Well-posedness of a Class of Non-homogeneous Boundary Value Problems of the Korteweg-de Vries Equation on a Finite Domain

Eugene Kramer
Department of Mathematics, Physics, and Computer Science
University of Cincinnati, Raymond Walters College
Cincinnati, Ohio 45236
eugene.f.kramer@uc.edu

Ivonne Rivas
Department of Mathematical Sciences,
University of Cincinnati
Cincinnati, Oh 45221
email: rivasie@mail.uc.edu

Bing-Yu Zhang
Department of Mathematical Sciences
University of Cincinnati
Cincinnati, Ohio 45221
email: zhangb@ucmail.uc.edu

Abstract

In this paper, we study a class of initial and boundary value problems proposed by Colin and Ghidalia for the Korteweg-de Vries equation posed on a bounded domain $(0, L)$. We show that the initial-value problem is locally well-posed in the classical Sobolev space $H^s(0, L)$ for $s > -\frac{3}{4}$, which provides a positive answer to one of the open questions of Colin and Ghidalia [18].

1 Introduction

In this paper we study a class of initial-boundary value problem (IBVP) for the Korteweg-de Vries (KdV) equation posed on a finite domain with nonhomogeneous boundary conditions,

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0, \qquad u(x,0) = \phi(x), \qquad x \in (0, L), \ t \in \mathbb{R}^+, \\ u(0,t) = h_1(t), \ u_x(L,x) = h_2(t), \ u_{xx}(L,t) = h_3(t). \end{cases} \tag{1.1}$$

This IBVP can be considered as a model for propagation of surface water waves in the situation where a wave-maker is putting energy in a finite-length channel from the left $(x = 0)$ while the right end $(x = L)$ of the channel is free (corresponding the case of $h_2 = h_3 = 0$) (see [16]). The problem was first proposed and studied by Colin and Ghidaglia in the late 1990s [16, 17, 18]. In particular, they investigated the well-posedness of the IBVP in the classical Sobolev space $H^s(0, L)$ and obtained the following results.

Theorem A ([18])

(i) *Given $h_j \in C^1([0, \infty))$, $j = 1, 2, 3$ and $\phi \in H^1(0, L)$ satisfying $h_1(0) = \phi(0)$, there exists a $T > 0$ such that the IBVP (1.1) admits a solution (in the sense of distribution)*

$$u \in L^\infty(0, T; H^1(0, L)) \cap C([0, T]; L^2(0, L)).$$

(ii) *The solution u of the IBVP (1.1) exists globally in $H^1(0, L)$ if the size of its initial value $\phi \in H^1(0, L)$ and its boundary values $h_j \in C^1([0, \infty))$, $j = 1, 2, 3$ are all small.*

In addition, they showed that the associate linear IBVP

$$\begin{cases} u_t + u_x + u_{xxx} = 0, \qquad u(x,0) = \phi(x) \quad x \in (0, L), \ t \in \mathbb{R}^+ \\ u(0,t) = 0, \ u_x(L,x) = 0, \ u_{xx}(L,t) = 0 \end{cases} \tag{1.2}$$

possesses the following smoothing property:

For any $\phi \in L^2(0, L)$, the linear IBVP (1.2) admits a unique solution

$$u \in C(\mathbb{R}^+; L^2(0, L)) \cap L^2_{loc}(\mathbb{R}^+; H^1(0, L)).$$

Aided by this smoothing property, Colin and Ghidaglia showed that the homogeneous IBVP (1.1) is locally well-posed in the space $L^2(0, L)$.

Theorem B ([18])

Assuming $h_1 = h_2 = h_3 \equiv 0$, then for any $\phi \in L^2(0, L)$, there exists a $T > 0$ such that the IBVP (1.1) admits a unique weak solution $u \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$.

The well-posedness results presented in Theorem A are not in the full strength of the well-posedness in the sense of Hadamard since both uniqueness and continuous dependence are missing, in particular, for the IBVP (1.1) with nonhomogeneous boundary conditions. To encourage further investigation, a series of open problems were proposed by Colin and Ghidaglia in [18], of which, two of them are listed below.

Problems

(1) *Is it possible to prove global existence of solutions of (1.1) for e.g. smooth solutions (as it in the case for both quarter plane and the whole line cases)?*

It is remarked by Colin and Ghidaglia in [18]: " *for these problems, uniqueness rely on a priori estimate in H^2 that we are not able to extend here and therefore establish the existence of more regular solutions.*"

(2) *Is it possible to establish the existence of solutions of (1.1) with their initial value in the space $H^s(0, L)$ for some $s < 0$ as in the case of the whole line?*

Colin and Ghidaglia expected the answer to be positive because of the the strong smoothing property of the associated linear IBVP (1.2).

In this paper, we will continue Colin and Ghidalia's work [16, 17, 18] to study the well-posedness problem of the IBVP (1.1) in the space $H^s(0, L)$. We aim at 1) establishing the well-posedness of the IBVP (1.1) in the full strength of Hadamard including *existence, uniqueness and continuous dependence* and 2) showing that the IBVP (1.1) is (locally) well-posed in the space $H^s(0, L)$ when $s \geq 0$ and $-\frac{3}{4} < s < 0$.

In order to describe precisely our results, we introduce the some notations.

For given $T > 0$ and $s \in \mathbb{R}$, let

$$\mathbb{H}^s(0,T) := H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s}{3}}(0,T) \times H^{\frac{s-1}{3}}(0,T),$$

$$D_{s,T} := H^s(0,L) \times \mathbb{H}^s(0,T)$$

and

$$Z_{s,T} = C([0,T]; H^s(0,L)) \cap L^2(0,T; H^{s+1}(0,L))$$

For the well-posedness of the IBVP (1.1), we intend to establish in this paper, some compatibility conditions relating the initial datum $\phi(x)$ and the boundary data $h_j(t), j = 1, 2, 3$ are needed. A simple computation shows that if u is a C^∞-smooth solution of the IBVP (1.1), then its initial data $u(x, 0) = \phi(x)$ and its boundary values $h_j(t),\ j = 1, 2, 3$ must satisfy the following compatibility conditions:

$$\phi_k(0) = h_1^{(k)}(0), \quad \phi_k'(L) = h_2^{(k)}(0), \quad \phi_k''(L) = h_3^{(k)}(0) \tag{1.3}$$

for $k = 0\ , 1, \cdots$, where $h_j^{(k)}(t)$ is the $k-$th order derivative of h_j and

$$\begin{cases} \phi_0(x) = \phi(x) \\ \phi_k(x) = -\left(\phi_{k-1}^{'''}(x) + \phi_{k-1}^{'}(x) + \sum_{j=0}^{k-1} \left(\phi_j(x)\phi_{k-j-1}(x)\right)'\right) \end{cases} \tag{1.4}$$

for $k = 1, 2, \cdots$. When the well-posedness of (1.1) is considered in the space $H^s(0, L)$ for some $s \geq 0$, the following $s-$compatibility conditions thus arise naturally.

Definition 1.1. *($s-$compatibility) Let $T > 0$ and $s \geq 0$ be given. A four-tuple*

$$(\phi, \vec{h}) = (\phi, h_1,\, h_2,\, h_3) \in D_{s,T}$$

is said to be $s-$compatible with respect to the IBVP (1.1) if

$$\phi_k(0) = h_1^{(k)}(0) \tag{1.5}$$

when $k = 0, 1, \dots [s/3] - 1$ and $s - 3[s/3] \leq 1/2$,

$$\phi_k(0) = h_1^{(k)}(0), \quad \phi_k'(1) = h_2^{(k)}(0) \tag{1.6}$$

when $k = 0, 1, \dots [s/3]$ and $1/2 < s - 3[s/3] \leq 3/2$ and

$$\phi_k(0) = h_1^{(k)}(0), \quad \phi_k'(1) = h_2^{(k)}(0) \quad \phi_k''(1) = h_3^{(k)}(0) \tag{1.7}$$

when $k = 0, 1, \dots [s/3] + 1$ and $3/2 \leq s - 3[s/3] \leq 9/2$. We adopt the convention that (1.5) is vacuous if $[s/3] - 1 < 0$.

As one of the main results in this paper, the following theorem states that the IBVP (1.1) is locally well-posed in the space $H^s(0, L)$ for any $s \geq 0$.

Theorem 1.2. *Let $s \geq 0$, $T > 0$ and $r > 0$ be given with*

$$s \neq \frac{2j-1}{2}, \quad j = 1, 2, \cdots .$$

There exists a $T^* \in (0, T]$ such that for any $s-$compatible

$$(\phi, \vec{h}) \in D_{s,T}$$

satisfying

$$\|(\phi, \vec{h})\|_{D_{s,T}} \leq r,$$

the IBVP (1.1) admits a unique solution

$$u \in C([0, T^*]; H^s(0, L)) \cap L^2(0, T^*; H^{s+1}(0, L)).$$

Moreover, the corresponding solution map is Liptschitz continuous [1].

To get the well-posedness of the IBVP (1.1) in the space $H^s(0, L)$ with $s < 0$, the following Bourgain spaces are needed (cf. [21, 26, 7]).

