generating field. If $a \in N^0$, then*

$$\mathcal{D}(a; 0) = a \cdot \mathcal{D}(1, 0)$$

and

$$\mathcal{I}(\Phi(a)) = a \cdot \mathcal{I}(\Phi(1)),$$

that is, the definitions of disk and of interior part are homogeneous respect to dilatation.

Proof Since a acts as a f.p.f. automorphism, following the previous lemma, and observing that the multiplication symbol \cdot in the field is always omitted, we have

$$\begin{aligned}
a\mathcal{D}(1;0) &= \\
&= a\{x \in \Phi(r) + c \mid 0 \in \Phi(r) + c, |(\Phi(r) + c) \cap \Phi(1)| = 1\} = \\
&= a\{x \in \Phi(r) + c \mid 0 \in \Phi(r) + c, |(\Phi(a^{-1}ar) + a^{-1}ac) \cap \Phi(a^{-1}a)| = 1\} = \\
&= a\{x \in \Phi(r) + c \mid 0 \in \Phi(a^{-1}ar) + a^{-1}ac, |a^{-1}(\Phi(ar) + ac) \cap a^{-1}\Phi(a)| = 1\} = \\
&= a\{x \in \Phi(a^{-1}ar) + a^{-1}ac \mid 0 \in \Phi(ar) + ac, |a^{-1}\{(\Phi(ar) + ac) \cap \Phi(a)\}| = 1\} = \\
&= a\{x \in a^{-1}(\Phi(ar) + ac) \mid 0 \in \Phi(ar) + ac, |(\Phi(ar) + ac) \cap \Phi(a)| = 1\} = \\
&= \{ax \in \Phi(ar) + ac \mid 0 \in \Phi(ar) + ac, |(\Phi(ar) + ac) \cap \Phi(a)| = 1\} = \\
&= \{y \in \Phi(\bar{r}) + \bar{c} \mid 0 \in \Phi(\bar{r}) + \bar{c}, |(\Phi(\bar{c}) + \bar{c}) \cap \Phi(a)| = 1\} = \\
&= \mathcal{D}(a;0),
\end{aligned}$$

and so the first equality. For the second we simply observe that

$$\begin{aligned}
\mathcal{I}(\Phi(a)) &= \mathcal{D}(a;0) \setminus \Phi(a) = a\mathcal{D}(1;0) \setminus a\Phi(1) = \\
&= a(\mathcal{D}(1;0) \setminus \Phi(1)) = a\mathcal{I}(\Phi(1)).
\end{aligned}$$

By the two previous lemmas we immediately have the following corollary.

Corollary 3.4 *With the same hypothesis of the previous lemma, we have*

$$\mathcal{D}(a;b) = a \cdot \mathcal{D}(1;0) + b = a \cdot \mathcal{D}(1;a^{-1} \cdot b)$$

and

$$\mathcal{I}(\Phi(a) + b) = a \cdot \mathcal{I}(\Phi(1)) + b = a \cdot \mathcal{I}(\Phi(1; a^{-1} \cdot b)).$$

We have just proved that there is a simple method to construct every disk, namely to construct the disk $\mathcal{D}(1;0)$, in particular in the following we will show how to further simplify this construction. Besides, we note that we can prove properties of disks that are kept for (field) multiplication and sum, only speaking about one disk; in particular it is now obvious that all disks have the same cardinality. Finally we must observe that by now we have not used neither the hypothesis of circularity nor of finiteness, that clearly will be fundamental for the following theorem, which is the main result of this section.

Theorem 3.5 *Let (N, Φ) be a finite circular field-generated Ferrero pair and let $|\Phi|$ be even, then for each $a, b \in N$, with $a \neq 0$, we have*

$$\mathcal{D}(a;b) = \sqcup E_{2^{-1}a}^{2^{-1}a} + b.$$

Proof At first we observe that Φ is a cyclic group of even order, so $-1 \in \Phi$. Since (N, Φ) is a Ferrero pair, from Definition 2.3 and N finite, it follows that $-(-1) + 1 = 1 + 1 = 2 : N \rightarrow N$, $2 : x \rightarrow x + x$ is a f.p.f. automorphism and so we can consider 2^{-1} (as automorphism). By the previous remark, we can consider, without loss of generality, only the disks like $\mathcal{D}(2a;0)$, $a \neq 0$.

