THE RATIONALITY OF THE MODULI SPACES OF TRIGONAL CURVES OF ODD GENUS

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ABSTRACT. The moduli spaces of trigonal curves of odd genus $g \geq 5$ are proven to be rational.

1. INTRODUCTION

The object of this article is to prove the following.

Theorem 1.1. *The moduli space $\mathcal{T}_g$ of trigonal curves of genus $g = 2n + 1$ with $n \geq 2$ is rational.*

By a *trigonal curve* we mean an irreducible smooth projective curve which admits a degree 3 morphism to $\mathbb{P}^1$. A trigonal curve of genus $g \geq 5$ has a unique g^1_3, so that the space $\mathcal{T}_g$ to be studied is regarded as a sublocus of $\mathcal{M}_g$, the moduli space of curves of genus g. Shepherd-Barron [5] proved the rationality of $\mathcal{T}_g$ for $g = 4n + 2$ with $n \geq 1$. Hence the space $\mathcal{T}_g$ is rational possibly except when the genus g is divisible by 4. For the one lower gonality, Katsylo and Bogomolov [4], [1] established the rationality of the moduli spaces of hyperelliptic curves.

The proof of Theorem 1.1 is based on the classical relation between trigonal curves and the Hirzebruch surfaces $\mathbb{F}_N = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(N))$. Recall that a canonically embedded trigonal curve $C \subset \mathbb{P}^{g-1}$ of genus $g \geq 5$ lies on a unique rational normal scroll S. The scroll S may obtained either as the intersection of quadrics containing C, or as the scroll swept out by the lines spanned by the fibers of the trigonal map. The surface S is the image of a Hirzebruch surface $\mathbb{F}_N$ by a linear system $|\mathcal{O}_\pi(1) \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(a)|$, $a > 0$, where $\pi : \mathbb{F}_N \to \mathbb{P}^1$ is the natural projection. The trigonal map of C is the restriction of π. When C is general in the moduli $\mathcal{T}_g$, we have $N = 0$ or 1 depending on whether g is even or odd. Thus, if $L_{3,b}$ denotes the line bundle $\mathcal{O}_\pi(3) \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(b)$ on $\mathbb{F}_1$ with $g = 2b + 1$, we have the birational equivalence

$$\mathcal{T}_g \sim |L_{3,b}|/\mathrm{Aut}(\mathbb{F}_1). \tag{1.1}$$

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Here $|L_{3,b}|/\mathrm{Aut}(\mathbb{F}_1)$ stands for a rational quotient of the linear system $|L_{3,b}|$ by the algebraic group $\mathrm{Aut}(\mathbb{F}_1)$. Then Theorem 1.1 is equivalent to the following assertion in invariant theory.

Theorem 1.2. *For the line bundle $L_{3,b}$ on the Hirzebruch surface $\mathbb{F}_1$ the quotient $|L_{3,b}|/\mathrm{Aut}(\mathbb{F}_1)$ is rational for $b \geq 2$.*

The rest of this article is devoted to the proof of this theorem. In Section 2 we construct an $\mathrm{Aut}(\mathbb{F}_1)$-equivariant map from $|L_{3,b}|$ to $S^b\mathbb{F}_1$, the symmetric product of $\mathbb{F}_1$, which plays crucial role in the proof. In Section 3 the rationality for $g \geq 9$ is established by using the rational normal curves. In Section 4 the rationality of $\mathcal{T}_7$ and $\mathcal{T}_5$ is proved,

Throughout this article we work over the field of complex numbers. We denote by $\pi : \mathbb{F}_1 \to \mathbb{P}^1$ the natural projection. The (-1)-curve on $\mathbb{F}_1$ is denoted by Σ. The line bundle $\mathcal{O}_\pi(a)\otimes\pi^*\mathcal{O}_{\mathbb{P}^1}(b)$ on $\mathbb{F}_1$ will be written as $L_{a,b}$. The bundle $\mathcal{O}_\pi(1)$ is the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$ by the blow-down $\mathbb{F}_1 \to \mathbb{P}^2$.

