Prime end rotation numbers of invariant separating continua of annular homeomorphisms

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ABSTRACT. Let f be a homeomorphism of the closed annulus A isotopic to the identity, and let $X \subset \mathrm{Int}A$ be an f-invariant continuum which separates A into two domains, the upper domain U_+ and the lower domain U_-. Fixing a lift of f to the universal cover of A, one defines the rotation set $\tilde\rho(X)$ of X by means of the invariant probabilities on X, as well as the prime end rotation number $\check\rho_\pm$ of $U_\pm$. The purpose of this paper is to show that $\check\rho_\pm$ belongs to $\tilde\rho(X)$ for any separating invariant continuum X.

1. Introduction

Let f be a homeomorphism of the closed annulus $A = S^1 \times [-1, 1]$, isotopic to the identity, i. e. f preserves the orientation and each of the boundary components $\partial_\pm A = S^1 \times \{\pm 1\}$. Suppose there is an f-invariant partition of A; $A = U_- \cup X \cup U_+$, where $U_\pm$ is a connected open set containing the boundary component $\partial_\pm A$ and X is a connected compact set. Let

$$\pi : \tilde A = \mathbb{R} \times [-1, 1] \to S^1 \times [-1, 1]$$

be the universal covering map and $T : \tilde A \to \tilde A$ a generator of the covering transformation group; $T(\xi, \eta) = (\xi + 1, \eta)$. Denote by $p : \tilde A \to \mathbb{R}$ the projection onto the first factor.

Fix once and for all a lift $\tilde f : \tilde A \to \tilde A$ of f. Then the function $p \circ \tilde f - p$ is T-invariant and can be looked upon as a function on the annulus A. Define the *rotation set* $\tilde\rho(X)$ as the set of values $\mu(p \circ \tilde f - p)$, where μ ranges over the f-invariant probability measures supported on X. The rotation set is a compact interval (maybe one point) in $\mathbb{R}$, which depends upon the choice of the lift $\tilde f$ of f.

The first example of an invariant continuum X such that the frontiers of $U_\pm$ satisfy $\mathrm{Fr}(U_+) = \mathrm{Fr}(U_-) = X$ and that the rotation set $\tilde\rho(X)$ is not a singleton is constructed by G. D. Birkhoff in his 1932 year paper [**B**], and is refered to as a *Birkhoff attractor*. It turns out that the Birkhoff attractor is an indecomposable continuum ([**C, L2**]). Furthermore it is shown by P. Le Calvez ([**L1**]) that for

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any rational number between the two prime end rotation numbers is realized by a correspoding periodic point of $\tilde{f}$.

Let $\hat{U}_\pm = U_\pm \cup \partial_\infty U_\pm$ be the prime end compactification of $U_\pm$, where $\partial_\infty U_\pm$ is the space of the prime ends ([**E, M, MN**]). The space $\partial_\infty U_\pm$ is homeomorphic to the circle and $\hat{U}_\pm$ to the closed annulus. As is well known, the homeomorphism f restricted to $U_\pm$ extends to a homeomorphism $\hat{f}_\pm : \hat{U}_\pm \to \hat{U}_\pm$. Denoting $I_+ = [0,1]$ and $I_- = [-1,0]$, define a homeomorphism

$$\Psi_\pm : \hat{U}_\pm \to S^1 \times I_\pm$$

such that $\Psi_\pm(\partial_\infty U_\pm) = S^1 \times 0$. By some abuse of notations denote by $\pi : \check{U}_\pm \to \hat{U}_\pm$ the universal covering map. Thus $\pi^{-1}(U_\pm)$ is considered to be a subspace of both $\tilde{A}$ and $\check{U}_\pm$. Let $\check{f}_\pm : \check{U}_\pm \to \check{U}_\pm$ be the lift of $\hat{f}_\pm$ such that $\check{f}_\pm = \tilde{f}$ on $\pi^{-1}(U_\pm)$. The rotation number of the restriction of $\check{f}_\pm$ to $\pi^{-1}(\partial_\infty U_\pm)$, denoted by $\check{\rho}_\pm$, is called the *prime end rotation number* of $U_\pm$.

The purpose of this paper is to show the following.