For any given $s \in \mathbb{R}$, $0 \leq b \leq 1$, $0 \leq \alpha \leq 1$ and function $w \equiv w(x, t) : \mathbb{R}^2 \to \mathbb{R}$, define

$$\Lambda_{s,b}(w) = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} < \tau - (\xi^3 + \xi) >^{2b} < \xi >^{2s} |\hat{w}(\xi, \tau)|^2 \, d\xi d\tau \right)^{\frac{1}{2}},$$

$$\lambda_\alpha(w) = \left(\int_{-\infty}^{\infty} \int_{|\xi| \leq 1} < \tau >^{2\alpha} |\hat{w}(\xi, \tau)|^2 \, d\xi d\tau \right)^{\frac{1}{2}} \tag{1.8}$$

[1]The solution map, is in fact, real analytic (cf. [48, 49, 50]

where $< \cdot >:= (1+|\cdot|^2)^{\frac{1}{2}}$. In addition, define

$$\mathcal{G}_s(w) = \left(\int_{-\infty}^{\infty} (1+|\xi|)^{2s} \left(\int_{-\infty}^{\infty} \frac{|\hat{w}(\xi,\tau)|}{1+|\tau-(\xi^3-\xi)|} d\tau \right)^2 d\xi \right)^{1/2},$$

$$\mathcal{Q}_{s,b}(w) = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1+|\xi|)^{2s} \frac{|\hat{w}(\xi,\tau)|^2}{\Big(1+|\tau-(\xi^3-\xi)|\Big)^{2b}} d\tau d\xi \right)^{1/2}$$

and

$$\mathcal{P}_\alpha(w) = \left(\int_{-\infty}^{\infty} \int_{|\xi|\le 1} \frac{|\hat{w}(\xi,\tau)|^2}{(1+|\tau|)^{2(1-\alpha)}} d\tau d\xi \right)^{1/2}.$$

Let $X_{s,b}$ be the space of all functions w satisfying

$$\|w\|_{X_{s,b}} := \Lambda_{s,b}(w) < \infty$$

while $Y_{s,b}$ is the space of all w satisfying

$$\|w\|_{Y_{s,b}} := \big(\mathcal{G}_s^2(w) + \mathcal{Q}_{s,b}^2(w)\big)^{1/2} < \infty.$$

In addition, let $X_{s,b}^\alpha$ be the space of all functions w satisfying

$$\|w\|_{X_{s,b}^\alpha} := \big(\Lambda_{s,b}^2(w) + \lambda_\alpha^2(w)\big)^{1/2} < \infty$$

and let $Y_{s,b}^\alpha$ be the space of all w satisfying

$$\|w\|_{Y_{s,b}^\alpha} := \big(\mathcal{P}_\alpha^2(w) + \mathcal{G}_s^2(w) + \mathcal{Q}_{s,b}^2(w)\big)^{1/2} < \infty.$$

The spaces $X_{s,b}$, $Y_{s,b}$, $X_{s,b}^\alpha$ and $Y_{s,b}^\alpha$ are all Banach spaces. Note that $X_{s,b}$ and $X_{s,b}^\alpha$ are equivalent when $b \ge \alpha$. The spaces $Y_{s,b}$ and $X_{s,-b}$ are also equivalent when $b < \frac{1}{2}$. Define also

$$\mathcal{X}_{s,b}^\alpha \equiv C(R; H^s(R)) \cap X_{s,b}^\alpha$$

with the norm

$$\|w\|_{\mathcal{X}_{s,b}^\alpha} = \left(\sup_{t\in R} \|w(\cdot,t)\|_{H^s(R)}^2 + \|w\|_{X_{s,b}^\alpha}^2 \right)^{1/2}.$$

The above Bourgain-type spaces are defined for functions posed on the whole plane $\mathbb{R}\times\mathbb{R}$. However, the IBVP (1.1) is posed on the finite domain $(0,L)\times(0,T)$. It is thus natural to define a restricted version of the Bourgain space $X_{s,b}$ to the domain $(0,L)\times(0,T)$ as follows:

$$X_{s,b}^T = X_{s,b}\Big|_{(0,L)\times(0,T)}$$

with the quotient norm

$$\|u\|_{X_{s,b}^T} \equiv \inf_{w\in X_{s,b}} \{\|w\|_{X_{s,b}} : \ w(x,t) = u(x,t) \text{ on } (0,L)\times(0,T)\}$$

for any given function $u(x,t)$ defined on $(0,L)\times(0,T)$. The spaces $Y_{s,b}^T$, $X_{s,b}^{\alpha,T}$, $Y_{s,b}^{\alpha,T}$ and $\mathcal{X}_{s,b}^{\alpha,T}$ are defined similarly.

Next theorem, another main result of this paper, provides a positive answer to Problem (2) listed earlier.

Theorem 1.3. *Let $s \in (-\frac{3}{4}, 0)$, $T > 0$ and $r > 0$ be given. There exist $T^* \in (0, T]$, $\alpha > \frac{1}{2}$ and $0 < b < \frac{1}{2}$ such that for any*

$$(\phi, \vec{h}) \in D_{s,T}$$

satisfying

$$\|(\phi, \vec{h})\|_{D_{s,T}} \le r,$$

the IBVP (1.1) admits a unique solution

$$u \in C([0, T^*]; H^s(0, L)) \cap X_{s,b}^{\alpha, T^*}.$$

Moreover, the corresponding solution map is Liptschitz continuous.

The following remarks are in order.

(i) According to Theorem 1.2, the IBVP (1.1) is well-posed in the space $H^s(0, L)$ for any $s \ge 0$, not just for $s = 0$ or $s = 1$. In particular, it demonstrates the existence of classical solutions and shows that the smoother of the initial value and boundary data, the smoother the corresponding solution.

(ii) In order to have solution u in the space $C(0, T]; H^s(0, L))$, Theorem 1.2 only requires that its initial value $\phi \in H^s(0, L)$ and its boundary data

$$h_1 \in H^{\frac{s+1}{3}}(0, T), \quad h_2 \in H^{\frac{s}{3}}(0, T), \quad h_3 \in H^{\frac{s-1}{3}}(0, T). \tag{1.9}$$

In particular, if $s = 1$, it is sufficient to require that

$$h_1 \in H^{\frac{1}{3}}(0, T), \quad h_2 \in L^2(0, T), \quad h_3 \in H^{-\frac{1}{3}}(0, T),$$

rather than $h_j \in C^1(0, T), j = 1, 2, 3$ as in Theorem A. Moreover, the condition (1.9) is optimal in order to have the corresponding solution $u \in C([0, T]; H^s(0, L))$.

(iii) Taking hint from the recent works of Bona, Sun and Zhang [9], Molinet [36], and Molinet and Vento [37], we conjecture that the IBVP (1.1) is locally well-posed in the space $H^s(0, L)$ for $-1 < s \le -\frac{3}{4}$, but ill-posed in the space $H^s(0, L)$ for any $s < -1$.

In the literature, there is another class of IBVP of the KdV equation posed on the finite domain $(0, L)$ as given below which has been well studied in the past few years [47, 5, 26, 9].

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0, & u(x, 0) = \phi(x) \qquad x \in (0, L),\ t \in \mathbb{R}^+, \\ u(0, t) = h_1(t),\ u(L, x) = h_2(t),\ u_x(L, t) = h_3(t). \end{cases} \tag{1.10}$$

It is interesting and constructive to compare the study of the IBVP (1.10) with that of the IBVP (1.1).

While the study of the IBVP (1.10) goes back as early as [12, 13], the nonhomogeneous IBVP (1.10) was first shown by Bona, Sun and Zhang [5] to be locally well-posed in the space $H^s(0, L)$ for any $s \ge 0$:

Let $s \ge 0$, $r > 0$ and $T > 0$ be given. There exists $T^* \in (0, T]$ such that for any $s-$compatible [2]

$$\phi \in H^s(0, L), \quad \vec{h} = (h_1, h_2, h_3) \in H^{\frac{s+1}{3}}(0, T) \times H^{\frac{s+1}{3}}(0, T) \times H^{\frac{s}{3}}(0, T)$$

[2] see [5] for the exact definition of $s-$compatibbility.

satisfying

$$\|\phi\|_{H^s(0,L)} + \|\vec{h}\|_{H^{\frac{s+1}{3}}(0,T)\times H^{\frac{s+1}{3}}(0,T)\times H^{\frac{s}{3}}(0,T)} \leq r,$$

the IBVP (1.10) admits a unique solution

$$u \in C([0,T^*]; H^s(0,L)) \cap L^2(0,T^*; H^{s+1}(0,L)).$$

Moreover, the corresponding solution map is Lipschitz continuous in the corresponding spaces.

Later Holmer [26] showed that the IBVP (1.10) is locally well-posed in the space $H^s(0,L)$ for any $-\frac{3}{4} < s < \frac{1}{2}$:

Let $s \in (-\frac{3}{4}, \frac{1}{2})$, $r > 0$ and $T > 0$ be given. here exists a $T^* \in (0,T]$ such that for any

$$\phi \in H^s(0,L), \quad \vec{h} = (h_1, h_2, h_3) \in H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s}{3}}(0,T)$$

satisfying

$$\|\phi\|_{H^s(0,L)} + \|\vec{h}\|_{H^{\frac{s+1}{3}}(0,T)\times H^{\frac{s+1}{3}}(0,T)\times H^{\frac{s}{3}}(0,T)} \leq r,$$

the IBVP (1.10) admits a unique mild solution [3]

$$u \in C([0,T^*]; H^s(0,L)).$$

Moreover, the corresponding solution map is Lipschitz continuous in the corresponding spaces.

More recently, Bona, Sun and Zhang [9] showed that the IBVP (1.10) is locally well-posed $H^s(0,L)$ for any $s > -1$.

Let $r > 0$, $-1 < s \leq 0$ and $T > 0$ be given. There exists a $T^* \in (0,T]$ such that for any

$$\phi \in H^s(0,L), \quad \vec{h} = (h_1, h_2, h_3) \in H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s}{3}}(0,T)$$

satisfying

$$\|\phi\|_{H^s(0,L)} + \|\vec{h}\|_{H^{\frac{s+1}{3}}(0,T)\times H^{\frac{s+1}{3}}(0,T)\times H^{\frac{s}{3}}(0,T)} \leq r,$$

the IBVP (1.10) admits a unique mild solution

$$u \in C([0,T^*]; H^s(0,L)).$$

Moreover, the corresponding solution map is Lipschitz continuous in the corresponding spaces.

Although there is only a slight difference between the boundary conditions of IBVP (1.1) and the IBVP (1.10), there is a big gap between their well-posedness results. For the IBVP (1.1), the well-posedness results presented in Theorem 1.2 and Theorem 1.3 are local in the sense that

[3] A function $u \in C([0,T^*]; H^s(0,L))$ is said to be a mild solution of the IBVP (1.10) if there exist a sequence

$$u_n \in C^1([0,T^*]; L^2(0,L)) \cap C[(0,T^*]; H^3(0,L)), \quad n = 1, 2, \cdots$$

solving the equation in (1.10) and as $n \to \infty$,

$$u_n \to u \quad in\ C([0,T^*]; H^s(0,L)),$$

$$h_{1,n} := u_n(0,\cdot) \to h_1, \qquad h_{2,n} := u(L,\cdot) \to h_2 \quad in H^{\frac{s+1}{3}}(0,T^*)$$

and

$$h_{3,n} := \partial_x u_n(L,\cdot) \to h_3 \quad in H^{\frac{s}{3}}(0,T^*).$$

the time interval $(0,T^*)$ in which the solution exists depends on r and, in general, the larger the r, the smaller the T^*. By contrast, the IBVP (1.10) is known to be globally well-posed in the space $H^s(0,L)$ for any $s\geq 0$ in the sense one always has $(0,T^*)=(0,T)$ no matter how large the r is (cf. [5, 23]). The cause of this difference is that the L^2-energy of the solution of the homogeneous IBVP (1.10) ($\vec{h}=0$) is decreasing:

$$\frac{d}{dt}\int_0^L u^2(x,t)dx = -\frac{3}{2}u_x^2(0,t) \ for \quad any \ t\geq 0.$$

But for the homogeneous IBVP (1.1), it is not clear at all, in general, whether the L^2-energy of its solution is increasing or decreasing since

$$\frac{d}{dt}\int_0^L u^2(x,t) = -\frac{3}{2}u_x^2(0,t)+3u^3(L,t) \ for \quad any \ t\geq 0.$$

The approach used in the proof of their results in [5, 22, 26] is very much different from what used in the proof of Theorem A, but more or less along the line used in the proof of Theorem B, in which the smoothing property of the associated linear system play an important role. In this paper, we will use the same approach as that developed in [5, 7] to prove our Theorem 1.2 and Theorem 1.3. The key ingredients of the approach are listed below.

