Finite-dimensional vertex algebra modules over fixed point commutative subalgebras

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Abstract

Let A be a connected commutative $\mathbb{C}$-algebra with derivation D, G a finite linear automorphism group of A which preserves D, and $R = A^G$ the fixed point subalgebra of A under the action of G. We show that if A is generated by a single element as an R-algebra and is a Galois extension over R in the sense of M. Auslander and O. Goldman, then every finite-dimensional vertex algebra R-module has a structure of twisted vertex algebra A-module.

Keywords: vertex algebra; Galois extension; commutative algebra

1 Introduction

Vertex algebras and modules over a vertex algebra were introduced by Borcherds in [4]. As an example, every commutative ring A with an arbitrary derivation D has a structure of vertex algebra, and every ring A-module naturally

*Partially supported by JSPS Grant-in-Aid for Scientific Research No. 20740002.

becomes a vertex algebra A-module. However, this does not imply that ring A-modules and vertex algebra A-modules are same. In fact, a vertex algebra $\mathbb{Z}[z, z^{-1}]$-module which is not a ring $\mathbb{Z}[z, z^{-1}]$-module was given in [4, Section 8], where $\mathbb{Z}[z, z^{-1}]$ is the ring of Laurent polynomials over $\mathbb{Z}$. This tells us that in general these two kind of A-modules are certainly different. From now on, for a commutative $\mathbb{C}$-algebra A with derivation D, we shall call a vertex algebra A-module a *vertex algebra* (A, D)*-module* to distinguish it from ring A-modules. It is a natural first step to investigate vertex algebra (A, D)-modules to understand modules over general vertex algebras. In [19, 20] for the polynomial ring $\mathbb{C}[s]$ and the field of rational functions $\mathbb{C}(s)$, the finite-dimensional vertex algebra modules which are not $\mathbb{C}$-algebra modules are classified.

Let A be a commutative $\mathbb{C}$-algebra with derivation D, G a finite linear automorphism group of A which preserves D, and $R = A^G$ the fixed point subalgebra of A under the action of G. In this paper, we shall investigate a relation between vertex algebra (R, D)-modules and twisted vertex algebra (A, D)-modules. In Theorem 1, I shall show that if A is a connected commutative $\mathbb{C}$-algebra generated by a single element as an R-algebra and is a finite Galois extension over R in the sense of [3, p.396], then every finite-dimensional indecomposable vertex algebra (R, D)-module becomes a g-twisted vertex algebra (A, D)-module for some $g \in G$. This is a generalization of [20, Theorem 1] and is related the following open conjecture on vertex operator algebras: let V be a vertex operator algebra and H a finite automorphism group of V. It is conjectured that under some conditions on V, every irreducible module over the fixed point vertex operator subalgebra V^H is contained in some irreducible h-twisted V-module for some $h \in H$ (cf.[7]). The conjecture is confirmed for some examples in [1, 8, 10, 11, 12, 21, 22]. However A is not a vertex operator algebra except in the case that $D = 0$ and $\dim_{\mathbb{C}} A < \infty$, Theorem 1 implies that the conjecture holds for all finite-dimensional vertex algebra R-modules in a stronger sense.

This paper is organized as follows: In Section 2 we recall some notation and properties of Galois extensions of rings, vertex algebras and their modules. In Section 3 we show that every finite-dimensional indecomposable vertex algebra R-module becomes a g-twisted vertex algebra (A, D)-module for some $g \in G$. In Section 4 we give the classification of the finite-dimensional vertex algebra $\mathbb{C}[s, s^{-1}]$-modules which are not $\mathbb{C}$-algebra $\mathbb{C}[s, s^{-1}]$-modules. In Section 5 for the $\mathbb{C}$-algebra $A = \mathbb{C}[s, s^{-1}][t]/(t^n - s)$, which is a Galois extension over $\mathbb{C}[s, s^{-1}]$ with Galois group the cyclic group of order n, and for all finite-dimensional indecomposable vertex algebra $\mathbb{C}[s, s^{-1}]$-modules (M, Y_M) obtained in Section 4, we study twisted vertex algebra (A, D)-module structures over (M, Y_M).

2 Preliminary

We assume that the reader is familiar with the basic knowledge on vertex algebras as presented in [4, 9, 17].

Throughout this paper all rings and algebras are commutative and associative and have identity elements, R denotes a ring, $R[Z]$ denotes the polynomial ring in one variable Z over R , G denotes a finite group, ζ_p denotes a primitive p-th root of unity for a positive integer p, and $(V, Y, \mathbf{1})$ denotes a vertex algebra. Recall that V is the underlying vector space, $Y(\cdot, x)$ is the linear map from V to $(\mathrm{End}\, V)[[x, x^{-1}]]$, and $\mathbf{1}$ is the vacuum vector. Let $\mathcal{D}$ be the endomorphism of V defined by $\mathcal{D}v = v_{-2}\mathbf{1}$ for $v \in V$.

First, we recall some results in [3, 5, 6, 15] for separable algebras over a ring. A ring R is called *connected* if R has no idempotent other than 0 and 1. An R-algebra A is called *separable* if A is a projective $A \otimes_R A$-module. An R-algebra A is called *strongly separable* if it is finitely generated, projective, and separable over R. Let us recall the Galois extension of R introduced in [3, p.396]. The following definition, which is equivalent to that in [3, p.396], is given in [5, Theorem 1.3].

Definition 1. Let A be a ring extension of R and let G be a finite group of automorphisms of A. We denote by A^G the fixed point subring of A under the action of G. The ring A is called a *Galois extension* of R with Galois group G, if the following three conditions hold:

(1) $A^G = R$.

(2) For each non-zero idempotent $e \in A$ and each $g \neq h$ in G, there is an element $x \in A$ with $g(x)e \neq h(x)e$.

(3) A is a separable R-algebra.

Note that if A is connected, then the condition (2) in Definition 1 is always satisfied. It follows from [5, Theorem 1.3] that if A is a Galois extension of R, then A is a strongly separable R-algebra.