Theorem 1. *The prime end rotation number $\check{\rho}_\pm$ belongs to $\tilde{\rho}(X)$.*

This result is already known for $X = \mathrm{Fr}(U_-) = \mathrm{Fr}(U_+)$ ([**BG**]), and for any X if the homeomorphism f is area preserving (Lemma 5.4, [**FL**]).

It is shown in Theorem 2.2 of [**F**] that any rational number in $\tilde{\rho}(X)$ is realized by a periodic point if X consists of nonwandering points. Notice that then X, consisting of chain recurrent points, is chain transitive since it is connected, and thus satisfies the condition of Theorem 2.2. As a corollary we have

Corollary 2. *If X consists of nonwandering points and if p/q lies in the closed interval bounded by $\check{\rho}_-$ and $\check{\rho}_+$, then there is a point $x \in \pi^{-1}(X)$ such that $\tilde{f}^q(x) = T^p(x)$.*

In what follows we also use the following notation. Let

$$\check{\Psi}_\pm : \check{U}_\pm \to \mathbb{R} \times I_\pm$$

be a lift of $\Psi_\pm$, and define $\check{p}_\pm : \check{U}_\pm \to \mathbb{R}$ by $\check{p}_\pm = p \circ \check{\Psi}_\pm$. The projection $\check{p}_\pm$ is within a bounded error of p on $\pi^{-1}(C)$ for a compact domain C of $U_\pm$. But they may be quite different on the whole $\pi^{-1}(U_\pm)$.

2. Proof

First of all let us state a deep and quite useful theorem of P. Le Calvez ([**L3**]) which plays a key role in the proof. A fixed point free and orientation preserving homeomorphism F of the plane $\mathbb{R}^2$ is called a *Brouwer homeomorphism.* A proper oriented simple curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ is called a *Brouwer line* for F if $F(\gamma) \subset R(\gamma)$ and $F^{-1}(\gamma) \subset L(\gamma)$, where $R(\gamma)$ (resp. $L(\gamma)$) is the right (resp. left) side complementary domain of γ, which is decided by the orientation of γ.

Theorem 2.1. *Let F be a Brouwer homeomorphism commuting with the elements of a group Γ which acts on $\mathbb{R}^2$ freely and properly discontinuously. Then there is a Γ-invariant oriented topological foliation of $\mathbb{R}^2$ whose leaves are Brouwer lines of F.*

The proof of Theorem 1 is by absurdity. Assume in way of contradiction that $\check{\rho}_- < p/q < \inf \tilde{\rho}(X)$. Considerng $\tilde{f}^q T^{-p}$ instead of $\tilde{f}$, it suffices to deduce a contradiction under the following assumption.

Assumption 2.2. $\check{\rho}_- < 0 < \inf \tilde{\rho}(X)$.

Since $\inf \tilde{\rho}(X) > 0$, the map $\tilde{f}$ does not admit a fixed point in $\pi^{-1}(X)$. The overall strategy of the proof is to modify the homeomorphism f away from X to a new one g without creating fixed points in A such that the restrictions of $\tilde{g}$ to the lifts of the both boundary circles $\pi^{-1}(\partial_\pm A)$ are nontrivial rigid translations by the same translation number. Then by glueing the two boundary circles we obtain a torus T^2 and a homeomorphism on T^2. Now we can apply Theorem 2.1 to the lift of the homeomorphism to the universal covering space. This yields a topological foliation on T^2, which has long been well understood. The proof will be done by analyzing the foliation. We first prepare a lemma which is necessary for the desired modification. We do not presume Assumption 2.2 in the following.