(1). An explicit solution formula will be derived for the following nonhomogeneous boundary value problem of the linear equation:

$$\begin{cases} v_t+v_x+v_{xxx}=0, \ x\in(0,L), \ t\geq 0, \\ v(x,0)=0, \\ v(0,t)=h_1(t), \quad v_x(L,t)=h_2(t), \\ \qquad v_{xx}(L,t)=h_3(t), \end{cases} \tag{1.11}$$

which not only enables us to establish the well-posedness of the IBVP (1.1) with the optimal regularity conditions imposed on the boundary data, but also plays an important roles in obtaining the well-posedness of the IBVP (1.1) in the space $H^s(0,L)$ with $-\frac{3}{4}<s<0$.

(2). The smoothing property of the associated linear problem

$$\begin{cases} v_t+v_x+v_{xxx}=f, \ x\in(0,L), \ t\geq 0, \\ v(x,0)=\phi(x), \\ v(0,t)=h_1(t), \quad v_x(L,t)=h_2(t), \quad v_{xx}(L,t)=h_3(t). \end{cases} \tag{1.12}$$

For given $0\leq s\leq 3$ *and* $T>0$*, there exists a constant* $C>0$ *such that the solution* v *of (1.12) satisfies*

$$\|v\|_{Z_{s,T}}\leq C\left(\|(\phi,\vec{h})\|_{D_{s,T}}+\|f\|_{W^{\frac{s}{3},1}(0,T;H^s(0,L))}\right)$$

for any $(\phi,\vec{h})\in D_{s,T}$ *and* $f\in W^{\frac{s}{3},1}(0,T;H^s(0,L))$. This property is an extension of the smoothing property obtained by Colin and Ghidalia to the nonhomogeneous problem.

(3). Following Bona, Sun and Shuming [7], the IBVP (1.1) will be converted to an integral equation posed on the whole line $\mathbb{R}$ which make it possible to conduct Bourgain spaces analysis to obtain the well-posedness of the IBVP (1.1) in $H^s(0,L)$ for $-\frac{3}{4}<s<0$.

This paper is organized as follow. In Section 2, we will study various linear problems associated to the IBVP (1.1). The Section 3 is devoted to the well-posedness of the nonlinear IBVP (1.1). The paper is ended with some concluding remarks given in Section 4. Some open questions will also be listed in Section 4 for further investigations.

2 Linear Problems

2.1 The boundary integral operators

Consider the nonhomogeneous boundary-value problem

$$\begin{cases} v_t + v_x + v_{xxx} = 0, \quad v(x,0) = 0, \quad x \in (0, L), \quad t \geq 0. \\ v(0,t) = h_1(t),\ v_x(L,t) = h_2(t),\ v_{xx}(L,t) = h_3(t). \end{cases} \tag{2.1}$$

We derive an explicit solution formula of the IBVP (2.1). (Without loss of generality, we assume that $L = 1$ in this subsection). Applying the Laplace transform with respect to t, (2.1) is converted to

$$\begin{cases} s\hat{v} + \hat{v}_x + \hat{v}_{xxx} = 0, \\ \hat{v}(0,s) = \hat{h}_1(s),\ \hat{v}_x(1,s) = \hat{h}_2(s),\ \hat{v}_{xx}(1,\xi) = \hat{h}_3(s), \end{cases} \tag{2.2}$$

where

$$\hat{v}(x,\xi) = \int_0^{+\infty} e^{-st} v(x,t) dt$$

and

$$\hat{h}_j(s) = \int_0^{\infty} e^{-st} h_j(t) dt, \quad j = 1, 2, 3.$$

The solution of (2.2) can be written in the form

$$\hat{v}(x,s) = \sum_{j=1}^{3} c_j(s) e^{\lambda_j(s) x}$$

where $\lambda_j(s), j = 1, 2, 3$ are solutions of the characteristic equation

$$s + \lambda + \lambda^3 = 0$$

and $c_j(s), j = 1, 2, 3$, solves the linear system

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 e^{\lambda_1} & \lambda_2 e^{\lambda_2} & \lambda_3 e^{\lambda_3} \\ \lambda_1^2 e^{\lambda_1} & \lambda_2^2 e^{\lambda_2} & \lambda_3^2 e^{\lambda_3} \end{pmatrix}}_{A} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \underbrace{\begin{pmatrix} \hat{h}_1 \\ \hat{h}_2 \\ \hat{h}_3 \end{pmatrix}}_{\hat{\vec{h}}.} \tag{2.3}$$

By Cramer's rule,

$$c_j = \frac{\Delta_j(s)}{\Delta(s)},\ j = 1, 2, 3$$

with Δ the determinant of A and Δ_j the determinant of the matrix A with the column j replaced by $\widehat{\vec{h}}$. Taking the inverse Laplace transform of $\widehat{v}$ and following the same arguments as that in [5] yield the representation

$$v(x,t) = \sum_{m=1}^{3} v_m(x,t)$$

with

$$v_m(x,t) = \sum_{j=1}^{3} v_{j,m}(x,t)$$

and

$$v_{j,m}(x,t) = v^+_{j,m}(x,t) + v^-_{j,m}(x,t)$$

where

$$v^+_{j,m}(x,t) = \frac{1}{2\pi i}\int_0^{+i\infty} e^{st}\frac{\Delta_{j,m}(s)}{\Delta(s)}\hat{h}_m(s)e^{\lambda_j(s)x}ds$$

and

$$v^-_{j,m}(x,t) = \frac{1}{2\pi i}\int_{-i\infty}^0 e^{st}\frac{\Delta_{j,m}(s)}{\Delta(s)}\hat{h}_m(s)e^{\lambda_j(s)x}ds$$

for $j,m = 1,2,3$. Here $\Delta_{j,m}(s)$ is obtained from $\Delta_j(s)$ by letting $\hat{h}_m(s) = 1$ and $\hat{h}_k(s) = 0$ for $k \neq m$, $k,m = 1,2,3$. Making the substitution $s = i(\rho^3 - \rho)$ with $1 < \rho < \infty$ in the the characteristic equation

$$s + \lambda + \lambda^3 = 0,$$

the three roots are given in terms of ρ by

$$\lambda_1^+(\rho) = i\rho, \quad \lambda_2^+(\rho) = \frac{\sqrt{3\rho^2-4} - i\rho}{2} \quad \lambda_3^+(\rho) = \frac{-\sqrt{3\rho^2-4} - i\rho}{2}. \tag{2.4}$$

Thus $v^+_{j,m}(x,t)$ has the form

$$v^+_{j,m}(x,t) = \frac{1}{2\pi}\int_1^{\infty} e^{i(\rho^3-\rho)t}\frac{\Delta^+_{j,m}(\rho)}{\Delta^+(\rho)}\hat{h}^+_m(\rho)e^{\lambda_j^+(\rho)x}(3\rho^2-1)d\rho$$

and

$$v^-_{j,m}(x,t) = \overline{v^+_{j,m}(x,t)}$$

where $\hat{h}^+_m(\rho) = \hat{h}_m(i(\rho^3 - \rho))$, $\Delta^+(\rho)$ and $\Delta^+_{j,m}(\rho)$ are obtained from $\Delta(s)$ and $\Delta_{j,m}(s)$ by replacing s with $i(\rho^3 - \rho)$ and $\lambda^+_j(\rho) = \lambda_j(i(\rho^3 - \rho))$.

For given $m, j = 1,2,3$, let $W_{j,m}$ be an operator on $H^s_0(\mathbb{R}^+)$ defined as follows: for any $h \in H^s_0(\mathbb{R}^+)$,

$$[W_{j,m}h](x,t) \equiv [U_{j,m}h](x,t) + \overline{[U_{j,m}h](x,t)} \tag{2.5}$$

with

$$\begin{aligned}
[U_{j,m}h](x,t) \quad &\equiv \frac{1}{2\pi}\int_1^{+\infty} e^{i(\rho^3-\rho)t}e^{-\lambda_j^+(\rho)(1-x)}(3\rho^2-1)[Q^+_{j,m}h](\rho)d\rho \\
&= \frac{1}{2\pi}\int_1^{+\infty} e^{i(\rho^3-\rho)t}e^{-\lambda_j^+(\rho)x'}(3\rho^2-1)[Q^+_{j,m}h](\rho)d\rho, \quad (x' = 1-x),
\end{aligned} \tag{2.6}$$

for $j = 1,2,\ m = 1,2,3$ and

$$[U_{3,m}h](x,t) \equiv \frac{1}{2\pi}\int_1^{+\infty} e^{i(\rho^3-\rho)t}e^{\lambda_3^+(\rho)x}(3\rho^2-1)[Q^+_{3,m}h](\rho)d\rho \tag{2.7}$$

for $m = 1,2,3$. Here

$$[Q^+_{3,m}h](\rho) := \frac{\Delta^+_{3,m}(\rho)}{\Delta^+(\rho)}\hat{h}^+(\rho), \qquad [Q^+_{j,m}h](\rho) = \frac{\Delta^+_{j,m}(\rho)}{\Delta^+(\rho)}e^{\lambda_j^+(\rho)}\hat{h}^+(\rho)$$

for $j = 1,2$ and $m = 1,2,3$, $\hat{h}^+(\rho) = \hat{h}(i(\rho^3 - \rho))$. Then the solution of the IBVP (2.1) has the following representation.