In [15, p.467], A polynomial $P(Z) \in R[Z]$ is called *separable* in case $P(Z)$ is monic and the factor ring $R[Z]/(P(Z))$ is a separable R-algebra. In this case, $R[Z]/(P(Z))$ is strongly separable since $R[Z]/(P(Z))$ is a free R-module of rank $\deg P(Z)$. For an R-algebra A, an element $\theta \in A$ is called a *primitive element* if $A = R[\theta]$, namely A is generated by a single element θ as an R-algebra. It is shown in [15, Theorem 2.9] that if A is a strongly separable R-algebra and if A has a primitive element, then there is a separable polynomial $P(Z)$ such that $A \cong R[Z]/(P(Z))$ as R-algebras.

Let R be a connected ring, $P(Z) \in R[Z]$ a separable polynomial, and suppose that the factor ring $A = R[Z]/(P(Z))$ is connected and is a Galois extension of R with Galois group G. Set $\theta = Z + P(Z) \in A$. Since $A = R[\theta]$, we have $g(\theta) \neq \theta$ for all $g \in G$ without the identity element. By [5, Lemma 4.1] and [15, Lemma 2.1], the order of G is equal to $\deg P(Z)$. Thus, G acts regularly on the set of all roots of the polynomial $P(Z)$ in A and hence $P(Z) = \prod_{g \in G}(Z - g(\theta))$. For an R-linear homomorphism f from A to an R-algebra B, [15, Lemma 2.1] says that $f(g(\theta)) \neq f(h(\theta))$ for all $g \neq h$ in G. This tells us that if B is an integral domain, f induces a bijection from $\{g(\theta) \mid g \in G\}$ to the set of all roots of $f(P(Z)) \in B[Z]$ in B. In particular, $f(P(Z))$ has no multiple root.

Next, we recall some results in [4] for a vertex algebra constructed from a commutative $\mathbb{C}$-algebra with a derivation.

Proposition 1. [4] *The following hold:*

(1) Let A be a commutative $\mathbb{C}$-algebra with identity element 1 and D a derivation of A. For $a \in A$, define $Y(a, x) \in (\operatorname{End} A)[[x]]$ by

$$Y(a, x)b = \sum_{i=0}^{\infty} \frac{1}{i!}(D^i a)bx^i$$

for $b \in A$. Then, $(A, Y, 1)$ is a vertex algebra.

(2) Let $(V, Y, \mathbf{1})$ be a vertex algebra such that $Y(u, x) \in (\operatorname{End} V)[[x]]$ for all $u \in V$. Define a multiplication on V by $uv = u_{-1}v$ for $u, v \in V$. Then, V is a commutative $\mathbb{C}$-algebra with identity element $\mathbf{1}$ and $\mathcal{D}$ is a derivation of V.

Throughout the rest of this section, A is a commutative $\mathbb{C}$-algebra with identity element 1 and D a derivation of A. Let $(A, Y, 1)$ be the vertex algebra constructed from A and D in Proposition 1 and let (M, Y_M) be a vertex algebra A-module. We call M a *vertex algebra (A, D)-module* to distinguish vertex algebra A-modules from $\mathbb{C}$-algebra A-modules as stated in Section 1.

Proposition 2. [4] *The following hold:*

(1) Let M be a $\mathbb{C}$-algebra A-module. For $a \in A$, define $Y_M(a, x) \in (\mathrm{End}_{\mathbb{C}} M)[[x]]$ by

$$Y(a, x)u = \sum_{i=0}^{\infty} \frac{1}{i!}(D^i a)ux^i$$

for $u \in M$. Then, (M, Y_M) is a vertex algebra (A, D)-module.

(2) Let (M, Y_M) be a vertex algebra (A, D)-module such that $Y(a, x) \in (\mathrm{End}_{\mathbb{C}} M)[[x]]$ for all $a \in A$. Define an action of A on M by $au = a_{-1}u$ for $a \in A$ and $u \in M$. Then, M is a $\mathbb{C}$-algebra A-module.

By Proposition 2, if there exists a vertex algebra (A, D)-module (M, Y_M) with $Y_M(a, x) \notin (\mathrm{End}_{\mathbb{C}} M)[[x]]$ for some element a in A, then vertex algebra (A, D)-modules and $\mathbb{C}$-algebra A-modules are different. However, no simple criterion for the existence of such a module (M, Y_M) is known.

For a $\mathbb{C}$-linear automorphism g of V of finite order p, set $V^r = \{u \in V \mid gu = \zeta_p^r u\}, 0 \leq r \leq p - 1\}$. We recall the definition of g-twisted V-modules.

Definition 2. A *g-twisted V-module* M is a vector space equipped with a linear map

$$Y_M(\,\cdot\,, x) : V \ni v \mapsto Y_M(v, x) = \sum_{i \in (1/p)\mathbb{Z}} v_i x^{-i-1} \in (\mathrm{End}_{\mathbb{C}} M)[[x^{1/p}, x^{-1/p}]]$$

which satisfies the following four conditions:

(1) $Y_M(u, x) = \sum_{i \in r/p + \mathbb{Z}} u_i x^{-i-1}$ for $u \in V^r$.

(2) $Y_M(u, x)w \in M((x^{1/p}))$ for $u \in V$ and $w \in M$.

(3) $Y_M(\mathbf{1}, x) = \mathrm{id}_M$.

(4) For $u \in V^r$, $v \in V^s$, $m \in r/T + \mathbb{Z}$, $n \in s/T + \mathbb{Z}$, and $l \in \mathbb{Z}$,

$$\sum_{i=0}^{\infty} \binom{m}{i} (u_{l+i}v)_{m+n-i} = \sum_{i=0}^{\infty} \binom{l}{i} (-1)^i \big(u_{l+m-i}v_{n+i} + (-1)^{l+1} v_{l+n-i}u_{m+i}\big).$$

For a g-twisted vertex algebra (A, D)-module (M, Y_M) and a linear automorphism h of A which preserves D, define $(M, Y_M) \circ h = (M \circ h, Y_{M \circ h})$ by $M \circ h = M$ as vector spaces and $Y_{M \circ h}(a, x) = Y_M(ha, x)$ for all $a \in A$. Then, $(M, Y_M) \circ h$ is a $h^{-1}gh$-twisted vertex algebra (A, D)-module.