Lemma 2.3. *Assume $\tilde{f}$ does not admit a fixed point in $\pi^{-1}(X)$. Then the prime end rotation number $\check{\rho}_\pm$ is nonzero.*

Proof: Consider the mapping $\tilde{f} - \mathrm{Id}$ defined on $\tilde{A}$. Since it is T-invariant, it yields a mapping from A, still denoted by the same letter. Then since there is no fixed point of $\tilde{f}$ in X, we have $(\tilde{f} - \mathrm{Id})(X) \subset \mathbb{R}^2 \setminus \{0\}$. Therefore there is an annular open neighbourhood V of X for which we get a mapping

$$\tilde{f} - \mathrm{Id} : V \to \mathbb{R}^2 \setminus \{0\}.$$

Clearly for any positively oriented essential simple closed curve γ in V, the degree of the map

$$\tilde{f} - \mathrm{Id} : \gamma \to \mathbb{R}^2 \setminus \{0\}$$

must be the same. If the curve γ is contained in $U_\pm$, then the degree can be studied by considering the map $\check{f}_\pm$ defined on the lift $\check{U}_\pm$ of the prime end compactification $\hat{U}_\pm$. If the prime end rotation number $\check{\rho}_\pm$ is nonzero, the degree is clearly 0. Notice that our definition of the degree differs from the usual definition of the index.

To analyze the case $\check{\rho}_\pm = 0$, we need the following form of the Cartwright-Littlewood theorem [**CL**].

Theorem 2.4. *If $\check{\rho}_+ = 0$ and if $\mathrm{Fix}(\tilde{f}) \cap \pi^{-1}(X) = \emptyset$, then the map $\hat{f}_+$ on $\partial_\infty U_+$ is Morse Smale and the attractors (resp. repellors) of $\hat{f}_+|_{\partial_\infty U_+}$ are attractors (resp. repellors) of the whole map $\hat{f}_+$.*

This is slightly stronger than the usual version in which it is assumed that $\mathrm{Fix}(f) \cap X = \emptyset$. However the proof works as well under the assumption of Theorem 2.4. See e. g. Sect. 3 of [**MN**].

Let us complete the proof of Lemma 2.3. Theorem 2.4 enables us to compute the degree of the curve δ in $U_\pm$ when $\check{\rho}_\pm = 0$. The degree is n if $\delta \subset U_-$ and $-n$ if $\delta \subset U_+$, where n is the number of the attractors. Since the degree must be the same in U_- and U_+, the conclusion follows. □

Now we have $\check{\rho}_- < 0$ and $\check{\rho}_+ \neq 0$ by Assumption 2.2 and Lemma 2.3. Let us start the modification of f.

Lemma 2.5. *Under Assumption 2.2, there exists a homeomorphism g of A such that*
(1) $g = f$ in some neighbourhood of X,
(2) $\tilde{g}$ does not admit a fixed point in $\tilde{A}$, where $\tilde{g}$ is the lift of g such that $\tilde{g} = \tilde{f}$ on $\pi^{-1}(X)$,

(3) $\tilde g$ is a negative rigid translation by the same translation number on $\pi^{-1}(\partial_\pm A)$, and
(4) $\check p_- \circ \check g_- - \check p_- \leq -c$ on $\hat U_-$ for some positive number c.

PROOF: The modification in U_- will be done in the following way. We identify $\hat U_-$ with $S^1 \times [-1,0]$ by the homeomorphism Ψ_- and the universal covering space $\check U_-$ with $\mathbb{R}\times[-1,0]$. Thus $\check p_-$ is just the projection onto the first factor; $\check p_-(\xi,\eta)=\xi$. Since $\check\rho_- < 0$, the lift

$$\check f_- : \mathbb{R}\times[-1,0] \to \mathbb{R}\times[-1,0]$$

of $\hat f_-$ satisfies that $\check p_- \circ \check f_-(\xi,0) < \xi - 2c$ for some $c > 0$. Therefore changing the coordinates of $[-1,0]$ if necessary, one may assume that $\check p_- \circ \check f_-(\xi,\eta) \leq \xi - c$ if $(\xi,\eta)\in\mathbb{R}\times[-1/2,0]$. Define a homeomorphism h of $S^1\times[-1,0]$ by

$$h(\xi,\eta) = (\xi + \varphi(\eta)\mathrm{mod}1, \eta),$$

where $\varphi : [-1,0] \to (-\infty,0]$ is a continuous function such that $\varphi([-1/2,0]) = 0$ and

$$\varphi(\eta) \leq -\sup\{(\check p_- \circ \check f_- - \check p_-)(\xi,\eta) \mid \xi \in S^1\} - c.$$

Define $g = f\circ h$. Then its lift $\check g_-$ satisfies

$$\check p_- \circ \check g_- - \check p_- \leq -c$$

on $\check U_- = \mathbb{R}\times[-1,0]$. Clearly condition (3) for $\pi^{-1}(\partial_- A)$ can be established by a further obvious modification.