2. Symmetric product of the Hirzebruch surface

Let $\mathbb{P}\mathcal{E}$ be the projective space bundle $\mathbb{P}\pi_*\mathcal{O}_\pi(2)$ on $\mathbb{P}^1$. The variety $\mathbb{P}\mathcal{E}$ parametrizes unordered pairs $q_+ + q_-$ of two points of $\mathbb{F}_1$ which lie on the same π-fiber. We have a rational map

$$\varphi_1 : |L_{3,b}| \dashrightarrow S^b(\mathbb{P}\mathcal{E}), \quad C \mapsto \sum_{i=1}^{b}(q_{i+} + q_{i-}) \tag{2.1}$$

defined as follows. If $C|_\Sigma = p_1 + \cdots + p_b$ and F_i is the π-fiber passing p_i, we set $q_{i+} + q_{i-} = C|_{F_i} - p_i$. The map φ_1 is clearly $\mathrm{Aut}(\mathbb{F}_1)$-equivariant. Next we define a rational map

$$\varphi_2 : S^b(\mathbb{P}\mathcal{E}) \dashrightarrow S^b\mathbb{F}_1, \quad \sum_{i=1}^{b}(q_{i+} + q_{i-}) \mapsto \sum_{i=1}^{b} q_i \tag{2.2}$$

as follows. If F_i is the π-fiber passing $\{q_{i+}, q_{i-}\}$ and $p_i = F_i \cap \Sigma$, there exists a unique involution ι_i of $F_i \simeq \mathbb{P}^1$ which fixes p_i and interchanges q_{i+} and q_{i-}. Then we let $q_i \in F_i$ be the fixed point of ι_i other than p_i. By the uniqueness of ι_i the map φ_2 is $\mathrm{Aut}(\mathbb{F}_1)$-equivariant. We study the composition map

$$\varphi = \varphi_2 \circ \varphi_1 : |L_{3,b}| \dashrightarrow S^b\mathbb{F}_1. \tag{2.3}$$

Lemma 2.1. *The map φ is dominant with a general fiber being an open set of a linear subspace of $|L_{3,b}|$.*

Proof. For a general point $q_1 + \cdots + q_b \in S^b\mathbb{F}_1$ let F_i be the π-fiber passing q_i and let $p_i = F_i \cap \Sigma$. We take an inhomogeneous coordinate x_i of $F_i \simeq \mathbb{P}^1$ in which p_i is $\{x_i = 0\}$ and q_i is $\{x_i = \infty\}$. The involution of F_i fixing p_i and q_i is given by $x_i \mapsto -x_i$. A smooth curve $C \in |L_{3,b}|$ is contained in

$\varphi^{-1}(q_1+\cdots+q_b)$ if and only if $C|_{F_i}$ has the equation $x_i(\alpha_i x_i^2+\beta_i)=0$ for each $i=1,\cdots,b$. Since these are $2b$ linear conditions on $|L_{3,b}|$, namely the vanishing of the coefficient of x_i^2 and the constant term for $C|_{F_i}$, the second assertion is proved. The dominancy of φ is a consequence of the dimension counting $\dim|L_{3,b}|=4b+9>2b$. □

Lemma 2.2. *The group* $\mathrm{Aut}(\mathbb{F}_1)$ *acts on* $S^b\mathbb{F}_1$ *almost freely if* $b\geq 4$.

Proof. First we treat the case $b\geq 5$. If a general point $p_1+\cdots+p_b\in S^b\mathbb{F}_1$ is fixed by a $g\in\mathrm{Aut}(\mathbb{F}_1)$, then g stabilizes a general $b\geq 5$ point set of the (-1)-curve Σ so that g acts trivially on Σ. Hence g fixes each p_i. As $\mathrm{Aut}(\mathbb{F}_1)$ acts almost freely on $(\mathbb{F}_1)^b$, it follows that $g=\mathrm{id}$.

Next we study the case $b=4$. Let $f:\mathbb{F}_1\to\mathbb{P}^2$ be the blow-down. For a general $p_1+\cdots+p_4\in S^4\mathbb{F}_1$ there exists a unique smooth conic Q passing $f(\Sigma)$ and $f(p_1),\cdots,f(p_4)$. Any $g\in\mathrm{Aut}(\mathbb{F}_1)$ fixing $p_1+\cdots+p_4$, regarded as an element of PGL_3, preserves Q and the five point set $f(\Sigma),f(p_1),\cdots,f(p_4)$ on it. Hence g acts trivially on Q, which implies that $g=\mathrm{id}$. □

We shall apply the no-name lemma (see [3], and also [2] for non-reductive groups) to the map φ when $b\geq 4$. For that we note the following.