Lemma 2.1. *Given h_1, h_2 and h_3, defining $\vec{h} = (h_1, h_2, h_3)$. The solution v of the IBVP (2.1) can be written in the form*

$$v(x,t) = [W_{bdr}\vec{h}](x,t) := \sum_{j,m=1}^{3} [W_{j,m}h_m](x,t). \tag{2.8}$$

2.2 Linear estimates

Consideration is first given to the IBVP of the linear equation:

$$\begin{cases} v_t + v_x + v_{xxx} = f, & x \in (0,L) \\ v(x,0) = \phi(x), \\ v(0,t) = 0, \ v_x(L,t) = 0, \ v_{xx}(L,t) = 0. \end{cases} \tag{2.9}$$

By the standard semigroup theory [39], for any $\phi \in L^2(0,L)$ and $f \in L^1(0,T;L^2(0,L))$, it admits a unique solution $v \in C([0,T];L^2(0,L))$, which can be written in the form

$$v(t) = W_0(t)\phi + \int_0^t W_0(t-\tau)f(\tau)d\tau$$

where W_0 is the C_0-Semigroup in the space $L^2(0,L)$ generated by the linear operator

$$Af = -f''' - f'$$

with the domain

$$\boldsymbol{\mathcal{D}}(A) = \{f \in H^3(0,L) : f(0) = f'(L) = f''(L) = 0\}.$$

Proposition 2.2. *Let $T > 0$ be given. There exists a constant C such that for any $\phi \in L^2(0,L)$ and $f \in L^1(0,T;L^2(0,L))$, the corresponding solution v of the IBVP (2.9) belongs to the space $Z_{0,T}$ and*

$$\|v\|_{Z_{0,T}} \le C\left(\|\phi\| + \|f\|_{L^1(0,T;L^2(0,L))}\right). \tag{2.10}$$

Proof. First multiplying the both sides of the equation in (2.9) by $2v$ and integrating over $(0,L)$ with respect to x yields that

$$\frac{d}{dt}\int_0^L v^2(x,t) + v^2(L,t) + v_x^2(0,t) = 2\int_0^L f(x,t)v(x,t)dx.$$

Then, multiplying the both sides of the equation in (2.9) by $2xv$ and integrating over $(0,L)$ with respect to x yields that

$$\frac{d}{dt}\int_0^L xv^2(x,t) + Lv^2(L,t) + 3\int_0^L v_x^2 dx = \int_0^L v^2 dx + \int_0^1 f(x,t)v(x,t)dx.$$

The estimate (2.10) follows easily. □

Next we consider the nonhomogeneous boundary-value problem

$$\begin{cases} v_t + v_x + v_{xxx} = 0, \quad x \in (0,L) \\ v(x,0) = 0, \\ v(0,t) = h_1(t), \ v_x(L,t) = h_2(t), \ v_{xx}(L,t) = h_3(t) \end{cases} \tag{2.11}$$

We have the following estimate for the solution of the IBVP (2.11)

Proposition 2.3. *For given $T > 0$, there exists a constant C such that for any $\vec{h} \in \mathbb{H}^s(0,T)$, the corresponding solution v of the (2.11) belongs to the space $Z_{0,T}$ and*

$$\|v\|_{Z_{0,T}} \leq C\|\vec{h}\|_{\mathbb{H}^s(0,T)}.$$

Proof: As

$$\lambda_1^+(\rho) = i\rho, \quad \lambda_2^+(\rho) = \frac{\sqrt{3\rho^2-4}-i\rho}{2} \quad \lambda_3^+(\rho) = \frac{-\sqrt{3\rho^2-4}-i\rho}{2},$$

the asymptotic behaviors of the ratios $\frac{\Delta_{j,m}^+(\rho)}{\Delta^+(\rho)}$ for $\rho \to +\infty$ are listed below.

$\frac{\Delta_{1,1}^+(\rho)}{\Delta^+(\rho)} \sim e^{-\frac{\sqrt{3}}{2}\rho}$	$\frac{\Delta_{2,1}^+(\rho)}{\Delta^+(\rho)} \sim e^{-\sqrt{3}\rho}$	$\frac{\Delta_{3,1}^+(\rho)}{\Delta^+(\rho)} \sim 1$
$\frac{\Delta_{1,2}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-1}$	$\frac{\Delta_{2,2}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-1}e^{-\frac{\sqrt{3}}{2}\rho}$	$\frac{\Delta_{3,2}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-1}$
$\frac{\Delta_{1,3}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-2}$	$\frac{\Delta_{2,3}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-2}e^{-\frac{\sqrt{3}}{2}\rho}$	$\frac{\Delta_{3,3}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-2}$

For $m = 1, 2, 3$ and $j = 1, 2$, set

$$\hat{h^*}_{3,m}^+(\rho) := [Q_{3,m}^+ h_m](\rho) = \frac{\Delta_{3,m}^+(\rho)}{\Delta^+(\rho)}\hat{h}_m^+(\rho)$$

and

$$\hat{h^*}_{j,m}^+(\rho) := [Q_{j,m}^+ h_m](\rho) = \frac{\Delta_{j,m}^+(\rho)}{\Delta^+(\rho)} e^{\lambda_j^+(\rho)}\hat{h}_m^+(\rho)$$

and view $h_{j,m}^*$ as the inverse Fourier transform of $\hat{h^*}_{j,m}^+$. It is straightforward to see that for any $s \in \mathbb{R}$,

$$\begin{cases} h_1 \in H_0^{(s+1)/3}(\mathbb{R}^+) & \Rightarrow h_{j,1}^* \in H^{\frac{s+1}{3}}(\mathbb{R}), \quad j = 1,2,3, \\ h_2 \in H_0^{s/3}(\mathbb{R}^+) & \Rightarrow h_{j,2}^* \in H^{\frac{s+1}{3}}(\mathbb{R}), \quad j = 1,2,3, \\ h_3 \in H_0^{(s-1)/3}(\mathbb{R}^+) & \Rightarrow h_{j,3}^* \in H^{\frac{s+1}{3}}(\mathbb{R}), \quad j = 1,2,3. \end{cases} \tag{2.12}$$

The proof is completed by using the same argument as that used in the proofs of Proposition 2.7, Proposition 2.8 and Proposition 2.9 in [5]. □

Combining Proposition 2.2 and Proposition 2.3 leads to the following estimates for solutions of the IBVP

$$\begin{cases} v_t + v_x + v_{xxx} = f, \quad x \in (0,L) \\ v(x,0) = \phi(x), \\ v(0,t) = h_1(t), \ v_x(L,t) = h_2(t), \ v_{xx}(L,t) = h_3(t) \end{cases} \tag{2.13}$$

Proposition 2.4. *Let $T > 0$ and $s \in [0,3]$ with $s \neq \frac{j}{2}, j = 1,3,5$ be given. There exists a constant $C > 0$ such that for any given $s-$compatible $(\phi,\vec{h}) \in D_{s,T}$ and $f \in W^{\frac{s}{3},1}(0,T;L^2(0,L))$, the IBVP (2.13) admits a unique solution $v \in Z_{s,T} \cap H^{\frac{s}{3}}(0,T;H^1(0,L))$ satisfying*

$$\|v\|_{Z_{s,T}\cap H^{\frac{s}{3}}(0,T;H^1(0,L))} \leq C\left(\|(\phi,\vec{h})\|_{D_{s,T}} + \|f\|_{W^{\frac{s}{3},1}(0,T;L^2(0,L))}\right).$$

Proof: We only prove it holds for $s = 0$ and $s = 3$. The other case of s follows by standard interpolation. Note that this proposition holds for $s = 0$ because of Proposition 2.2 and Proposition 2.3. To see it holds for $s = 3$, let $w = v_t$. Then w solves

$$\begin{cases} w_t + w_x + w_{xxx} = f_t, \quad x \in (0, L) \\ w(x, 0) = \phi^*(x), \\ w(0, t) = h_1'(t),\ w_x(L, t) = h_2'(t),\ w_{xx}(L, t) = h_3'(t) \end{cases}$$

with

$$\phi^*(x) = f_t(x, 0) - \phi'''(x) - \phi'(x).$$

Thus

$$\|v_t\|_{Z_{0,T}} = \|w\|_{Z_{0,T}} \le C\left(\|f\|_{W^{1,1}(0,T;L^2(0,L))} + \|(\phi, \vec{h})\|_{X_{3,T}} \right).$$

Since

$$v_{xxx} = f - v_t - v_{xxx} - v_x,$$

we obtain further that

$$\|v\|_{Z_{3,T}} \le C\left(\|f\|_{W^{1,1}(0,T;L^2(0,L))} + \|(\phi, \vec{h})\|_{D_{3,T}} \right).$$

The proof is complete. □

Proposition 2.4 will be sufficient for us to obtain the local well-posedness of the IBVP (1.1) int the space $H^s(0, L)$ for $s \ge 0$. However, to obtain its well-posedness in the space $H^s(0, L)$ with $s < 0$, we need to extend the problem posed on the finite domain $(0, L) \times (0, T)$ to an equivalent problem posed on the whole plane $\mathbb{R} \times \mathbb{R}$ in order to use Bourgain space analysis.

First recall the solution of the following linear KdV equation,

$$\begin{cases} v_t + v_x + v_{xxx} = 0, \qquad x \in \mathbb{R},\ t \in \mathbb{R}^+ \\ v(x, 0) = \psi, \end{cases} \tag{2.14}$$

has the explicit form

$$v(x, t) = W_{\mathbb{R}}(t)]\psi(x) = c\int_{\mathbb{R}} e^{i(\xi^3 - \xi)t} e^{ix\xi}\hat{\psi}(\xi)d\xi \tag{2.15}$$

Here $\hat{\psi}$ denotes the Fourier transform of ψ.