Now to modify f in U_+, we do the same thing as in U_-. If the prime end rotation number $\check\rho_+$ is negative, then with an auxiliary modification we are done. If it is positive insert a time one map of the Reeb flow. □

Consider the torus T^2 which is obtained from A by glueing the two boundary curves $\partial_- A$ and $\partial_+ A$. Then the condition (3) above shows that g induces a homeomorphism of T^2, again denoted by g. The universal cover of T^2 is $\mathbb{R}^2$ and $\tilde A = \mathbb{R}\times[-1,1]$ is a subset of $\mathbb{R}^2$. The lift $\tilde g : \tilde A \to \tilde A$ can be extended uniquely to a lift $\tilde g : \mathbb{R}^2 \to \mathbb{R}^2$ of $g : T^2 \to T^2$. The covering transformation group Γ is isomorphic to $\mathbb{Z}^2$, generated by the horizontal translation T and the vertical translation by 2, denoted by S. Since $\tilde g$ is a Brouwer homeomorphism which commutes with Γ, there is a Γ-invariant oriented foliation on $\mathbb{R}^2$ whose leaves are Brouwer lines for $\tilde g$. This yields an oriented foliation $\mathcal{F}$ on the torus T^2. The proof is divided into several cases according to the topological type of the foliation $\mathcal{F}$. We are going to deduce a contradiction in each case. But before going into detail we need another lemma.

Lemma 2.6. *For any $C > 0$ there is $n > 0$ such that $p\circ\tilde g^n - p \geq C$ on X.*

PROOF: If not, there would be a point $x_n \in X$ for any $n > 0$ such that

$$(p\circ\tilde g^n - p)(x_n) = \sum_{j=0}^{n-1}(p\circ\tilde g - p)(g^j(x_n)) < C$$

for some $C > 0$, and the averages of Dirac masses

$$\mu_n = \frac{1}{n}\sum_{j=0}^{n-1} g_*^j \delta_{x_n}$$

would satisfy $\mu_n(p\circ\tilde g - p) < C/n$. Therefore an accumulation point μ of μ_n would have the property that $\mu(p\circ\tilde g - p) \le 0$, contradicting the assumption $\inf\tilde\rho(X) > 0$. □

CASE 1. *The foliation $\mathcal F$ does not admit a compact leaf.* Then $\mathcal F$ is conjugate either to a linear foliation or to a Denjoy foliation, both of irrational slope. The lift $\tilde{\mathcal F}$ of $\mathcal F$ to the open annulus $\mathbb R^2/\langle T\rangle$ is conjugate to a foliation by vertical lines. The space of leaves of $\tilde{\mathcal F}$ is homeomorphic to S^1 and there is a projection from $\mathbb R^2/\langle T\rangle$ to S^1 along the leaves of the foliation. This lifts to a projection $q:\mathbb R^2\to\mathbb R$.

Now q restricted to $\tilde A$ is within a bounded error of the first factor projection $p:\tilde A\to\mathbb R$ that we have used for the definition of the rotation set $\tilde\rho(X)$. In fact both p and q are lifts of degree one maps from $\mathbb R^2/\langle T\rangle$ to S^1 and their difference is bounded on the preimage $\tilde A = \pi^{-1}(A)$ of a compact subset A. Thus Lemma 2.6 shows that $q\circ\tilde g^n(x)\to\infty$ $(n\to\infty)$ for $x\in\pi^{-1}(X)$. That is, the foliation $\tilde{\mathcal F}$ is oriented upward. But this shows that $q\circ\tilde g(x) > q(x)$ even for a point $x\in\pi^{-1}(\partial_-A)$. On the other hand by condition (3) of Lemma 2.5, $\tilde g$ is a negative translation on $\pi^{-1}(\partial_-A)$. A contradiction.

CASE 2.1. *The foliation $\mathcal F$ admits a compact leaf L of nonzero slope and does not admit a Reeb component.* In this case the lifted foliation $\tilde{\mathcal F}$ is also conjugate to the vertical foliation and the argument of Case 1 applies.