3 Finite-dimensional vertex algebra modules over fixed point commutative subalgebras

Throughout this section, R is a connected commutative $\mathbb{C}$-algebra, A is a commutative $\mathbb{C}$-algebra generated by a single element as an R-algebra and is a Galois extension of R with Galois group G . It follows from [15, Theorem 2.9] that $A \cong R[Z]/(P(Z))$ as R-algebras for some separable polynomial $P(Z) \in R[Z]$. Let D be a derivation of A which is invariant under the action of G. For a finite-dimensional vertex algebra (R, D)-module (M, Y_M), $g \in G$ of order p, and a linear map $\tilde{Y}(\cdot, x)$ from A to $(\mathrm{End}_{\mathbb{C}} M)((x^{1/p}))$, we call $(M, \tilde{Y}_M)$ a *g-twisted vertex algebra (A, D)-module structure over* (M, Y_M) if $(M, \tilde{Y}_M)$ is a g-twisted vertex algebra (A, D)-module and if $\tilde{Y}(\cdot, x)|_R = Y(\cdot, x)$.

In this section, we shall show that every finite-dimensional indecomposable vertex algebra (R, D)-module has a g-twisted vertex algebra (A, D)-module structure over (M, Y_M) for some $g \in G$. We use the following notation in [20, Section 3]. For a commutative ring C, let $\mathrm{Mat}_n(C)$ denote the set of all $n \times n$ matrices with entries in C. Let E_n denote the $n \times n$ identity matrix and let E_{ij} denote the matrix whose (i, j) entry is 1 and all other entries are 0. Define $\Delta_k(C) = \{(x_{ij}) \in \mathrm{Mat}_n(C) \mid x_{ij} = 0 \text{ if } i + k \neq j\}$ for $0 \leq k \leq n$. Then, for $a \in \Delta_k(C)$ and $b \in \Delta_l(C)$, we have $ab \in \Delta_{k+l}(C)$. For $X = (x_{ij}) \in \mathrm{Mat}_n(C)$ and $k = 0, \ldots, n-1$, define the matrix $X^{(k)} = \sum_{i=1}^{n} x_{i,i+k} E_{i,i+k} \in \Delta_k(C)$. For a upper triangular matrix X, we see that $X = \sum_{k=0}^{n-1} X^{(k)}$.

Let A be a commutative $\mathbb{C}$-algebra, D a derivation of A, g a $\mathbb{C}$-linear automorphism of A of finite order p. For a vector space W over $\mathbb{C}$ and a linear map $Y_W(\cdot,x)$ from A to $(\mathrm{End}_{\mathbb{C}} W)[[x^{1/p},x^{-1/p}]]$, we denote by $\mathcal{A}_W(A)$ the subalgebra of $\mathrm{End}_{\mathbb{C}} W$ generated by all coefficients of $Y_W(a,x)$ where a ranges over all elements of A. Let M be a finite-dimensional g-twisted vertex algebra (A,D)-module. Then, $\mathcal{A}_M(A)$ is a commutative $\mathbb{C}$-algebra and M is a finite-dimensional $\mathcal{A}_M(A)$-module. Note that every $\mathcal{A}_M(A)$-module becomes g-twisted vertex algebra (A,D)-module. Let $\mathcal{J}_M(A)$ denote the Jacobson radical of $\mathcal{A}_M(A)$. Recall that the module $\mathrm{top}\, M = M/\mathcal{J}_M(A)M$ is called *the top of* M, which is completely reducible (cf. [2, Chapter I]). Since $\mathcal{A}_M(A)$ is a finite-dimensional commutative $\mathbb{C}$-algebra, the Wedderburn–Malcev theorem (cf.[18, Section 11.6]) says that $\mathcal{A}_M(A) = \oplus_{i=1}^m \mathbb{C}e_i \oplus \mathcal{J}_M(A)$ where $e_1,\ldots,e_m$ are primitive orthogonal idempotents of $\mathcal{A}_M(A)$. For $U \in \mathcal{A}_M(A)((x))$, we denote by $U^{[0]}$ the image of U under the projection $\mathcal{A}_M(A)((x)) = \oplus_{i=1}^m \mathbb{C}((x))e_i \oplus \mathcal{J}_M(A)((x)) \to \oplus_{i=1}^m \mathbb{C}((x))e_i \cong \mathbb{C}((x))^{\oplus m}$. We denote by $\psi[A,(M,Y_M)]$ the $\mathbb{C}$-algebra homomorphism $Y_M(\cdot,x)^{[0]}$ from A to $\mathbb{C}((x))^{\oplus m}$, which corresponds to the module $\mathrm{top}\, M$. Note that $\mathcal{J}_M(A)^n((x)) = 0$, where $n = \dim_{\mathbb{C}} M$. Since $\mathcal{A}_M(A)$ is commutative, we shall sometimes identify $\mathrm{End}_{\mathbb{C}} M$ with $\mathrm{Mat}_n(\mathbb{C})$ by fixing a basis of M so that all elements of $\mathcal{A}_M(A)$ are upper triangular matrices. Under this identification, for $U \in \mathcal{A}_M(A)((x))$ we see that $U^{[0]} = U^{(0)}$.