Suppose that $|\Phi| = 2n$, let φ be a generator of Φ and $c \in \Phi(a)$, we now show that $\Phi(a) + c \subseteq \mathcal{D}(2a; 0)$, that is we prove that

$$0 \in \Phi(a) + c; \quad |(\Phi(a) + c) \cap \Phi(2a)| = 1.$$

Since $c \in \Phi(a)$, there exists $1 \leq l \leq 2n$ so that $c = \varphi^l(a)$; observing that of course $\varphi^n = -1$, we get

$$0 = \varphi^{l+n}(a) + c \in \Phi(a) + c$$

and

$$\varphi^l(2a) = 2\varphi^l(a) = \varphi^l(a) + \varphi^l(a) \in \Phi(a) + c$$

and finally

$$\varphi^l(2a) \in ((\Phi(a) + c) \cap \Phi(2a)).$$

Now we must prove $|(\Phi(a) + c) \cap \Phi(2a)| = 1$. Without loss of generality, we suppose that $l = 0$ (i.e. $c = a$), in such way our thesis becomes $(\Phi(a) + a) \cap \Phi(2a) = \{2a\}$.

By contradiction let $m \not\equiv 0 \pmod{2n}$ such that $\varphi^m(2a) \in \Phi(a) + a$. Then there exists $1 \leq h \leq 2n - 1$ so that $\varphi^m(2a) = \varphi^h(a) + a$, or equivalently $\varphi^{-h}(\varphi^m(2a)) = \varphi^{-h}(\varphi^h(a) + a)$, which means that

$$\varphi^{m-h}(2a) = \varphi^{-h}(a) + a.$$

Then also $\varphi^{m-h}(2a) \in \Phi(a) + a$ and such element is different from $2a$ and $\varphi^m(2a)$ because every element of Φ is f.p.f. Therefore $|(\Phi(a) + a) \cap \Phi(2a)| \geq 3$ and this is a contradiction by the hypothesis of circularity. So we have proved that for every $c \in \Phi(a)$, it is $\Phi(a) + c \subseteq \mathcal{D}(2a; 0)$, that means

$$\sqcup E_a^a \subseteq \mathcal{D}(2a; 0).$$

Now let $f \in \mathcal{D}(2a; 0)$, then f belongs to a circle passing through 0 and, without loss of generality, tangent to $\Phi(2a)$ in $2a$; therefore let $\Phi(s) + s$ be such a circle. If $\varphi^t(s) + s = 2a$, then

$$\varphi^{-t}(\varphi^t(s) + s) = s + \varphi^{-t}(s) = \varphi^{-t}(2a) \in (\Phi(s) + s) \cap \Phi(2a) = \{2a\}$$

for which $t = 0$, and $2s = 2a$, that is $s = a$. In conclusion we have proved that

$$\mathcal{D}(2a; 0) \subseteq \sqcup E_a^a,$$

and with the previous inclusion, we have the thesis.

Corollary 3.6 *In the same hypothesis of the previous theorem, the center is interior to the circle, in particular every disk is nonempty.*

Theorem 3.7 *In the same hypothesis of the previous theorem, if $c \in \mathcal{D}(a; b)$, then $\Phi(c) + b \subseteq \mathcal{D}(a; b)$. More precisely, if $|\Phi| = 2n$, then every disk is union of $n + 1$ circles (one of which degenerates), that is there are $c_1, \dots, c_{n-1} \in N^0 \setminus \Phi(a)$, with $\Phi(c_i) \neq \Phi(c_j)$ if $i \neq j$, so that*

$$\mathcal{D}(a; b) = \{\Phi(0) \cup \Phi(c_1) \cup \dots \cup \Phi(c_{n-1}) \cup \Phi(a)\} + b.$$

It follows that every disk has exactly $2n^2 + 1$ points.

Proof Without loss of generality, we can consider $b = 0$. If $c \in \mathcal{D}(a; 0) = \sqcup E_r^r$, then $C \in E_r^r$ exists so that $c \in C$. By Lemma 2.13, we have that for any $\varphi \in \Phi$, $\varphi(C) \in E_r^r$ and so $\varphi(c) \in \mathcal{D}(a; 0)$. For the generality we have chosen φ , the first part of the thesis follows.

With the above notation, let $c = \varphi(r) + r \in C \in E_r^r$, then $\varphi^{-1}(c) = \varphi^{-1}(\varphi(r) + r) = r + \varphi^{-1}(r) \in C$, so $C \cap \Phi(c) = \{c, \varphi^{-1}(c)\}$ and $|C \cap \Phi(c)| = 1$ if and only if $c = 0$ or $\varphi = 1$ (namely if c is the center of the circle or belongs to it). Therefore, the $n - 2$ points of $C \setminus (\Phi(a) \cup \{0\})$ belong two by two to the same circle of center 0, and so it follows the second part of the thesis. The last part is now an obvious corollary.