CASE 2.2. *The foliation $\mathcal F$ admits a Reeb component R of nonzero slope.* The Brouwer property of leaves implies that $g(R)\subset \mathrm{Int}(R)$ or $g^{-1}(R)\subset\mathrm{Int}(R)$. That is, a point of the boundary of R is wandering under g. Therefore ∂_-A, consisting of nonwandering points of g according to (3) of Lemma 2.5, cannot intersect the boundary of R, which is however impossible since the slope of R is nonzero.

CASE 2.3. *The foliation $\mathcal F$ admits a compact leaf of slope 0.* Hereafter we only consider the dynamics and the foliation on the open annulus $\mathbb R^2/\langle T\rangle$. Recall that A is a subset of $\mathbb R^2/\langle T\rangle$, and the homeomorphism g on A is extended to the whole $\mathbb R^2/\langle T\rangle$, again denoted by g, in such a way that g commutes with the vertical translation S, while the foliation is denoted by $\tilde{\mathcal F}$ as before.

Now the foliation $\tilde{\mathcal F}$ yields a partition $\mathcal P$ of the open annulus $\mathbb R^2/\langle T\rangle$ into compact leaves, interiors of Reeb components and foliated I-bundles. The set $\mathcal P$ is totally ordered by the height. The minimal element which intersects X cannot be a compact leaf by the Brouwer line property. Let R be the closure of the minimal element. Thus R is either a Reeb component or a foliated I-bundle such that $\mathrm{Int}(R)\cap X\neq\emptyset$ and $\partial_-R\cap X=\emptyset$, where ∂_-R is the lower boundary curve of R.

Assume for a while that ∂_-R is oriented from the right to the left. Thus the homeomorphism g carries ∂_-R into the upper complement of ∂_-R.

CASE 2.3.1 *R is a Reeb component.* First notice that $g(R)\subset\mathrm{Int}R$ and that the interior leaves of R are oriented upwards by the assumption $\inf\tilde\rho(X)>0$ and the fact that $g(X\cap R)\subset X\cap R$. Choose a simple arc

$$\alpha:[0,1]\to\pi^{-1}(R)$$

such that $\alpha(0)\in\pi^{-1}(\partial_-R)$, $\alpha(1)=\tilde g(\alpha(0))$, and $\alpha((0,1))\subset\mathrm{Int}(\pi^{-1}(R))\setminus\tilde g(\pi^{-1}(R))$. Since $g^{-1}(\pi(\alpha))$ is below $\mathrm{Int}R$, $\tilde g^{-1}(\alpha)$, and hence α, is contained in $\pi^{-1}(U_-)$.

Concatenating nonnegative iterates of α, we obtain a simple path $\gamma:[0,\infty)\to\pi^{-1}(R\cap U_-)$ such that $\tilde g\circ\gamma(t)=\gamma(t+1)$ for any $t\ge 0$. Let $q:\pi^{-1}(\mathrm{Int}(R))\to\mathbb R$

be the lift of the projection along the leaves. Since $\gamma([1,\infty))$ is contained in the lift of a compact subset $\tilde g(R)\subset \mathrm{Int}(R)$ and the leaves in $\mathrm{Int}(R)$ is oriented upward, we have $q\circ\gamma(t)\to\infty$ as $t\to\infty$. We also have $p\circ\gamma(t)\to\infty$ because q is within bounded error of p on $\gamma([1,\infty))$.

On the other hand by condition (4) of Lemma 2.5, we have $\check p\circ\gamma(t)\to-\infty$ as $t\to\infty$. In particular the curve γ is proper both in $\tilde A$ and in $\check U_-$ pointing toward the opposite direction. By joining the point $\gamma(0)$ to an appropriate point in $\pi^{-1}(\partial_- A)$, we obtain a simple curve δ in $\pi^{-1}(U_-)$ starting at a point on $\pi^{-1}(\partial_- A)$ which extends γ.

Notice that there is a point of $\pi^{-1}(X)$ on the left of a proper oriented curve δ in $\tilde A$, because the map p is bounded from below on δ and a high iterate of T^{-1} carries a point in $\pi^{-1}(X)$ beyond that bound. (There might be a point of $\pi^{-1}(X)$ on the right of δ however.)