Let M be a finite-dimensional indecomposable vertex algebra (R,D)-module. Since $\mathcal{A}_M(R)$ is local, we see that $\mathcal{A}_M(R) = \mathbb{C}\,\mathrm{id} \oplus \mathcal{J}_M(R)$. In this case we shall often identify the subalgebra $\mathbb{C}((x))\,\mathrm{id}$ in $\mathcal{A}_M(A)((x))$ with $\mathbb{C}((x))$. Let $(M,\tilde{Y}_M)$ be a g-twisted vertex algebra (A,D)-module structure over (M,Y_M). Since $\mathcal{A}_M(R)$ is a subalgebra of $\mathcal{A}_M(A)$, we see that M is an indecomposable $\mathcal{A}_M(A)$-module. Therefore, $\mathcal{A}_M(A)$ is local since $\mathcal{A}_M(A)$ is commutative. Thus, $\mathcal{A}_M(A) = \mathbb{C}\,\mathrm{id} \oplus \mathcal{J}_M(A)$ and hence $\psi[A,(M,\tilde{Y}_M)]|_R = \psi[R,(M,Y_M)]$. It follows from Nakayama's lemma (cf. [2, Lemma 2.2]) that $\mathcal{J}_M(A)M \neq M$ and hence $\mathcal{J}_M(A)M = \mathcal{J}_M(R)M$ is a proper $\mathcal{A}_M(A)$-submodule of M. This tells us that $\mathrm{top}\, M = M/\mathcal{J}_M(R)M$ has a g-twisted vertex algebra (A,D)-module structure over $(\mathrm{top}\, M, Y_{\mathrm{top}\, M})$. We conclude that a g-twisted vertex algebra (A,D)-module structure $(M,\tilde{Y}_M)$ over (M,Y_M) induces a g-twisted vertex algebra (A,D)-module structure $(\mathrm{top}\, M, \tilde{Y}_{\mathrm{top}\, M})$ over $(\mathrm{top}\, M, Y_{\mathrm{top}\, M})$.

Now we state our main theorem.

Theorem 1. *Let A be a connected commutative $\mathbb{C}$-algebra which is a Galois* *extension of R with Galois group G and let D be a derivation of A which is invariant under the action of G. Suppose A is generated by a single element as an R-algebra. Then, for every non-zero finite-dimensional indecomposable vertex algebra (R, D)-module (M, Y_M), we have the following results:*

(1) M has a g-twisted vertex algebra (A, D)-module structure over (M, Y_M) for some $g \in G$.

(2) Let $g \in G$. If $\operatorname{top} M$ has a g-twisted vertex algebra (A, D)-module structure over $(\operatorname{top} M, Y_{\operatorname{top} M})$, then M has a unique g-twisted vertex algebra (A, D)-module structure $(M, \tilde{Y}_M)$ over (M, Y_M) such that $\operatorname{top} M \cong M/\mathcal{J}_M(A)M$ as g-twisted vertex algebra (A, D)-modules.

(3) Let $g \in G$ and let $(M, \tilde{Y}_M)$ be a g-twisted vertex algebra (A, D)-module structure over (M, Y_M). Then, $\tilde{Y}_M \circ h, h \in G$, are all distinct homomorphisms from A to $(\operatorname{End}_{\mathbb{C}} M)((x^{1/|g|}))$.

(4) For each $k = 1, 2$, let g_k be an element in G and let $(M, \tilde{Y}_M^k)$ be a g_k-twisted vertex algebra (A, D)-module structure over (M, Y_M). Then, $(M, \tilde{Y}_M^1) \circ h \cong (M, \tilde{Y}_M^2)$ for some $h \in G$.

Proof. Set $n = \dim_{\mathbb{C}} M$ and $N = |G|$. Let the notation be as above. By [15, Theorem 2.9], we may assume $A = R[Z]/(P(Z))$ where $P(Z) = \sum_{i=0}^{N} P_i Z^i \in R[Z]$ is a separable polynomial. We denote by R_0 the image of the homomorphism $\psi[R, (M, Y_M)] : R \to \mathbb{C}((x))$, by $Q(R_0)$ the quotient field of R_0 in $\mathbb{C}((x))$, by θ the primitive element $Z + (P(Z)) \in A$, by $\hat{P}(Z) \in (\mathcal{A}_M(R)((x)))[Z]$ the image of $P(Z)$ under the map $Y_M(\cdot, x)$, and by $\hat{P}^{[0]}(Z) \in \mathbb{C}((x))[Z]$ the image of $P(Z)$ under the map $\psi[R, (M, Y_M)]$. We write $\hat{P}(Z) = \sum_{i=0}^{N} \hat{P}_i(x) Z^i$, $\hat{P}_i(x) \in \mathcal{A}_M(R)((x))$. We use [20, Lemma 4] by setting $\mathcal{B} = R \cup \{\theta\}$.

It is well known that any finite extension of $\mathbb{C}((x))$ is $\mathbb{C}((x^{1/j}))$ for some positive integer j and $\Omega = \cup_{j=1}^{\infty} \mathbb{C}((x^{1/j}))$ is the algebraic closure of $\mathbb{C}((x))$ (cf. [13, Corollary 13.15]). The field $\mathbb{C}((x^{1/j}))$ becomes a Galois extension of $\mathbb{C}((x))$ whose Galois group is the cyclic group generated by the automorphism sending $x^{1/j}$ to $\zeta_j x^{1/j}$. Let K_0 denote the splitting field of $\hat{P}^{[0]}(Z)$ in Ω.

(1) Since K_0 is a finite extension of $Q(R_0)$ and $Q(R_0)$ is a subfield of $\mathbb{C}((x))$, we see that $K_0 \mathbb{C}((x)) = \mathbb{C}((x^{1/p}))$ for some positive integer p. It follows from the isomorphism $\operatorname{Gal}(\mathbb{C}((x^{1/p}))/\mathbb{C}((x))) \cong \operatorname{Gal}(K_0/(K_0 \cap \mathbb{C}((x))))$ that $\operatorname{Gal}(K_0/(K_0 \cap \mathbb{C}((x))))$ has an element σ of order p. Since K_0 is