Taking advantage of this simplicity as it is done in [5], we rewrite $W_0(t)$ in term of $W_{\mathbb{R}}(t)$ and $W_{bdry}(t)$ as follows. For any $\phi \in H^s(0, L)$, let $\phi^* \in H^s(\mathbb{R})$ be its standard extension from $(0, L)$ to $\mathbb{R}$. Let $v = v(x, t)$ is the solution of

$$\begin{cases} v_t + v_x + v_{xxx} = 0, \quad x \in \mathbb{R},\ t \ge 0 \\ v(x, 0) = \phi^*, \end{cases}$$

and set $g_1(t) = v(0, t)$, $g_2(t) = v_x(L, t)$ and $g_3(t) = v_{xx}(L, t)$, $\vec{g} = (g_1, g_2, g_3)$ and

$$v_{\vec{g}} = v_{\vec{g}}(x, t) = [W_{bdr}(t)\vec{g}](x),$$

which is the corresponding solution of the nonhomogeneous boundary-value problem 2.11 with boundary data $h_j(t) = g_j(t)$ for $j = 1, 2, 3$ and $t \ge 0$. Then $v(x, t) - v_{\vec{g}}$ solves the IBVP (2.9). This leads us thus a particular representation of $W_0(t)$ in terms of $W_{bdr}(t)$ and $W_{\mathbb{R}}(t)$.

Let $B : H^s(0, L) \to H^s(\mathbb{R})$ be a standard extension operator from $H^s(0, L)$ to $H^s(\mathbb{R})$.

Lemma 2.5. *Given $s \in \mathbb{R}$ and $\phi \in H^s(0,L)$, let $\phi^* = B\phi$. Then*

$$W_0(t)\phi = W_{\mathbb{R}}(t)\phi^* - W_{bdr}(t)\vec{g} \tag{2.16}$$

for any $t > 0$ and $x \in (0,L)$, where $\vec{g}$ is obtained from the trace of $W_{\mathbb{R}}(t)\phi^*$ at $x = 0, L$.

The solution of the non-homogeneous initial boundary-value problem

$$\begin{cases} v_t + v_x + v_{xxx} = f(x,t), \qquad x \in (0,L),\ t \geq 0 \\ v(x,0) = 0, \\ v(0,t) = 0, \qquad v_x(L,t) = 0, \qquad v_{xx}(L,t) = 0 \end{cases} \tag{2.17}$$

can also be expressed in terms of $W_{\mathbb{R}}(t)$ and $W_{bdr}(t)$.

Lemma 2.6. *If $f^*(.,t) = Bf(.,t)$, with B as was defined before the extension of f from $[0,L] \times \mathbb{R}^+ \to \mathbb{R} \times \mathbb{R}^+$, then the solution u of the extended problem (2.17) is*

$$v(x,t) = \int_0^t W_0(t-\tau)f(\tau)d\tau = \int_0^t W_{\mathbb{R}}(t-\tau)f^*(.,\tau)d\tau - W_{bdr}(t)\vec{v}$$

for any $x \in (0,L)$ and $t \geq 0$ where $\vec{v} \equiv \vec{v}(t) = (v_1(t), v_2(t), v_3(t))$ is the appropriate boundary traces of

$$q(x,t) = \int_0^t W_{\mathbb{R}}(t-\tau)f^*(\tau)d\tau$$

at $x = 0, L$ i.e.

$$v_1(t) = q(0,t),\ v_2(t) = q_x(1,t),\ v_3(t) = q_{xx}(L,t)$$

Lemma 2.5 and Lemma 2.6 are validated when $x \in (0,L)$ and $t \geq 0$ since some of the operators that we have constructed are defined only in this interval, moreover the only operator that is defined in the whole line is $W_{\mathbb{R}}(t)$ for any values of x and t. In the equation (2.16), the left hand side is defined for all $x \in \mathbb{R}$ but the right hand side is defined just in $(0,L)$. Since we want to use the Bourgain Spaces, we need to extend the operator of the right hand side.

Recall that

$$W_{bdr}(t)\vec{h} = \sum_{j,m=1}^{3} W_{j,m}h_j$$

and each $W_{j,m}h_j$ is either of the form (see Lemma 2.1)

$$[U^1_{bdr}(t)]h(x) = \frac{1}{2\pi}\mathfrak{Re}\int_1^\infty e^{it(\mu^3-\mu)}e^{\frac{-\sqrt{3\mu^2-4}-i\mu}{2}x}(3\mu^2-1)\hat{h}(\mu)d\mu \tag{2.18}$$

or of the form

$$[U^2_{bdr}(t)]h(x) = \frac{1}{2\pi}\mathfrak{Re}\int_1^\infty e^{it(\mu^3-\mu)}e^{i\mu x}(3\mu^2-1)\hat{h}(\mu)d\mu \tag{2.19}$$

where $\hat{h}(\mu) = h(i(\mu^3-\mu))$. Therefore by the extension method introduced in [7], the operator $W_{bdr}(t)$ can be extended as $\mathcal{W}_{bdr}(t)$ with

$$[\mathcal{W}_{bdr}(t)\vec{h}](x,t)$$

defined for any $t,\ x \in \mathbb{R}$ and

$$[\mathcal{W}_{bdr}(t)\vec{h}](x,t) = [W_{bdr}\vec{h}](x,t) \ for\ any\ (x,t) \in (0,L) \times (0,T).$$

Moreover, the following estimates hold.

Proposition 2.7. *For given $\alpha > \frac{1}{2}$ and (b, s) such that $s \le 0$ and $b < 1/2$ satisfying*

$$0 \le b < 1/2 - s/3,$$

there exists a constant C such that for any $T > 0$ and any $\hat{h} \in \mathcal{H}^s(0, T)$,

$$\mathcal{W}_{bdr}\hat{h} \in C([0,T]; H^s(0,L)) \cap X^{\alpha,T}_{s,b}$$

and

$$\|\mathcal{W}_{bdr}\hat{h}\|_{C([0,T];H^s(0,L))\cap X^{\alpha,T}_{s,b}} \le C\|\vec{h}\|_{\mathcal{H}^s(0,T)}.$$

The following lemmas are important in establishing the well-posedness of of the IBVP (1.1) in $H^s(0, L)$ with $s < 0$ whose proofs can be found in [31, 21, 26, 7].

Lemma 2.8. *Let $-\infty < s < \infty$, $0 < b \le 1$, $\frac{1}{2} < \alpha \le 1$, and $\psi \in C_0^\infty(\mathbb{R})$ be given. There exists a constant C depending only on s, α, b and ψ such that*

$$\|\psi(t)W_{\mathbb{R}}(t)\phi\|_{\mathcal{X}^\alpha_{s,b}} \le C\|\phi\|_{H^s(\mathbb{R})} \tag{2.20}$$

and

$$\left\|\psi(t)\int_0^t W_{\mathbb{R}}(t-t')f(t')dt'\right\|_{\mathcal{X}^\alpha_{s,b}} \le C_\delta\|f\|_{Y^{1-\alpha}_{s,1-b}} \tag{2.21}$$

Next we present the spatial trace estimates of $W_{\mathbb{R}}(t)\phi$ and $\int_0^t W_{\mathbb{R}}(t-t')f(\cdot,t')dt'$ whose proofs can be found in [21, 26]

Lemma 2.9. *Let $s \in [-1, 2]$ be given. There exists a constant C depending only on s such that*

$$\sup_{x\in\mathbb{R}} \|W_{\mathbb{R}}(t)\phi\|_{H_t^{\frac{s+1}{3}}(\mathbb{R})} \le \|\phi\|_{H^s}(\mathbb{R}), \tag{2.22}$$

$$\sup_{x\in\mathbb{R}} \|\partial_x W_{\mathbb{R}}(t)\phi\|_{H_t^{\frac{s}{3}}(\mathbb{R})} \le \|\phi\|_{H^s}(\mathbb{R}) \tag{2.23}$$

and

$$\sup_{x\in\mathbb{R}} \|\partial_{xx} W_{\mathbb{R}}(t)\phi\|_{H_t^{\frac{s-1}{3}}(\mathbb{R})} \le \|\phi\|_{H^s}(\mathbb{R}) \tag{2.24}$$

Lemma 2.10. *Let $0 \le b < 1/2$, $-1 \le s \le 2$, $\psi \in C_0^\infty(\mathbb{R})$ and*

$$w(x,t) = \int_0^t W_{\mathbb{R}}(t-t')f(\cdot,t')dt'$$

there exists C depending only on b, s and ψ such that

$$\sup_{x\in\mathbb{R}} \|\psi(\cdot)w(x,.)\|_{H_t^{\frac{s+1}{3}}(\mathbb{R})} \le C\|f\|_{Y_{s,b}},$$

$$\sup_{x\in\mathbb{R}} \|\psi(\cdot)w_x(x,.)\|_{H_t^{\frac{s}{3}}(\mathbb{R})} \le C\|f\|_{Y_{s,b}}$$

and

$$\sup_{x\in\mathbb{R}} \|\psi(\cdot)w_{xx}(x,\cdot)\|_{H_t^{\frac{s-1}{3}}(\mathbb{R})} \le C\|f\|_{Y_{s,b}}$$

The following bilinear estimate is crucial in establishing the well-posedness of the IBVP (1.1 whose proof can be found in [31, 21, 26]).

Lemma 2.11. *Given $s > -\frac{3}{4}$, there exist $b = b(s) < \frac{1}{2}$, $\alpha = \alpha(s) > \frac{1}{2}$ and C, $\mu > 0$ such that*

$$\|\partial_x(uv)\|_{Y^\alpha_{s,b}} \le CT^\mu\|u\|_{X^\alpha_{s,b}}\|v\|_{X^\alpha_{s,b}} \tag{2.25}$$

for any u, $v \in X^\alpha_{s,b}$ with compact support in $[-T, T]$.

3 Nonlinear Problem

In this section, we consider the well-posedness of the following nonlinear problem in the space $H^s(0,L)$.

$$\begin{cases} v_t + v_x + vv_x + v_{xxx} = 0, & x \in (0,L),\ t > 0 \\ v(x,0) = \phi(x), \\ v(0,t) = h_1(t),\ v_x(L,t) = h_2(t),\ v_{xx}(L,t) = h_3(t) & t \geq 0. \end{cases} \tag{3.1}$$

First we consider its well-posedness in the space $H^s(0,L)$ for $s \geq 0$. Recall that for given $s \geq 0$ and $T > 0$,

$$D_{s,T} := H^s(0,L) \times H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s}{3}}(0,T) \times H^{\frac{s-1}{3}}(0,T)$$

and

$$Z_{s,T} := \mathbb{C}([0,T]; H^s(0,L)) \cap L^2(0,T; H^{s+1}(0,L)).$$

In addition, let

$$\mathcal{Z}_{s,T} := Z_{s,T} \cap H^{\frac{s}{3}}(0,T; H^1(0,L)).$$

The proof of the following lemma can be found in [5, 33].