Lemma 2.3. *Every line bundle on* $\mathbb{F}_1$ *admits an* $\mathrm{Aut}(\mathbb{F}_1)$*-linearization.*

Proof. We have canonical $\mathrm{Aut}(\mathbb{F}_1)$-linearizations on the bundles $K_{\mathbb{F}_1}=L_{-2,-1}$, $\pi^*K_{\mathbb{P}^1}=L_{0,-2}$, and $f^*K_{\mathbb{P}^2}=L_{-3,0}$ where $f:\mathbb{F}_1\to\mathbb{P}^2$ is the blow-down of Σ. These induce $\mathrm{Aut}(\mathbb{F}_1)$-linearizations on $L_{1,0}$ and $L_{0,1}$. Since $\mathrm{Pic}(\mathbb{F}_1)$ is freely generated by $L_{1,0}$ and $L_{0,1}$, the lemma is proved. □

By Lemma 2.3 the $\mathrm{Aut}(\mathbb{F}_1)$-action on $|L_{3,b}|$ is induced by an $\mathrm{Aut}(\mathbb{F}_1)$-representation on $H^0(L_{3,b})$. Then Lemma 2.1 shows that $|L_{3,b}|$ is $\mathrm{Aut}(\mathbb{F}_1)$-birational to the projectivization of an $\mathrm{Aut}(\mathbb{F}_1)$-linearized vector bundle on an open set of $S^b\mathbb{F}_1$. By Lemma 2.2 we may apply the no-name lemma to see the

Proposition 2.4. *For* $b\geq 4$ *we have a birational equivalence*

$$|L_{3,b}|/\mathrm{Aut}(\mathbb{F}_1)\sim\mathbb{P}^{2b+9}\times(S^b\mathbb{F}_1/\mathrm{Aut}(\mathbb{F}_1)). \tag{2.4}$$

Thus the rationality of $|L_{3,b}|/\mathrm{Aut}(\mathbb{F}_1)$ for $b\geq 4$ is reduced to a stable rationality of $S^b\mathbb{F}_1/\mathrm{Aut}(\mathbb{F}_1)$.

3. Projection of rational normal curve

In this section we prove a stable rationality of the quotient $S^b\mathbb{F}_1/\mathrm{Aut}(\mathbb{F}_1)$ to derive Theorem 1.2 for $b\geq 4$. For an integer $d\geq 0$ we consider the universal curve $f:\mathcal{H}_d\to|L_{1,d}|$ over the linear system $|L_{1,d}|$. The variety $\mathcal{H}_d$

is defined as a divisor on $\mathbb{F}_1 \times |L_{1,d}|$, and f is the restriction of the second projection $\mathbb{F}_1 \times |L_{1,d}| \to |L_{1,d}|$. The bundle $L_{0,1}$ on $\mathbb{F}_1$ induces a relative hyperplane bundle for f which we denote by $\mathcal{O}_f(1)$. Let

$$\mathcal{H}_{d,b} = \mathbb{P} f_* \mathcal{O}_f(b). \tag{3.1}$$

An open set of $\mathcal{H}_{d,b}$ parametrizes pairs $(H, q_1 + \cdots + q_b)$ where $H \in |L_{1,d}|$ is smooth and $q_1, \cdots, q_b$ are b points on H. Note that a smooth $H \in |L_{1,d}|$ is a section of π.

Lemma 3.1. *For $4 \leq b \leq 2d + 2$ we have a birational equivalence*

$$\mathcal{H}_{d,b}/\mathrm{Aut}(\mathbb{F}_1) \sim \mathbb{P}^{2d+2-b} \times (S^b \mathbb{F}_1/\mathrm{Aut}(\mathbb{F}_1)). \tag{3.2}$$

Proof. Consider the evaluation map

$$\psi : \mathcal{H}_{d,b} \dashrightarrow S^b \mathbb{F}_1, \quad (H, q_1 + \cdots + q_b) \mapsto q_1 + \cdots + q_b. \tag{3.3}$$

The fiber $\psi^{-1}(q_1 + \cdots + q_b)$ over a general $q_1 + \cdots + q_b$ is an open set of the sub linear system of $|L_{1,d}|$ of curves passing $q_1, \cdots, q_b$. Since $\dim |L_{1,d}| = 2d + 2 \geq b$, $\psi^{-1}(q_1 + \cdots + q_b)$ is non-empty and of dimension $2d + 2 - b$. In particular, ψ is dominant. Then we may apply the no-name lemma for ψ as like the proof of Proposition 2.4 to deduce the equivalence (3.2). □