a field, there is $a_0 \in K_0$ such that $\sigma a_0 = \zeta_p^j a_0$ with $(j,p) = 1$. It follows from $a_0^p \in K_0 \cap \mathbb{C}((x))$ that a_0 is a root of the polynomial $Z^p - a_0^p \in \mathbb{C}((x))[Z]$. Thus, a_0 is an element of $x^{-r/p}\mathbb{C}((x))$ for some integer r. We have $(r,p) = 1$ since $a_0^i \notin K_0^{\langle\sigma\rangle}$ for all $i = 1, \ldots, p-1$. Let γ, δ be integers with $\gamma r + \delta p = 1$. By replacing a_0 by a_0^γ, we have $\sigma a_0 = \zeta_p^{\gamma j} a_0$ and $a_0 \in x^{-1/p}\mathbb{C}((x))$. Since $(\gamma j, p) = 1$, by replacing σ by a suitable power of σ, we have $\sigma a_0 = \zeta_p a_0$ and $a_0 \in x^{-1/p}\mathbb{C}((x))$. For all $b_0 \in K_0$ with $\sigma b_0 = \zeta_p^i b_0$, we have $\sigma(a_0^{-i} b_0) = a_0^{-i} b_0$ and hence $b_0 \in x^{-i/p}\mathbb{C}((x))$.

Let $T(x)^{[0]} \in K_0$ be a root of $\hat{P}^{[0]}(Z)$. We have a $\mathbb{C}$-algebra homomorphism ρ from $A = R[Z]/(P(Z))$ to K_0 with $\rho(\theta) = T(x)^{[0]}$. Since σ fixes all elements in $Q(R_0) \subset K_0 \cap \mathbb{C}((x))$, $\sigma(T(x)^{[0]})$ is a root of $\hat{P}^{[0]}(Z)$. Since $A = R[\theta]$ and ρ induces a bijection from $\{g(\theta) \mid g \in G\}$ to the set of all roots of $\hat{P}^{[0]}(Z)$ in K_0 as explained just before Proposition 1, $T(x)^{[0]}$ is a primitive element of K_0 over $Q(R_0)$ and there exists a unique $g \in G$ with $\rho(g(\theta)) = \sigma(T(x)^{[0]}) = \sigma(\rho(\theta))$. These results tell us that $\rho g = \sigma\rho$ and hence the order of g is equal to p.

Set $\hat{P}^{[1]}(Z) = \hat{P}(Z) - \hat{P}^{[0]}(Z)\,\mathrm{id} \in \mathcal{J}_M(R)((x))[Z]$ and $\hat{P}^{[k]}(Z) = 0$ for all $k \geq 2$. We write $\hat{P}^{[k]}(Z) = \sum_{i=0}^N \hat{P}_i(x)^{[k]} Z^i$, $\hat{P}_i(x)^{[k]} \in \mathcal{J}_M(R)^k((x))$, for all $k \geq 0$.

Since $\hat{P}^{[0]}(Z)$ has no multiple root in Ω, we see that $(d\hat{P}^{[0]}/dZ)(T(x)^{[0]}) \neq 0$. For $k = 1, 2, \ldots, n-1$ we inductively define $T(x)^{[k]} \in \mathcal{J}_M(R)^k((x^{1/p}))$ by

$$T(x)^{[k]} = -(\frac{d\hat{P}^{[0]}}{dZ}(T(x)^{[0]}))^{-1} \times \sum_{i=0}^{N} \sum_{j_0=0}^{k} \sum_{\substack{0 \leq j_1,\ldots,j_i < k \\ j_0+j_1+\cdots+j_i = k}} \hat{P}_i(x)^{[j_0]} T(x)^{[j_1]} \cdots T(x)^{[j_i]}. \qquad (3.1)$$

Set $T(x) = \sum_{k=0}^{n-1} T(x)^{[k]} \in \mathcal{A}_M(R)((x^{1/p}))$. Since $\mathcal{J}_M(R)^n((x)) = 0$, we have

$$\begin{aligned}
\hat{P}(T(x)) &= \hat{P}^{[0]}(T(x)^{[0]}) \\
&\quad + \sum_{k=1}^{n-1}\sum_{i=0}^{N}\sum_{\substack{0\le j_0,j_1,\ldots,j_i\\ j_0+j_1+\cdots+j_i=k}} \hat{P}_i(x)^{[j_0]}T(x)^{[j_1]}\cdots T(x)^{[j_i]} \\
&= 0+\sum_{k=1}^{n-1}\Big(T(x)^{[k]}\frac{d\hat{P}^{[0]}}{dZ}(T(x)^{[0]}) \\
&\quad + \sum_{i=0}^{N}\sum_{j_0=0}^{k}\sum_{\substack{0\le j_1,\ldots,j_i<k\\ j_0+j_1+\cdots+j_i=k}} \hat{P}_i(x)^{[j_0]}T(x)^{[j_1]}\cdots T(x)^{[j_i]}\Big) \\
&= 0.
\end{aligned}$$

This result enables us to define a homomorphism $\tilde{Y}_M(\cdot,x)$ from $A = R[Z]/(P(Z))$ to $\mathcal{A}_M(R)((x^{1/p}))$ sending θ to $T(x)$. Since $\mathcal{A}_M(R)$ is commutative, the subalgebra $\mathcal{A}_M(A)$ of $\mathrm{End}_{\mathbb{C}} M$ obtained by $\tilde{Y}_M(\cdot,x)$ is commutative.