Lemma 3.1. *(i) For $s \geq 0$ there exists a $C \geq 0$ such that for any $T > 0$ and $u,\ v \in Z_{s,T}$,*

$$\int_0^T \|uv_x\|_{H^s(0,L)}\, d\tau \leq C(T^{\frac{1}{2}} + T^{\frac{1}{3}})\|u\|_{Z_{s,T}}\|v\|_{Z_{s,T}} \tag{3.2}$$

(ii) For $0 \leq s \leq 3$ there exists a $C \geq 0$ such that for any $T > 0$ and $u,\ v \in \mathcal{Z}_{s,T}$,

$$\|uv_x\|_{W^{\frac{s}{3},1}(0,T;L^2(0,1))} \leq C(T^{\frac{1}{2}} + T^{\frac{1}{3}})\|u\|_{\mathcal{Z}_{s,T}}\|v\|_{\mathcal{Z}_{s,T}} \tag{3.3}$$

Theorem 3.2. *Let $T > 0$, $r > 0$ and $s \geq 0$ be given with $s \neq \frac{2j+1}{2}$ for $j = 0,1,2,\cdots$. There exists a $T^* \in (0,T]$ such that for any $s-$compatible $(\phi,\vec{h}) \in X_{s,T}$, the IBVP (3.1) admits a unique solution*

$$v \in Z_{s,T^*}.$$

Moreover, the corresponding solution map is Lipschitz continuous.

Proof: Since the proof is similar to that presented in [5], we only provide a sketch and refer to [5] for more detail.

1). We first consider the case of $0 \leq s \leq 3$. Let $r > 0$ and $0 < \theta \leq \max\{1,T\}$ be constants to be determined. Set

$$S_{\theta,\eta} = \{w \in \mathcal{Z}_{s,\theta}, \quad \|w\|_{\mathcal{Z}_{s,\theta}} \leq \eta\}.$$

For given $(\phi,\vec{h}) \in D_{s,T}$ with

$$\|(\phi,\vec{h})\|_{D_{s,T}} \leq r,$$

define a nonlinear map on $S_{\theta,\eta}$ by

$$v = \Gamma(w)$$

being the unique solution of the IBVP

$$\begin{cases} v_t + v_x + v_{xxx} = -ww_x, & x \in (0,L),\ t>0 \\ v(x,0) = \phi(x), \\ v(0,t) = h_1(t),\ v_x(L,t) = h_2(t),\ v_{xx}(L,t) = h_3(t) & t \geq 0 \end{cases}$$

for $w \in \mathcal{Z}_{\theta,\eta}$. Using Proposition 2.4 and Lemma 3.1, one can show that there exist $\eta > 0$ and $\theta > 0$ depending only on s, r and T such that the map Γ is a contraction on the metric space $S_{\theta,\eta}$ whose fixed point is the desired solution for the IBVP (3.1) Thus the theorem hods for $0 \leq s \leq 3$.

2). Next we consider the following IBVP of the linear KdV equation with variable coefficients.

$$\begin{cases} v_t + v_x + v_{xxx} + (av)_x = f, & x \in (0,L),\ t>0 \\ v(x,0) = \phi(x), \\ v(0,t) = h_1(t),\ v_x(L,t) - v(L,t) = h_2(t),\ v_{xx}(L,t) - v(L,t) = h_3(t) & t \geq 0 \end{cases} \tag{3.4}$$

As in the step 1, using the contraction mapping principle, one can show the following proposition holds.

Proposition 3.3. *Let $T > 0$ and $0 \leq s \leq 3$ be given and assume that $a \in \mathcal{Z}_{s,T}$. Then for any $s-$compatible $(\phi, \vec{h}) \in D_{s,T}$ and $f \in W^{\frac{s}{3},1}(0,T;L^2(0,L))$, the IBVP (3.4) admits a unique solution $v \in \mathcal{Z}_{s,T}$. Moreover, there exists a constant $C > 0$ depending only on T and $\|a\|_{\mathcal{Z}_{s,T}}$ such that*

$$\|v\|_{\mathcal{Z}_{0,T}} \leq C\left(\|(\phi,\vec{h})\|_{D_{s,T}} + \|f\|_{W^{\frac{s}{3},1}(0,T;H^s(0,L))} \right).$$

3). Now we prove the theorem hold for $3 \leq s \leq 6$. The other of $s > 6$ follows similarly. First of all, according to Step 2), the IBVP (3.1) admits a unique solution $uv \in \mathcal{Z}_{3,T^*}$. We just need to prove this solution v also belong to the space $\mathcal{Z}_{s,T^*}$. To see that, let $z = v_t$. Then z solves the following linearized IBVP

$$\begin{cases} z_t + z_x + (a(x,t)z)_x + z_{xxx} = 0, \\ z(x,0) = \phi_1(x), \\ z(0,t) = h_1^{(1)}(t), \quad z_x(L,t) = h_2^{(1)}(t), \quad z_{xx}(L,t) = h_3^{(1)}(t) \end{cases}$$

where $a(x,t) = v(x,t) \in \mathcal{Z}_{3,T^*}$ and

$$\phi_1 \in H^{s-3}(0,L),\ h_1^{(1)} \in H^{\frac{s-2}{3}}(0,T^*), \quad ,\ h_2^{(1)} \in H^{\frac{s-3}{3}}(0,T^*), \quad h_3^{(1)} \in H^{\frac{s-4}{3}}(0,T^*).$$

It thus follows from Proposition 3.3 that

$$z = v_t \in \mathcal{Z}_{s-3,T^*}$$

and therefore

$$v \in \mathcal{Z}_{s,T^*}$$

since

$$v_{xxx} = -v_t - v_x - vv_x.$$

□

Next we consider the well-posedness of the IVP (3.1) in the space $H^s(0,L)$ with $s<0$. We first rewrite the IBVP (3.1) in its integral form;

$$v(t) = W_0(t)\phi + W_{bdr}(t)\vec{h} - \int_0^t W_0(t-\tau)(vv_x)(\tau)d\tau. \tag{3.5}$$

Theorem 3.4. *Let $T>0$, $r>0$ and $-\frac{3}{4}<s<0$ be given. There exists a $T^* \in (0,T]$ and $b \in (0,\frac{1}{2})$ such that for any $(\phi,\vec{h}) \in X_{s,T}$, (3.5) admits a unique solution*

$$v \in C([0,T^*], H^s(0,L)) \cap Y_{s,b}^{T^*}.$$

Moreover, the corresponding solution map is Lipschitz continuous.

The following lemmas are needed to prove Theorem 3.4. Let

$$\mathcal{X}_{s,b}^{\alpha,T} := C([0,T]; H^s(0,L)) \cap X_{s,b}^{\alpha,T}.$$

Lemma 3.5. *Let $T>0$, $s<0$, $\frac{1}{2}<\alpha\le 1$ and $b\in(0,1)$ be given satisfying*

$$0 < b < \frac{1}{2} - \frac{s}{3}.$$

For any $\phi \in H^s(0,L)$, $W_0(t)\phi \in \mathcal{X}_{s,b}^{\alpha,T}$ and

$$\|W_0(t)\phi\|_{\mathcal{X}_{s,b}^{\alpha,T}} \le C\|\phi\|_{H^s(0,L)}$$

where $C>0$ is independent of ϕ.

Proof: It follows from Lemma 2.5, Lemma 2.8, Lemma 2.9, and Proposition 2.7.

Lemma 3.6. *Assume that $-1\le s<1$, $\frac{1}{2}<\alpha\le 1$ and $0<b<\frac{1}{2}$. For any $T>0$, there is a constant C such that for any $f \in Y_{s,b}^{1-\alpha,T}$,*

$$u = \int_0^t W_0(t-\tau)f(\tau)d\tau \in \mathcal{X}_{s,b}^{\alpha,T}$$

and satisfies the inequality

$$\|u\|_{\mathcal{X}_{s,b}^{\alpha,T}} \le C\|f\|_{Y_{s,b}^{1-\alpha,T}}. \tag{3.6}$$

In addition, there exists a $b^* \in (0,\frac{1}{2})$ such that if $f \in Y_{s,b^*}^{1-\alpha,T}$, then u belongs to the space $\mathcal{X}_{s,\frac{1}{2}}^{\alpha,T}$ and satisfies the bound

$$\|u\|_{\mathcal{X}_{s,\frac{1}{2}}^{\alpha,T}} \le C\|f\|_{Y_{s,b^*}^{1-\alpha,T}}. \tag{3.7}$$

Proof: It follows from Lemma 2.6, Lemma 2.8, Lemma 2.10, and Proposition 2.7.

Lemma 3.7. *Given $T>0$, $s>-\frac{3}{4}$, there exist $b=b(s)<\frac{1}{2}$, $\alpha=\alpha(s)>\frac{1}{2}$ and C, $\mu>0$ such that*

$$\|\partial_x(uv)\|_{Y_{s,b}^{\alpha,T}} \le CT^\mu \|u\|_{\mathcal{X}_{s,b}^{\alpha,T}} \|v\|_{\mathcal{X}_{s,b}^{\alpha,T}} \tag{3.8}$$

for any u, $v \in \mathcal{X}_{s,b}^{\alpha,T}$.

Proof: It follows from Lemma 2.11 directly.

Now we at the stage to present of the proof of Theorem 3.4

Proof of Theorem 3.4:

For given $(\phi, \vec{h}) \in D_{s,T}$ and $s \in (-\frac{3}{4}, 0)$, let $\theta \in (0,1]$ to be determined. Define $\Gamma : \mathcal{X}^{\alpha,\theta}_{s,\frac{1}{2}} \to \mathcal{X}^{\alpha,\theta}_{s,\frac{1}{2}}$ such that

$$\Gamma(\omega) := W_0(t)\phi + W_{bdr}(t)\vec{h} - \int_0^t W_0(t-\tau)\big(\omega\omega_x\big)(\tau)d\tau$$

By Lemmas 3.5, Lemma 3.6 and Lemma 3.7, we have

$$\begin{aligned}
\|\Gamma(\omega)\|_{\mathcal{X}^{\alpha,\theta}_{s,\frac{1}{2}}} &\leq \|W_0(t)\phi\|_{\mathcal{X}^{\alpha,\theta}_{s,\frac{1}{2}}} + \|W_{bdr}(t)\vec{h}\|_{\mathcal{X}^{\alpha,\theta}_{s,\frac{1}{2}}} \\
&\quad + \|\int_0^t W_0(t-\tau)\big(\omega\omega_x\big)(\tau)\|_{\mathcal{X}^{\alpha,\theta}_{s,\frac{1}{2}}} \\
&\leq C\|(\phi,\vec{h})\|_{D_{s,T}} + C\theta^{\mu}\|\omega\|^2_{\mathcal{X}^{\alpha,\theta}_{s,\frac{1}{2}}}
\end{aligned}$$

Let $r = 2C\|(\phi,\vec{h})\|_{D_{s,T}}$ and the ball

$$B_r : \{w \in \mathcal{Y}^{\theta}_{s,b}) : \|w\|_{\mathcal{X}^{\alpha,\theta}_{s,\frac{1}{2}}} \leq r\}$$

$$\begin{aligned}
\|\Gamma(\omega)\|_{\mathcal{X}^{\alpha,\theta}_{s,\frac{1}{2}}} &\leq r/2 + C\theta^{\mu}r^2 \\
&\leq r\left(1 + C\theta^{\mu}r\right) \\
&\leq r/2 + r/2 = r.
\end{aligned}$$

when we select $T^* = \theta > 0$ and $2C(T^*)^{\mu}r < 1$.