Definition 3.8 Let (N, Φ) be a circular Ferrero pair and $a, b \in N$, with $a \neq b$. We call the pair $\{a, b\}$ segment with endpoints a and b , and then, as in the Euclidean case, we define the axis of segment $\{a, b\}$

$$\Delta^{a,b} = \bigcup_{r \in N} \{(\Phi(r) + a) \cap (\Phi(r) + b)\},$$

and the set of middle-points of $\{a, b\}$

$$M^{a,b} = \{x \in N \mid \{x\} = (\Phi(r) + a) \cap (\Phi(r) + b), r \in N\}.$$

First we recall that this definition of segment is very different from that in [13], but we observe that we consider circular nearrings, and so we have a plane with only circles and points. We also note that in the general case we can not define the middle-point of the segment $\{a, b\}$ (that is, in general $|M^{a,b}| \neq 1$). For example let consider the circular pair $(\mathbb{Z}_{163}, \langle 38 \rangle)$, then $M^{0,1} = \{20, 22, 34, 47, 117, 130, 142, 144\}$. Nevertheless we can realize some geometrical constructions as in the Euclidean plane and, in some cases, we find interesting properties.

By the definition of axis, it follows immediately that if (N, Φ) is a finite circular Ferrero pair and $a, b, c \in N$ are all distinct, we can simply realize these geometrical constructions:

1. to determinate the circle passing through a , b and c : we only need to calculate $\Delta^{a,b}$ and $\Delta^{b,c}$; if $\Delta^{a,b} \cap \Delta^{b,c} = \emptyset$, then such a circle does not exist, if $\Delta^{a,b} \cap \Delta^{b,c} = \{d\}$, then d is the center of the circle we were looking for;
2. to determinate the bundle of circles through a and b : it suffices to consider all circles passing through a (or equivalently b) and whose center belongs to $\Delta^{a,b}$.

Note that in the first construction, if a , b and c are all distinct, then $\Delta^{a,b}$ and $\Delta^{b,c}$ could share at most one element, in fact this follows immediately from the circularity of N . The second one is based on the following lemma.

Lemma 3.9 Let (N, Φ) be a finite circular Ferrero pair and $a, b \in N$ with $a \neq b$. Then $x \in \Delta^{a,b}$ if, and only if, $\Phi(a - x) = \Phi(b - x)$.

Proof Let $x \in \Delta^{a,b}$, then $r \in N$ exists such that $x \in (\Phi(r) + a) \cap (\Phi(r) + b)$, or, equivalently, there exist $r \in N$, $0 \leq s, t \leq |\Phi| - 1$ and a generator φ of Φ such that

$$\begin{cases} x = \varphi^s(r) + a, \\ x = \varphi^t(r) + b. \end{cases}$$

From the former equation it follows that $\Phi(r) = \Phi(\varphi^s(r)) = \Phi(\varphi^s(r) + a - a) = \Phi(x - a)$, while from the latter we have $x - b = \varphi^t(r) \in \Phi(r) = \Phi(x - a)$, and then $\Phi(a - x) = \Phi(b - x)$. The other implication is straightforward as if $\Phi(a - x) = \Phi(b - x)$, then $x \in \Phi(a - x) + a \cap \Phi(b - x) + b = \Phi(a - x) + b$.

Lemma 3.9 can also be expressed in the following way: in a finite circular Ferrero pair the axis of a segment is the locus of the centers of the circles passing through the endpoints of the segment. Moreover it is in some way the dual of Definition 3.8, and this holds also for the definition of the set of middle points, as the following theorem shows.

Theorem 3.10 *Let (N, Φ) be a finite circular Ferrero pair, let $|\Phi|$ be even and $a, b \in N$ with $a \neq b$. If $x \in M^{a,b}$ then the circle $\Phi(a - x) + x$ is tangent to $\Phi(a - b) + b$ in a and $\Phi(b - x) + x$ is tangent to $\Phi(b - a) + a$ in b .*

Proof Without loss of generality we can suppose $b = 0$ and observe that from Lemma 3.9, $\Phi(a - x) = \Phi(-x)$. Moreover let $|\Phi| = 2n$ and φ be a generator of Φ . Let $x \in M^{a,0}$, we have to prove that

$$\begin{cases} (\Phi(a - x) + x) \cap \Phi(a) = \{a\} \\ (\Phi(-x) + x) \cap (\Phi(-a) + a) = \{0\} \end{cases}$$

We start considering the first equality, clearly $a \in \Phi(a)$ and $a \in \Phi(a - x) + x$, and from $\Phi(a - x) = \Phi(-x)$ it follows that $1 \leq h \leq 2n - 1$ exists such that

$$a = \varphi^h(-x) + x. \quad (3.1)$$

From $x \in M^{a,0}$ it follows that there exist $r \in N$ and unique $0 \leq s, t \leq 2n - 1$ such that

$$\begin{cases} x = \varphi^s(r) + a \\ x = \varphi^t(r) \end{cases} \quad (3.2)$$