For all $b \in A$ with $gb = \zeta_p^i b$, we shall show that $\tilde{Y}_M(b,x) \in x^{-i/p}(\mathrm{End}_{\mathbb{C}} M)((x))$. Set $B(x) = \tilde{Y}_M(b,x)$ and $Q(x) = B(x)^p \in \mathcal{A}_M(R)((x))$. We identify $\mathrm{End}_{\mathbb{C}} M$ with $\mathrm{Mat}_n(\mathbb{C})$ by fixing a basis of M so that all elements of $\mathcal{A}_M(R)$ are upper triangular matrices. We use the expansion $B(x) = \sum_{k=0}^{n-1} B(x)^{(k)}, B(x)^{(k)} \in \Delta_k(\mathrm{End}_{\mathbb{C}} M)((x^{1/p}))$. Since $\zeta_p^i\rho(b) = \rho(gb) = \sigma(\rho(b))$, we have already seen that $B(x)^{(0)} = \rho(b) \in x^{-i/p}(\mathrm{End}_{\mathbb{C}} M)((x))$. By $B(x)^p = Q(x)$, for all $k = 1,\ldots,n-1$ we have

$$\begin{aligned}
B(x)^{(k)} &= -p^{-1}(B(x)^{(0)})^{-p+1} \\
&\quad \times \Big(Q(x)^{(k)} + \sum_{\substack{0\le j_1,\ldots,j_p<k\\ j_1+\cdots+j_p=k}} B(x)^{(j_1)}\cdots B(x)^{(j_p)}\Big).
\end{aligned}$$

It follows by induction on k that $B(x)^{(k)} \in x^{-i/p}(\mathrm{End}_{\mathbb{C}} M)((x))$ and hence $B(x) \in x^{-i/p}(\mathrm{End}_{\mathbb{C}} M)((x))$.

It follows from $P(\theta) = 0$ that $0 = D(P(\theta)) = \sum_{i=0}^{N}(DP_i)\theta^i + (dP/dZ)(\theta)(D\theta)$. Note that $(d\hat{P}(Z)/dZ)(T(x))$ is an invertible element in $\mathcal{A}_M(R)((x^{1/p}))$ since $(d\hat{P}^{[0]}(Z)/dZ)(T(x)^{[0]}) \neq 0$. Since $Y_M(DP_i,x) = dY_M(P_i,x)/dx$ for all i, we have $\tilde{Y}_M(D\theta,x) = d\tilde{Y}_M(\theta,x)/dx$.

We conclude that $(M, \tilde{Y}_M)$ is a g-twisted vertex algebra (A, D)-module structure over (M, Y_M).

(2) We denote the order of g by p. Let $(\operatorname{top} M, \tilde{Y}_{\operatorname{top} M})$ be a g-twisted vertex algebra (A, D)-module structure over $(\operatorname{top} M, Y_{\operatorname{top} M})$. Let us denote by φ the map $\tilde{Y}_{\operatorname{top} M}(\cdot, x) : A \to \Omega$, namely $\varphi = \psi[A, (\operatorname{top} M, \tilde{Y}_{\operatorname{top} M})]$. Note that $\varphi|_R = \psi[R, (M, Y_M)]$ and $\varphi(\theta)$ is a root of $\hat{P}^{[0]}(Z)$ in Ω. By the same argument as in (1), we can construct a root $T(x) \in \mathcal{A}_M(R)((x^{1/p}))$ of $\hat{P}(Z)$ whose semisimple part $T(x)^{[0]}$ is equal to $\varphi(\theta)$. The linear homomorphism from A to $\mathcal{A}_M(R)((x^{1/p}))$ sending θ to $T(x)$ induces a g-twisted vertex algebra (A, D)-module structure $(M, \tilde{Y}_M)$ over (M, Y_M). Since θ is a primitive element of A over R, we see that $\psi[A, (M, \tilde{Y}_M)] = \varphi$.

We shall show the uniqueness of the g-twisted vertex algebra (A, D)-module structure over (M, Y_M) which satisfies the conditions. Let $(M, \tilde{Y}^1_M)$ be a g-twisted vertex algebra (A, D)-module structure over (M, Y_M) with $\psi[(A, (M, \tilde{Y}^1_M)] = \varphi$. We identify $\operatorname{End}_{\mathbb{C}} M$ with $\operatorname{Mat}_n(\mathbb{C})$ by fixing a basis of M so that all elements of $\mathcal{A}_M(A)$ are upper triangular matrices. Set $U(x) = \tilde{Y}^1_M(\theta, x) \in (\operatorname{Mat}_n(\mathbb{C}))((x^{1/p}))$. We use the expansion $U(x) = \sum_{k=0}^{n-1} U(x)^{(k)}$ and $\hat{P}_i(x) = \sum_{k=0}^{n-1} \hat{P}_i(x)^{(k)}$, where $U(x)^{(k)}, \hat{P}_i(x)^{(k)} \in \Delta_k(\operatorname{End}_{\mathbb{C}} M)((x^{1/p}))$. Set $\hat{P}^{(0)}(Z) = \sum_{i=0}^{N} \hat{P}_i(x)^{(0)} Z^i$. Under the identification of $\operatorname{End}_{\mathbb{C}} M$ with $\operatorname{Mat}_n(\mathbb{C})$, we have $\hat{P}^{[0]}(Z) = \hat{P}^{(0)}(Z)$. Note that $U(x)^{(0)} = \varphi(\theta)$ and we do not assume $U(x) \in \mathcal{A}_M(R)((x^{1/p}))$. We have

$$\begin{aligned}
0 &= \hat{P}(U(x)) \\
&= \hat{P}^{(0)}(U(x)^{(0)}) \\
&\quad + \sum_{k=1}^{n-1} \sum_{i=0}^{N} \sum_{\substack{0 \le j_0, j_1, \dots, j_i \\ j_0 + j_1 + \dots + j_i = k}} \hat{P}_i(x)^{(j_0)} U(x)^{(j_1)} \cdots U(x)^{(j_i)} \\
&= 0 + U(x)^{(k)} \frac{d\hat{P}^{(0)}}{dZ}(U(x)^{(0)}) \\
&\quad + \sum_{k=1}^{n-1} \sum_{i=0}^{N} \sum_{j_0=0}^{k} \sum_{\substack{0 \le j_1, \dots, j_i < k \\ j_0 + j_1 + \dots + j_i = k}} \hat{P}_i(x)^{(j_0)} U(x)^{(j_1)} \cdots U(x)^{(j_i)}
\end{aligned}$$

and hence

$$U(x)^{(k)} = -(\frac{d\hat{P}^{[0]}}{dZ}(\varphi(\theta))^{-1} \times \sum_{i=0}^{N} \sum_{j_0=0}^{k} \sum_{\substack{0 \le j_1,\dots,j_i < k \\ j_0+j_1+\cdots+j_i=k}} \hat{P}_i(x)^{(j_0)} U(x)^{(j_1)} \cdots U(x)^{(j_i)}.$$

It follows by induction on k that $U(x) = \sum_{k=0}^{n-1} U(x)^{(k)}$ is uniquely determined by $\varphi(\theta)$ and $\hat{P}(Z)$. We conclude that M has a unique g-twisted vertex algebra (A, D)-module structure $(M, \tilde{Y}_M)$ over (M, Y_M) such that $\psi[(A, (M, \tilde{Y}_M)] = \varphi$.