Therefore,

$$\Gamma(B_r) \subset B_r.$$

Similarly, taking $v, \omega \in \mathcal{Y}^{T^*}_{s,b}$,

$$\begin{aligned}
\|\Gamma(v) - \Gamma(\omega)\|_{\mathcal{X}^{\alpha,T^*}_{s,b}} &\leq C\theta^{\mu}\|v-\omega\|_{\mathcal{X}^{\alpha,T^*}_{s,b}}\|v+\omega\|_{\mathcal{X}^{\alpha,T^*}_{s,b}} \\
&\leq C\theta^{\mu}\|v-\omega\|_{\mathcal{X}^{\alpha,T^*}_{s,b}}\Big(\|v\|_{\mathcal{X}^{\alpha,T^*}_{s,b}} + \|\omega\|_{\mathcal{X}^{\alpha,T^*}_{s,b}}\Big) \\
&\leq 2rC\theta^{\mu}\|v-\omega\|_{\mathcal{Y}^{T^*}_{s,b}} \\
&\leq \beta\|v-\omega\|_{\mathcal{X}^{\alpha,T^*}_{s,b}}
\end{aligned}$$

with $\beta = 2C(T^*)^{\mu}r < 1$ as we have defined before. Then, by the contraction mapping theorem, the fixed point u is the unique solution of (3.5).

4 Concluding remarks

The focus of our discussion has been the well-posedness of the initial value problem of the KdV equation posed on the finite interval $(0, L)$:

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0, \qquad x \in (0, L), \ t > 0, \\ u(x, 0) = \phi(x), \\ u(0, t) = h_1(t), \ u_x(L, x) = h_2(t), \ u_{xx}(L, t) = h_3(t). \end{cases} \tag{4.1}$$

It is considered with the initial data $\phi \in H^s(0, L)$ and the boundary data $\vec{h} = (h_1, h_2, h_3)$ belongs to the space $D_{s,T} := H^{\frac{s+1}{3}}(0, T) \times H^{\frac{s}{3}}(0, T) \times H^{\frac{s-1}{3}}(0, T)$ with $s > -\frac{3}{4}$. Using the approaches developed in [5, 7] we have succeeded in showing that the IBVP (4.1) is locally well-posed in the space $H^s(0, L)$ for any $s > -\frac{3}{4}$ with $s \neq \frac{2j+1}{2}$, $j = 0, 1, 2, \cdots$, which extends and improve the earlier works of Colin and Ghidalia [16, 17, 18].

However, the well-posedness results presented in Theorem 1.1 and Theorem 1.3 are *conditional* in the sense that the uniqueness holds in a stronger Banach space than that of $C([0.T]; H^s(0, L))$. In particular, in the case of $s \geq 0$, according to Theorem 1.1, the uniqueness holds in the space

$$C([0, T]; H^s(0, L)) \cap L^2(0, T; H^{s+1}(0, L))$$

rather than in the space $C([0, T]; H^s(0, L))$. Also when $-\frac{3}{4} < s < 0$, Theorem 1.3 states that the uniqueness holds in the space

$$C([0, T]; H^s(0, L)) \cap X^{\alpha,T}_{s,\frac{1}{2}}$$

which is a stronger subspace of $C([0, T]; H^s(0, L))$. A question arises naturally:

Does the uniqueness hold in the space $C([0, T]; H^s(0, L))$?

If the uniqueness does hold in the space $C([0, T]; H^s(0, L))$, then the corresponding well-posedness is called *unconditional well-posedness.* (The interesting readers are referred to Bona, Sun and Zhang [6] for conditional and unconditional well-posedness of evolution equations.)

By using the usual energy estimate method, one can show that the uniqueness does hold for the IBVP (4.1) in the space $C([0, T]; H^s(0, L))$ when $s > \frac{3}{2}$. Thus the IBVP (4.1) is unconditionally (locally) well-posed in the space $H^s(0, L)$ for any $s > \frac{3}{2}$ with

$$s \neq \frac{2j+1}{2}, \quad j = 1, 2, \cdots$$

The following question remains open.

Question 4.1: *Is the IBVP (4.1) unconditionally well-posed in the space $H^s(0, L)$ for some $s < \frac{3}{2}$?*

By contrast, the IBVP

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0, \qquad x \in (0, L), \ t > 0, \\ u(x, 0) = \phi(x), \\ u(0, t) = h_1(t), \ u(L, x) = h_2(t), \ u_x(L, t) = h_3(t). \end{cases} \tag{4.2}$$

is known to be *unconditionally well-posed* in the space $H^s(0, L)$ for any $s > -1$. This is because that the IBVP (4.2) is known to be globally well-posed in the space $H^s(0, L)$ for any $s \geq 0$. In particular, its classical solutions exist globally. However, the IBVP (4.1) is only known to be locally well-posed. Whether it is globally well-posed is still an open question.

Question 4.2: *Is the IBVP (4.1) globally well-posed in the space $H^s(0,L)$ for some $s \geq 0$?*

This is the same question asked earlier by Colin and Ghidalia [18]. They showed that that if $\phi \in H^1(0,L)$ and $h_j \in C^1(\mathbb{R}^+)$, $j = 1,2,3$ are small enough, then the corresponding solution u of (4.1) exists globally:

$$u \in L^\infty(\mathbb{R}^+; H^1(0,L)).$$

Recently, Rivas, Usman and Zhang [41] showed that the solutions of the IBVP (4.1) exist globally (in time) in the space $H^s(0,L)$ for any $s \geq 0$ as long as its auxiliary data $(\phi, \vec{h})$ is small in the space D^s_T. In addition, they have shown that those small amplitude solutions decay exponentially if their boundary value $\vec{h}(t)$ decays exponentially. In particular, those solutions satisfying homogenous boundary conditions decay exponentially in the space $H^s(0,L)$ if their initial values are small in $H^s(0,L)$.

Note that a positive answer to Question 4.2 leads to a positive answer to Question 4.1 using the general approach developed by Bona, Sun and Zhang [6] for establishing unconditional well-posedness of nonlinear evolution equations.

Recently, Bona, Sun and Zhang [9] showed that the IBVP (4.2) is locally (unconditionally) well-posed in the space $H^s(0,L)$ for any $s > -1$. One of the key steps in their approach is to transfer the IBVP (4.2) of the KdV equation to an equivalent IBVP of the KdV-Burgers equation. Precisely, let

$$u(x,t) = e^{2t-x}v(x,t).$$

Then u is a solution of the IBVP (4.2) if and only if v is a solution of the following IBVP of the KdV-Burgers equation posed on the finite interval $(0,L)$:

$$\begin{cases} v_t + 4v_x - 3v_{xx} + v_{xxx} + e^{2t-x}(vv_x - v^2) = 0, \ x \in (0,L), \ t \geq 0, \\ v(x,0) = \phi(x)e^x, \\ v(0,t) = e^{-2t+L}h_1(t), \quad V(L,t) = e^{-2t+L}h_2(t), \\ \quad v_x(L,t) = e^{-2t+L}h_3(t) + h_1(t)e^{-2t+L}. \end{cases} \tag{4.3}$$

Consequently, one can adapt the approach of Molinet [35] in dealing with the pure initial value problems of the KdV-Burgers equation posed either on the whole line $\mathbb{R}$ or on a periodic domain $\mathbb{T}$ to show that the IBVP (4.3) is locally well-posed in the space $H^s(0,L)$ for any $s > -1$. However, the same transformation converts the IBVP(4.1) to the following IBVP of the KdV-Burgers equation

$$\begin{cases} v_t + 4v_x - 3v_{xx} + v_{xxx} + e^{2t-x}(vv_x - v^2) = 0, \ x \in (0,L), \ t \geq 0, \\ v(x,0) = \phi(x)e^x, \\ v(0,t) = e^{-2t+L}h_1(t), \quad v_x(L,t) - v(L,t) = e^{-2t+L}h_2(t), \\ \qquad v_{xx}(L,t) - v(L,t) = e^{-2t+L}(2h_2(t) + h_3(t)). \end{cases} \tag{4.4}$$

Note that the boundary conditions of (4.4) are different from those of (4.3). That brings a challenge to show that the IBVP (4.4) to be locally well-posed in $H^s(0,L)$ for $s > -1$. The following question thus remains to be open.

Question 4.3: *Is the IBVP well-posed in the space $H^s(0,L)$ for $-1 < s \leq -\frac{3}{4}$?*

Finally we would like to point out that the KdV equation including, in particular, the IBVP (4.2) has been extensively studied in the past twenty years from control point of view (cf [34, 45, 46, 42, 47, 51, 20, 14, 15, 40, 38, 43] and the reference therein). The interested readers are specially referred to Rosier and Zhang [44] for a recent survey on this subject. By contrast, the study of the IBVP (4.1) is still widely open. It will be very interesting to see if there are any differences between the IBVP (4.1) and the the IBVP (4.2) from control point view.

Acknowledgments. Ivonne Rivas was partially supported by the Taft Memorial Fund at the University of Cincinnati through Graduate Dissertation Fellowship. Bing-Yu Zhang was partially supported by the Taft Memorial Fund at the University of Cincinnati. The work was partially conducted while the second author (IR) and the third author (BZ) were participating the trimester program, *Control of Partial Differential Equations and Applications*, held at the Institut Henri Poincaré (Paris) from October 1, 2010 to December 18, 2010. They thank the Institute for its hospitality and financial support.