By a comparison of Proposition 2.4 and Lemma 3.1, it suffices for the proof of Theorem 1.2 for $b \geq 4$ to show the rationality of $\mathcal{H}_{d,b}/\mathrm{Aut}(\mathbb{F}_1)$ for *one* d in the range $b \leq 2d + 2 \leq 3b + 9$. We begin with the

Lemma 3.2. *For $d \geq 5$ we have a birational equivalence*

$$\mathcal{H}_{d,b}/\mathrm{Aut}(\mathbb{F}_1) \sim \mathbb{P}^b \times (|L_{1,d}|/\mathrm{Aut}(\mathbb{F}_1)). \tag{3.4}$$

Proof. This lemma is an application of the no-name method for the fibration $\mathcal{H}_{d,b} \to |L_{1,d}|$. Since the bundle $L_{0,1}$ on $\mathbb{F}_1$ admits an $\mathrm{Aut}(\mathbb{F}_1)$-linearization, so is the bundle $\mathcal{O}_f(1)$ on the universal curve $\mathcal{H}_d$. Hence the sheaf $f_*\mathcal{O}_f(b)$ on $|L_{1,d}|$ is $\mathrm{Aut}(\mathbb{F}_1)$-linearized. It remains to check the almost freeness of the $\mathrm{Aut}(\mathbb{F}_1)$-action on $|L_{1,d}|$ for $d \geq 5$. For a general $H \in |L_{1,d}|$ the intersection $H \cap \Sigma$ is a general d point set of $H \simeq \mathbb{P}^1$. If a $g \in \mathrm{Aut}(\mathbb{F}_1)$ stabilizes H, then we have $g(H \cap \Sigma) = H \cap \Sigma$ so that g acts trivially on H. This is enough for concluding that $g = \mathrm{id}$. □

Blowing-down $\mathbb{F}_1$ to $\mathbb{P}^2$, we see that the quotient $|L_{1,d}|/\mathrm{Aut}(\mathbb{F}_1)$ is birational to the PGL_3-quotient of the space $\mathcal{X}_d$ of rational plane curves of degree $d+1$ having an ordinary d-fold point. Let $\widetilde{\mathcal{X}}_d$ be the space of morphisms $\phi : \mathbb{P}^1 \to \mathbb{P}^2$ such that $\phi^*\mathcal{O}_{\mathbb{P}^2}(1) \simeq \mathcal{O}_{\mathbb{P}^1}(d + 1)$ and $\phi(\mathbb{P}^1) \in \mathcal{X}_d$. We have

$$|L_{1,d}|/\mathrm{Aut}(\mathbb{F}_1) \sim \mathrm{PGL}_2 \backslash \widetilde{\mathcal{X}}_d/\mathrm{PGL}_3. \tag{3.5}$$

Let $\mathbb{P}V_{d+1} = |\mathcal{O}_{\mathbb{P}^1}(d + 1)|^\vee$ and $\Gamma_{d+1} \subset \mathbb{P}V_{d+1}$ be the rational normal curve $\phi_0(\mathbb{P}^1)$ where ϕ_0 is the embedding associated to $\mathcal{O}_{\mathbb{P}^1}(d+1)$. Recall that every morphism $\phi : \mathbb{P}^1 \to \mathbb{P}^2$ with $\phi^*\mathcal{O}_{\mathbb{P}^2}(1) \simeq \mathcal{O}_{\mathbb{P}^1}(d+1)$ is the composition of (1) the isomorphism $\phi_0 : \mathbb{P}^1 \to \Gamma_{d+1}$, (2) the projection $\Gamma_{d+1} \to \mathbb{P}(V_{d+1}/W)$ from a $(d-2)$-plane $\mathbb{P}W \subset \mathbb{P}V_{d+1}$ which is disjoint from Γ_{d+1}, and (3) an isomorphism $\mathbb{P}(V_{d+1}/W) \to \mathbb{P}^2$. The group PGL_3 acts on $\widetilde{\mathcal{X}}_d$ by transformation of an isomorphism $\mathbb{P}(V_{d+1}/W) \to \mathbb{P}^2$. Hence the quotient $\widetilde{\mathcal{X}}_d/\mathrm{PGL}_3$ is naturally birational to the locus $\mathcal{Y}_d$ in the Grassmannian $\mathbf{G}(d-2, \mathbb{P}V_{d+1})$ consisting of $(d-2)$-planes $\mathbb{P}W$ such that (i) $\mathbb{P}W \cap \Gamma_{d+1} = \emptyset$ and (ii) there exists a $(d-1)$-plane $\mathbb{P}U$ containing $\mathbb{P}W$ with $\mathbb{P}U \cap \Gamma_{d+1}$ being a d point set. For such a $\mathbb{P}W$, the $(d-1)$-plane $\mathbb{P}U$ is spanned by the point set $\mathbb{P}U \cap \Gamma_{d+1}$ because of the fact that any distinct d points on a rational normal curve in $\mathbb{P}^{d+1}$ are linearly independent. Also $\mathbb{P}U$ is uniquely determined by $\mathbb{P}W$ for an irreducible plane curve of degree $d+1$ has at most one singularity of multiplicity d. These two facts imply that $\mathcal{Y}_d$ is identified with an open set of the locus