(3) Let $h \in G$ with $h \neq 1$. Since θ and $h(\theta)$ are distinct roots of $P(Z)$ in A, [15, Lemma 2.1] says that $\theta - h(\theta)$ is an invertible element of A. Since $\tilde{Y}_{M \circ h}(\theta, x) = \tilde{Y}_M(h\theta, x) \neq \tilde{Y}_M(\theta, x)$, we see that $\tilde{Y}_{M \circ h}(\cdot, x)$ is distinct from $\tilde{Y}_M(\cdot, x)$. This says that $\tilde{Y}_M \circ h, h \in G$, are all distinct homomorphisms from A to $(\operatorname{End}_{\mathbb{C}} M)((x^{1/|g|}))$.

(4) For each $k = 1, 2$, let g_k be an element in G and let $(M, \tilde{Y}_M^k)$ be a g_k-twisted vertex algebra (A, D)-module structure over (M, Y_M). We denote $\psi[A, (M, \tilde{Y}_M^k)]$ by ψ_k and $\psi[R, (M, Y_M)]$ by ψ briefly. Since each ψ_k induces a bijection from $\{g(\theta) \mid g \in G\}$ to the set of all roots of $\hat{P}^{[0]}(Z)$ in K_0 as explained just before Proposition 1, there is an element $h \in G$ with $\psi_1(h(\theta)) = \psi_2(\theta)$. This tells us that $(\operatorname{top} M, \tilde{Y}^1_{\operatorname{top} M}) \circ h \cong (\operatorname{top} M, \tilde{Y}^2_{\operatorname{top} M})$ and hence $(M, \tilde{Y}_M^1) \circ h \cong (M, \tilde{Y}_M^2)$ by (2).

□

4 Finite-dimensional vertex algebra $\mathbb{C}[s, s^{-1}]$-modules

Let $\mathbb{C}[s, s^{-1}]$ be the algebra of Laurent polynomials in one variable s over $\mathbb{C}$. In this section we shall classify the finite-dimensional vertex algebra $\mathbb{C}[s, s^{-1}]$-modules. We use the notation introduced in Section 3. It is easy to see that every non-zero derivation D of $\mathbb{C}[s, s^{-1}]$ can be expressed as $D = (p(s)/s^{N_q})d/ds$ so that the polynomials $p(s)$ and s^{N_q} in $\mathbb{C}[s]$ are coprime.

The following lemma easily follows from [20, Lemma 4].

Lemma 3. *Let the notation be as above. Let M be a finite-dimensional vector space and let $S(x) = \sum_{i\in\mathbb{Z}} S_{(i)}x^i$ be a non-zero element of $(\operatorname{End}_{\mathbb{C}} M)((x))$. Then, there exists a vertex algebra $(\mathbb{C}[s, s^{-1}], D)$-module (M, Y_M) with $Y_M(s, x) = S(x)$ if and only if the following three conditions hold:*

(i) $S(x)$ *is an invertible element in* $(\operatorname{End}_{\mathbb{C}} M)((x))$.

(ii) For all $i, j \in \mathbb{Z}$, $S_{(i)}S_{(j)} = S_{(j)}S_{(i)}$.

(iii) $S(x)^{N_q} dS(x)/dx = p(S(x))$.

In this case, for $u(s) \in \mathbb{C}[s, s^{-1}]$ *we have* $Y_M(u(s), x) = u(S(x))$ *and hence* (M, Y_M) *is uniquely determined by* $S(x)$.

Proof. If (M, Y_M) is a vertex algebra $(\mathbb{C}[s, s^{-1}], D)$-module, then [20, Lemma 4] tells us that the conditions (i)–(iii) are clearly hold and $Y_M(u(s), x) = u(S(x))$ for all $u(s) \in \mathbb{C}[s, s^{-1}]$.

Conversely, suppose that (M, Y_M) satisfies the conditions (i)–(iii). We use [20, Lemma 4] by setting $\mathcal{B} = \{s, s^{-1}\}$. For $u(s) \in \mathbb{C}[s]$, set $Y_M(u(s), x) = u(S(x))$. Since $S(x)$ is an invertible element in $(\operatorname{End}_{\mathbb{C}} M)((x))$, this induces a $\mathbb{C}$-algebra homomorphism from $\mathbb{C}[s, s^{-1}]$ to $(\operatorname{End}_{\mathbb{C}} M)((x))$. Since $S(x)^{-1}$ is a polynomial in $S(x)$, we see that $\mathcal{A}_M(\mathbb{C}[s, s^{-1}])$ is commutative. Since

$$\begin{aligned} Y_M(D(s^{-1}), x) &= Y_M(-(Ds)(s^{-2}), x) \\ &= -Y_M(Ds, x)Y_M(s, x)^{-2} \\ &= \frac{d}{dx}(Y_M(s, x))^{-1}), \end{aligned}$$

we conclude that (M, Y_M) is a vertex algebra $(\mathbb{C}[s, s^{-1}], D)$-module. □

Let (M, Y_M) be a finite-dimensional indecomposable vertex algebra $\mathbb{C}[s, s^{-1}]$-module. We identify $\operatorname{End}_{\mathbb{C}} M$ with $\operatorname{Mat}_n(\mathbb{C})$ by fixing a basis of M so that all elements of $\mathcal{A}_M(\mathbb{C}[s, s^{-1}])$ are upper triangular matrices. Let J_n denote the following $n \times n$ matrix:

$$J_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

We denote $Y_M(s,x)$ by $S(x)$. We use the expansion $S(x) = \sum_{k=0}^{n-1} S(x)^{(k)}, S(x)^{(k)} \in \Delta_k(\mathrm{End}_{\mathbb{C}} M)((x))$, as in Section 3. Recall that under this identification, the semisimple part $S(x)^{[0]}$ of $S(x)$ is equal to $S(x)^{(0)}$.