References

[1] Benjamin, T. B., Bona, J. L., & Mahony, J. J. *Model equations for long waves in nonlinear dispersive systems*, Proc. R. Soc. Lond. A., **272** (1972), 47–78.

[2] Bona, J. L., Pritchard, W. G. & Scott, L. R., *An evaluation of a model equation for water waves*, Philos. Trans. Roy. Soc. London Ser. A. **302** (1981), 457–510.

[3] Bona, J. L., Sun, S. M. & Zhang, B.-Y., *A nonhomogeneous boundary- value problem for the Korteweg-de Vries equation in a quarter plane*, Trans. American Math. Soc. **354** (2002), 427–490.

[4] Bona, J. L., Sun, S. M. & Zhang, B.-Y., *Forced Oscillations of a Damped Korteweg-de Vries Equation in a quarter plane*, Comm. in PDEs **5** (2003), 369–400.

[5] Bona, J. L., Sun, S. M. & Zhang, B.-Y., *A nonhomogeneous boundary-value problem for the Korteweg-de Vries Equation on a finite domain*, Comm. in PDEs. **28**: 1391–1436, 2003.

[6] Bona, J. L., Sun, S. M. & Zhang, B.-Y., *Conditional and unconditional well posedness of nonlinear evolution equations*, Adv. Differential Eq. **9** (2004), 241–265.

[7] Bona, J. L., Sun, S. M. & Zhang, B.-Y., *Boundary Smoothing Properties of the Korteweg-de Vries Equation in a Quarter Plane and Applications*, Dynamics of PDEs. **3** (2006), 1–69.

[8] Bona, J. L., Sun, S. M. and Zhang, B.-Y., Nonhomogeneous problems for the Korteweg-de Vries *and the Korteweg-de Vries-Burgers equations in a quarter plane, Ann. Inst. H. Poincaré Anal. Non Linéaire*, **25** (2008), 1145–1185.

[9] Bona, J. L., Sun, S. M. and Zhang, B.-Y., Nonhomogeneous problem for the Korteweg-de Vries equation in a bounded domain II, *J. Diff. Eqns*, **247** (2009), 2558C2596.

[10] Bourgain, J., *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, part I: Shrödinger equations*, Geom. Funct. Anal. **3**(1993), 107–156.

[11] Bourgain, J., *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, part II: the KdV-equation*, Geom. Funct. Anal. **3**(1993), 209–262.

[12] Bubnov, B. A., *Generalized boundary value problems for the Korteweg-de Vries equation in bounded domain*, Differential Equations **15**(1979), 17–21.

[13] Bubnov, B. A., *Solvability in the large of nonlinear boundary-value problem for the Korteweg-de Vries equations*,Differential Equations, **16**(1980), 24–30.

[14] Cerpa, E., *Exact controllability of a nonlinear Korteweg-de Vries equation on a critical spatial domain,* SIAM J. Control Optim., **46** (2007), 877–899.

[15] Cerpa, E. and Crépeau, E., *Boundary controllability for the nonlinear Korteweg-de Vries equation on any critical domain*, Ann. I.H. Poincaré - AN, **26**(2009), 457-475.

[16] Colin, T.; Ghidaglia, J.-M. , Un probléme aux limites pour l'équation de Korteweg-de Vries sur un intervalle borné. (French) [A boundary value problem for the Korteweg-de Vries equation on a bounded interval] Journes "quations aux Drives Partielles" (Saint-Jean-de-Monts, 1997), Exp. No. III, 10 pp., école Polytech., Palaiseau, 1997.

[17] Colin, T.; Ghidaglia, J.-M., Un problme mixte pour l'éuation de Korteweg-de Vries sur un intervalle borné. (French) [A mixed initial-boundary value problem for the Korteweg-de Vries equation on a bounded interval] C. R. Acad. Sci. Paris Sér. I Math. **324** (1997), 599C603.

[18] Colin, T., Ghidaglia, J.-M., *An initial-boundary-value problem fo the Korteweg-de Vries Equation posed on a finite interval*, Adv. Differential Equations **6** (2001), 1463-1492.

[19] Colin, T.; Gisclon, M., An initial-boundary-value problem that approximate the quarter-plane problem for the Korteweg-de Vries equation, Nonlinear Anal. **46** (2001), no. 6, Ser. A: Theory Methods, 869C892.

[20] Coron, J.-M. and Crépeau, E., *Exact boundary controllability of a nonlinear KdV equation with a critical length,* J. Eur. Math. Soc., **6** (2004), 367–398.

[21] Colliander, J. E., Kenig, C., *The generalized Korteweg-de Vries equation on the half line*, Comm. Partial Differential Eq. **27** (2002), 2187–2266.

[22] Faminskii, A. V., *On an initial boundary value problem in a bounded domain for the generalized Korteweg-de Vries equation,* International Conference on Differential and Functional Differential Equations (Moscow, 1999), Funct. Differ. Equ. **8** (2001), 183C194.

[23] Faminskii, A. V., *Global well-posedness of two initial-boundary-value problems for the Korteweg-de Vries equation*, Differential Integral Equations **20** (2007), 601C642.

[24] Ghidaglia, J.-M., *Weakly damped forced Korteweg-de Vries equations behave as a finite-dimensional dynamical system in the long time*, J. Differential Eqns. **74** (1988), 369–390.

[25] Ghidaglia, J.-M., *A note on the strong convergence towards attractors of damped forced KdV equations*, J. Differential Eqns. **110**(1994), 356–359.

[26] Holmer, J., *The Initial-Boundary Value Problem for the Korteweg-de Vries Equation*, Comm. Partial Differential Eq. **31**(2006), 1151 - 1190.

[27] Kappeler, T., Topalov, P.,*Global well-posedness of KdV in $H^{-1}(T, R)$*, Duke Math. J., **135** (2006), 327–360.

[28] Kato, T., *On the Korteweg-de Vries Equation*, manuscripta mathematica, **28** (1979), 89–99.

[29] Kato, T., *On the Cauchy problem for the (generalized) Korteweg-de Vries equations*, Advances in Mathematics Supplementary Studies, Studies in Applied Math. **8**(1983), 93–128.

[30] Kenig, C.,Ponce, G. & Vega, L., *Well-Posedness of the Initial Value Problem for the Korteweg-de Vries Equation*, J. Amer. Math. Soc. **4**(1991), 323–347.

[31] Kenig, C.,Ponce, G. & Vega, L., *Well-Posedness and scattering results for the generalized Korteweg-de Vries equations via the contraction principle*, Comm. Pure Appl. Math. **46** (1993), 527–620

[32] Kenig, C.,Ponce, G. & Vega, L., *A Bilinear Estimate with Applicatios to the KdV Equation*, J. Amer. Math. Soc., **9** (1996), 573–603.

[33] Kramer, E. F. and Zhang, B.-Y., *Nonhomogeneous boundary value problems for the Korteweg-de Vries equation on a bounded domain*, J. Syst. Sci. Complex, **23** (2010), 499–526.

[34] Komornik, V., Russell, D. L. & Zhang, B.-Y., *Stabilization de l'equation de Korteweg-de Vries*, C. R. Acad. Sci. Paris **312** (1991), 841–843.

[35] Molinet, L. and Ribaud, F., *On the low regularity of the Korteweg-de Vries-Burgers equation*, Int. Math. Res. Not. 2002, no. 37, 1979–2005.

[36] Monilet, L., *A note on ill-posedness for the KdV equation* , arXive:10043133V2 [Math AP], 15 September 2010.

[37] Molinet, L. and Vento, S., *Sharp ill-posedness and well-posedness results for the KdV-Burgers equation: the periodic case*, arXive:10054805V1 [Math AP], 26 May 2010.

[38] Pazoto, A. F., *Unique continuation and decay for the Korteweg-de Vries equation with localized damping,* ESAIM Control Optim. Calc. Var., **11** (2005), 473–486.

[39] Pazy, A., *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, **44**, Springer-Verlag, New York, 1983.

[40] Perla-Menzala, G., Vasconcellos, C. F. and Zuazua, E., *Stabilization of the Korteweg-de Vries equa- tion with localized damping,* Quart. Appl. Math., **60**(2002), 111–129.

[41] Rivas, I., Usman, M. and Zhang, B.-Y., *Global well-posedness and asymptotic behavior of a class of initial-boundary-value problem of the Korteweg-de Vries equation on a finite domain*, arXiv:1010.4658[math.AP] 22 Oct. 2010.

[42] Rosier, L., *Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain.*, ESAIM: COCV, www.emath.fr/cocv/, **2**(1997), 33–55.

[43] Rosier, L. and Zhang, B.-Y., *Global stabilization of the generalized Korteweg-de Vries equation,* SIAM J. Control Optim., **45**(2006), 927–956.

[44] Rosier, L. and Zhang, B.-Y., *Control and stabilization of the Korteweg-de Vries equation: recent progresses*, J. Syst. Sci. Complex, **22** (2009), 647–682.

[45] Russell, D. L. and Zhang, B.-Y.,*Controllability and stabilizability of the third-order linear dispersion equation on a periodic domain*, SIAM J. Control Optim. **31**(1993), 659–676.

[46] Russell, D. L. and Zhang, B.-Y., *Smoothing and decay properties of solutions of the Korteweg-de Vries equation on a periodic domain with point dissipation*, J. Math. Anal. Appl., **190** (1995), 449–488.

[47] Zhang, B.-Y., *Boundary stabilization of the Korteweg-de Vries equations*, Proc. of International Conference on Control and Estimation of Distributed Parameter Systems: Nonlinear Phenomena, Vorau (Styria, Austria), July 18-24, 1993, International Series of Numerical Mathematics, **118**, 371 – 389, 1994.

[48] Zhang, B.-Y., *A remark on the Cauchy problem for the Korteweg de-Vries equation on a periodic domain*, Diff. and Integral Eq., **8** (1995), 1191–1204.

[49] Zhang, B.-Y., *Analyticity of solutions for the generalized Korteweg de-Vries equation with respect to their initial datum* , SIAM J. Math. Anal., **26** (1995), 1488–1513.

[50] Zhang, B.-Y., *Taylor series expansion for solutions of the Korteweg-de Vries equation with respect to their initial values*, J. Func. Anal.,**129** (1995), 293–324.

[51] Zhang, B.-Y., Exact boundary controllability of the Korteweg-de Vries equation, *SIAM J. Cont. Optim.*, 37 (1999), 543–565.