Let x be a point in $\pi^{-1}(\partial_- A)$ left to the initial point of δ. Then there is a simple path $\beta:[0,\infty)\to\pi^{-1}(U_-)$ such that $\beta(0)=x$, $\lim_{t\to\infty}\beta(t)\in\pi^{-1}(X)$, and β is disjoint from δ. The path β, extendable in $\pi^{-1}(A)$ is also extendable in $\check U_-$, the lift of the prime end compactification. (See e. g. Lemma 2.5 of [**MN**].) This implies that β defines a simple path in $\check U_-$ joining x to a prime end in $\pi^{-1}(\partial_\infty U_-)$ without intersecting δ, which is impossible since $\pi^{-1}(\partial_\infty U_-)$ is contained in the right side of the proper path δ in $\check U_-$ since $\check p_-\delta(t)\to-\infty$, while x is on the left side. A contradiction.

Case 2.3.2 *R is a foliated I-bundle.* Thus the upper boundary curve ∂_+R of R is also oriented from the right to the left, and its image by g lies on the upper complement of R. The interior leaves of R are oriented upward.

Recall that the boundary component $\partial_- A$ consisting of nonwandering points cannot intersect a compact leaf. Moreover $\partial_- A$ lies in a Reeb component or a foliated I-bundle whose interior leaves are oriented downward since $p\tilde g^n(x)\to-\infty$ as $n\to\infty$ for $x\in\pi^{-1}(\partial_- A)$. Let C be the annulus in $\mathbb{R}^2/\langle T\rangle$ bounded by $\partial_- A$ and $\partial_+ R$, the upper boundary curve of R. Notice that $\mathrm{Int}(C)$ contains $\partial_- R$.

Case 2.3.2.1 *The intersection $X\cap C$ has a component which separates $\partial_- A$ from $\partial_+ A$.* One can derive a contradiction by the same argument as in Case 2.3.1, since the like defined path γ cannot evade R.

Case 2.3.2.2 *There is a simple path in U_- joining a point in $\partial_- A$ with a point in $\partial_+ R$.* Notice first of all that $g^{-1}(C)\subset C$. Let $\mathcal{Y}$ be the family of the connected components of $\pi^{-1}(X\cap C)$. Then any element $Y\in\mathcal{Y}$ is compact, and intersects $\pi^{-1}(\partial_+ R)$ since otherwise Y would be a connected component of $\pi^{-1}(X)$ itself.

Choose a simple curve $\gamma:[0,1]\to\pi^{-1}(C)$ such that
(1) $\gamma(0)\in\pi^{-1}(\partial_- A)$,
(2) $\gamma(1)\in\pi^{-1}(X\cap C)$, and
(3) $\gamma([0,1))\subset\pi^{-1}(U_-\cap C)$.

Let Y be an element of $\mathcal{Y}$ which contains $\gamma(1)$. Then there are two unbounded connected components of the complement $\pi^{-1}(C)\setminus(Y\cup\gamma)$, one $L(Y\cup\gamma)$ on the left, and the other $R(Y\cup\gamma)$ on the right.

Notice that for any $n>0$, $\tilde g^{-n}\gamma$ is a path in C, and that $p\tilde g^{-n}(\gamma(1))\to-\infty$ and $p\tilde g^{-n}(\gamma(0))\to\infty$ as $n\to\infty$. That is, for any large n, $\tilde g^{-n}(\gamma(1))\in L(Y\cup\gamma)$ and $\tilde g^{-n}(\gamma(0))\in R(Y\cup\gamma)$, showing that $\tilde g^{-n}(\gamma)$ intersects γ. On the other hand in $\check U_-$, γ defines a curve from a point in $\pi^{-1}(\partial_- A)$ to a prime end in $\pi^{-1}(\partial_\infty U_-)$.

But by condition (4) of Lemma 2.5, γ cannot intersect $\tilde{g}^{-n}(\gamma)$ for any large n. A contradiction.

Finally the case where $\partial_- R$ is oriented from the left to the right can be dealt with similarly by reversing the time. This completes the proof of Theorem 1.

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