From the former equality $r \in \Phi(x)$, so there exists unique $0 \leq z \leq 2n - 1$ such that $r = \varphi^z(x)$, and applying (3.1) it follows that the equation

$$\varphi^{s'}(x) + a = \varphi^{t'}(x) \quad (3.3)$$

has unique solution (in \mathbb{Z}_{2n}) $s' = h$ and $t' = 0$. Let us now suppose by absurd that $a' \neq a$ exists such that $a' \in \Phi(a) \cap (\Phi(-x) + x)$. Then $0 \leq u, v \leq 2n - 1$ exist such that

$$\begin{cases} a' = \varphi^u(a) \\ a' = \varphi^v(-x) + x \end{cases} \quad (3.4)$$

with $u \neq 0$ and $v \neq h$ as $a' \neq a$. From (3.4) it follows $\varphi^u(a) = \varphi^v(-x) + x$, or equivalently $\varphi^{v-u}(x) + a = \varphi^{-u}(a)$, that has unique solution $v - u = h$ and $-u = 0$ and this is a contradiction as $a' \neq a$. Analogous considerations prove that $(\Phi(-x) + x) \cap (\Phi(-a) + a) = \{0\}$.

Corollary 3.11 *Let (N, Φ) be a finite circular field-generated Ferrero pair with $|\Phi|$ even. If $a, b \in N$, $a \neq b$, then $|M^{a,b}| = 1$.*

3.2 Combinatorial properties

We are now able to prove, in a special case, an important combinatorial property of the incidence structures obtained considering the sets off all disks, namely

$$\mathcal{B}^{\mathcal{D}} = \{\mathcal{D}(a; b) \mid a, b \in N, a \neq 0\} = \{\sqcup(E_r^r + d) \mid r, d \in N, r \neq 0\}.$$

Theorem 3.12 *Let (N, Φ) be a finite circular field-generated Ferrero pair with $|N| = v$ and $|\Phi| = 2n$. Then $(N, \mathcal{B}^{\mathcal{D}}, \in)$ is a BIBD of parameters v , $b = v(v-1)/2n$, $k = 2n^2 + 1$, $r = (v-1)(2n^2 + 1)/2n$, $\lambda = n(2n^2 + 1)$.*

Proof The values of v and b are obvious, while from Theorem 3.7 it follows $k = 2n^2 + 1$. Now suppose $b \in N$, then $b \in \mathcal{D}(a; b) + c$ if and only if $b \in \mathcal{D}(a; 0) + b + c$ if and only if $-c \in \mathcal{D}(a; 0)$, that is, for $2n^2 + 1$ distinct values of c . Moreover since $\mathcal{D}(a; b) + c_1 = \mathcal{D}(a; b) + c_2$ if and only if $c_1 = c_2$, then there exist exactly $2n^2 + 1$ disks of radius a containing b , and being the possible radii in number of $|\mathcal{B}^*| = (|N| - 1)/|\Phi| = (v-1)/2n$, then b belongs to exactly $(v-1)(2n^2 + 1)/2n$ distinct disks, and so, for the generality we have chosen b , we have $r = (v-1)(2n^2 + 1)/2n$. Now let $x, y \in N$, $x \neq y$ and $x \neq 0$; then

$$x, y \in \mathcal{D}(a; b) \Leftrightarrow x - x, y - x \in \mathcal{D}(a; b) - x \Leftrightarrow 0, y - x \in \mathcal{D}(a; b - x),$$

and therefore $\llbracket x, y \rrbracket = \llbracket 0, y - x \rrbracket$. Now if $z \in N \setminus \{0, 1\}$, we obtain

$$0, z \in \mathcal{D}(a; b) \Leftrightarrow 0z^{-1}, zz^{-1} \in \mathcal{D}(a; b)z^{-1} \Leftrightarrow 0, 1 \in \mathcal{D}(az^{-1}; bz^{-1}),$$

that is, $\llbracket 0, z \rrbracket = \llbracket 0, 1 \rrbracket$. Therefore $\llbracket x, y \rrbracket = \llbracket 0, y - x \rrbracket = \llbracket 0, 1 \rrbracket$ and $(N, \mathcal{B}^{\mathcal{D}}, \in)$ is a BIBD. The value of λ follows by Proposition 2.10.

4 Future work

In this paper we have obtained some quantitative results in the even case ($|\Phi|$ even), leaving open the odd case. This is not very surprising, in fact it seems that the odd case is always harder than the even one, see for example ([11], §IV.5.19). Nevertheless, we are able to give some reasonable conjectures, in particular what we believe is that if (with the usual notation) $|\Phi| = 2n + 1$, then $|M^{a,b}| = 2n$. From this, we could easily deduce that in the field-generated case the formula

$$\mathcal{D}(a; b) = \bigcup_{r \in M^{a,b}} \{x \in \Phi(r) + c \mid \Phi(r) + c \in E_{-r}^r + b\}$$

holds.

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