$$\mathcal{Z}_d \subset \mathbf{G}(d-2, \mathbb{P}V_{d+1}) \times |\mathcal{O}_{\mathbb{P}^1}(d)| \tag{3.6}$$

of pairs $(\mathbb{P}W, \mathbf{p})$ such that $\mathbf{p} = p_1 + \cdots + p_d$ is a distinct d point set on $\mathbb{P}^1$ and $\mathbb{P}W$ is a hyperplane of the $(d-1)$-plane $\mathbb{P}U_{\mathbf{p}} = \langle \phi_0(p_1), \cdots, \phi_0(p_d) \rangle$. We arrived at the birational equivalence

$$\mathcal{X}_d/\mathrm{PGL}_3 \sim \mathcal{Z}_d/\mathrm{PGL}_2. \tag{3.7}$$

Now we prove the

Proposition 3.3. *If $d \geq 5$ is odd, the PGL_2-quotient of $\mathcal{Z}_d$ is rational. Hence $|L_{1,d}|/\mathrm{Aut}(\mathbb{F}_1)$ is rational too.*

Proof. The morphism

$$\mathcal{Z}_d \to |\mathcal{O}_{\mathbb{P}^1}(d)|, \quad (\mathbb{P}W, \mathbf{p}) \mapsto \mathbf{p} \tag{3.8}$$

is dominant with the fiber over a general $\mathbf{p}$ being $\mathbb{P}U_{\mathbf{p}}^\vee$. The vector space $U_{\mathbf{p}}$ is a subspace of $V_{d+1} = H^0(\mathcal{O}_{\mathbb{P}^1}(d+1))^\vee$. Since $d+1$ is even, the bundle $\mathcal{O}_{\mathbb{P}^1}(d+1)$ is PGL_2-linearized so that the PGL_2-action on $\mathbb{P}V_{d+1}$ is induced by a PGL_2-representation on V_{d+1}. Therefore $\mathcal{Z}_d$ is PGL_2-isomorphic to the projectivization of a PGL_2-linearized vector bundle on an open set of $|\mathcal{O}_{\mathbb{P}^1}(d)|$. As PGL_2 acts almost freely on $|\mathcal{O}_{\mathbb{P}^1}(d)|$, the no-name method applied to the fibration (3.8) shows that

$$\mathcal{Z}_d/\mathrm{PGL}_2 \sim \mathbb{P}^{d-1} \times (|\mathcal{O}_{\mathbb{P}^1}(d)|/\mathrm{PGL}_2). \tag{3.9}$$

The quotient $|\mathcal{O}_{\mathbb{P}^1}(d)|/\mathrm{PGL}_2$ is rational by Katsylo [4]. □

Proof of Theorem 1.2 for $b \geq 4$. We may take an odd $d \geq 5$ in the range $b \leq 2d + 2 \leq 3b + 9$. By Proposition 2.4, Lemma 3.1, and Lemma 3.2 we have

$$|L_{3,b}|/\mathrm{Aut}(\mathbb{F}_1) \sim \mathbb{P}^{4b+7-2d} \times (|L_{1,d}|/\mathrm{Aut}(\mathbb{F}_1)). \tag{3.10}$$

Then $|L_{1,d}|/\mathrm{Aut}(\mathbb{F}_1)$ is rational by Proposition 3.3. □

4. The case $g \le 7$

4.1. **The rationality of $\mathcal{T}_7$.** We consider the $\mathrm{Aut}(\mathbb{F}_1)$-equivariant map $\varphi : |L_{3,3}| \dashrightarrow S^3\mathbb{F}_1$ defined in (2.3). The group $\mathrm{Aut}(\mathbb{F}_1)$ acts almost transitively on $S^3\mathbb{F}_1$, with the stabilizer G of a general point $q_1 + q_2 + q_3$ being isomorphic to $\mathfrak{S}_3$ by the permutation action on the set $\{q_1, q_2, q_3\}$. As proved in Lemma 2.1, the fiber $\varphi^{-1}(q_1+q_2+q_3)$ is an open set of a sub linear system $\mathbb{P}V \subset |L_{3,3}|$. Then by the slice method (see [3]) we have the birational equivalence

$$|L_{3,3}|/\mathrm{Aut}(\mathbb{F}_1) \sim \mathbb{P}V/G. \tag{4.1}$$

The G-action on $\mathbb{P}V$ is induced by a G-representation on V because the bundle $L_{3,3}$ admits an $\mathrm{Aut}(\mathbb{F}_1)$-linearization. It is well-known that for any linear representation V' of $\mathfrak{S}_3$ the quotient $\mathbb{P}V'/\mathfrak{S}_3$ is rational (apply the no-name method for the irreducible decomposition). Hence the quotient $\mathbb{P}V/G$ is rational, and Theorem 1.2 is proved for $b = 3$.

4.2. **The rationality of $\mathcal{T}_5$.** We consider the $\mathrm{Aut}(\mathbb{F}_1)$-equivariant map $\varphi_1 : |L_{3,2}| \dashrightarrow S^2(\mathbb{P}\mathcal{E})$ defined in (2.1).

Lemma 4.1. *The group* $\mathrm{Aut}(\mathbb{F}_1)$ *acts almost transitively on* $S^2(\mathbb{P}\mathcal{E})$ *with the stabilizer* G *of a general point* $\mathbf{q} = (q_{1+}+q_{1-})+(q_{2+}+q_{2-})$ *being isomorphic to* $\mathfrak{S}_2 \ltimes (\mathfrak{S}_2 \times \mathfrak{S}_2)$.

Proof. Since $\mathrm{Aut}(\mathbb{F}_1)$ and $S^2(\mathbb{P}\mathcal{E})$ have the same dimention, it suffices to calculate the stabilizer G. If $p_{i\pm} \in \mathbb{P}^2$ is the image of $q_{i\pm}$ by the blow-down $\mathbb{F}_1 \to \mathbb{P}^2$, the group G is identified with the group of those $g \in \mathrm{PGL}_3$ such that for each $i = 1, 2$ we have $g(\{p_{i+}, p_{i-}\}) = \{p_{j+}, p_{j-}\}$ for some $1 \le j \le 2$. □

Let F_i be the π-fiber passing $q_{i\pm}$ and let $p_i = F_i \cap \Sigma$. The fiber $\varphi_1^{-1}(\mathbf{q})$ is an open set of the sub linear system $\mathbb{P}V \subset |L_{3,2}|$ of curves passing $q_{1+}, \cdots, q_{2-}$ and p_1, p_2. Similarly as Section 4.1, the slice method applied to the map φ_1 implies that

$$|L_{3,2}|/\mathrm{Aut}(\mathbb{F}_1) \sim \mathbb{P}V/G, \tag{4.2}$$

where the G-action on $\mathbb{P}V$ is induced by a G-representation on V. Let $\mathbb{P}W \subset \mathbb{P}V$ be the sub linear system defined by

$$\mathbb{P}W = 2F_1 + 2F_2 + 2\Sigma + |L_{1,0}|. \tag{4.3}$$

Since the group G preserves the curves $F_1 + F_2$ and Σ, the subspace $\mathbb{P}W$ is invariant under the G-action. Since G is finite, we have a G-decomposition $V = W \oplus W^{\perp}$ where $W^{\perp}$ is a G-invariant subspace. The group G acts almost freely on the linear system $|L_{1,0}|$. Hence we may apply the no-name lemma for the projection $\mathbb{P}V \dashrightarrow \mathbb{P}W$ from $\mathbb{P}W^{\perp}$ to see that

$$\mathbb{P}V/G \sim \mathbb{C}^9 \times (\mathbb{P}W/G). \tag{4.4}$$

The quotient $\mathbb{P}W/G$, being of dimension 2, is rational by Castelnuovo's theorem. This completes the proof of rationality of $\mathcal{T}_5$.

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