For all $H(x) = \sum_{i=L}^{\infty} H_i x^i \in (\mathrm{End}_{\mathbb{C}} M)((x))$ with $H_L \neq 0$, we denote L by $\mathrm{ld}(H(x))$ and H_L by $\mathrm{lc}(H(x))$. Note that if $\mathrm{ld}(S(x)^{[0]}) > 0$, then $\mathrm{ld}((S(x)^{-1})^{[0]}) = \mathrm{ld}((S(x)^{[0]})^{-1}) < 0$. This implies that if $\mathrm{ld}(S(x)^{[0]}) \neq 0$, then vertex algebra $(\mathbb{C}[s,s^{-1}], D)$-module (M, Y_M) is not a $\mathbb{C}$-algebra $\mathbb{C}[s,s^{-1}]$-module.

Theorem 2. *Let α be a non-zero complex number and $D = (p(s)/s^{N_q})d/ds$ a non-zero derivation of $\mathbb{C}[s,s^{-1}]$ such that the polynomials $p(s)$ and s^{N_q} of $\mathbb{C}[s]$ are coprime. We write $p(s) = \sum_{i=L_p}^{N_p} p_i s^i$ where p_{L_p}, p_{N_p} are non-zero complex numbers. Then, the following results hold:*

(1) Every finite-dimensional indecomposable vertex algebra $(\mathbb{C}[s,s^{-1}], D)$-module M with $\mathrm{ld}(S(x)^{[0]}) = 0$ is a $\mathbb{C}$-algebra A-module.

(2) There exists a non-zero finite-dimensional indecomposable vertex algebra $(\mathbb{C}[s,s^{-1}], D)$-module M with $\mathrm{ld}(S(x)^{[0]}) > 0$ and with $\mathrm{lc}(S(x)^{[0]}) = \alpha$ if and only if $N_q = 0$ and $p(0) = \alpha$. Moreover, in this case $\mathrm{ld}(S(x)^{[0]}) = 1$.

(3) There exists a non-zero finite-dimensional indecomposable vertex algebra $(\mathbb{C}[s,s^{-1}], D)$-module M with $\mathrm{ld}(S(x)^{[0]}) < 0$ and with $\mathrm{lc}(S(x)^{[0]}) = \alpha$ if and only if $N_p = N_q + 2$ and $\alpha = -1/p_{N_p}$. Moreover, in this case $\mathrm{ld}(S(x)^{[0]}) = -1$.

In the case of (2) and (3), for every positive integer n, there exists a unique n-dimensional indecomposable vertex algebra $(\mathbb{C}[s,s^{-1}], D)$-module which satisfies the conditions up to isomorphism.

Proof. We use Lemma 3. Let (M, Y_M) be a non-zero finite-dimensional indecomposable vertex algebra $(\mathbb{C}[s,s^{-1}], D)$-module with $\mathrm{lc}(S(x)^{[0]}) = \alpha$. Since M is indecomposable, we see that $S(x)^{(0)} \in \mathbb{C}((x))E_n$. Since $S(x)$ is invertible, we have $S(x)^{(0)} \neq 0$ and

$$\begin{aligned} S(x)^{-1} &= (S(x)^{(0)} + \sum_{k=1}^{n-1} S(x)^{(k)})^{-1} \\ &= \sum_{i=0}^{n-1} (-1)^i (S(x)^{(0)})^{-1-i} (\sum_{k=1}^{n-1} S(x)^{(k)})^i. \end{aligned} \tag{4.1}$$

By Lemma 3, we have

$$S(x)^{N_q}\frac{dS(x)}{dx} = p(S(x)). \tag{4.2}$$

and hence

$$(S(x)^{(0)})^{N_q}\frac{dS(x)^{(0)}}{dx} = p(S(x)^{(0)}). \tag{4.3}$$

We shall give a formula for $S(x)^{(k)} = \sum_{i\in\mathbb{Z}} S^{(k)}_{(i)}x^i \in \Delta_k((\mathrm{End}_{\mathbb{C}}\, M)((x)))$ for $k = 1, 2 \ldots, n-1$. By standard Jordan canonical form theory, we may assume $S_{(0)} = S^{(0)}_{(0)} + S^{(1)}_{(0)}$, that is, $S^{(j)}_{(0)} = 0$ for all $j = 2, \ldots, n-1$. We have the following expansions of $(dS(x)/dx)S(x)^{N_q}$ and $p(S(x))$:

$$\begin{aligned}
&\frac{dS(x)}{dx}S(x)^{N_q}\\
&= \sum_{j_0=0}^{n-1}\frac{dS(x)^{(j_0)}}{dx}\Big(\sum_{0\le j_1,\ldots,j_{N_q}\le n-1} S(x)^{(j_1)}\cdots S(x)^{(j_{N_q})}\Big)\\
&= \sum_{0\le j_0,j_1,\ldots,j_{N_q}\le n-1}\frac{dS(x)^{(j_0)}}{dx}S(x)^{(j_1)}\cdots S(x)^{(j_{N_q})}\\
&= \sum_{k=0}^{n-1}\sum_{\substack{0\le j_0,j_1,\ldots,j_{N_q}\\ j_0+j_1+\cdots+j_{N_q}=k}}\frac{dS(x)^{(j_0)}}{dx}S(x)^{(j_1)}\cdots S(x)^{(j_{N_q})}\\
&= \frac{dS(x)^{(0)}}{dx}(S(x)^{(0)})^{N_q}\\
&\quad + \sum_{k=1}^{n-1}\Big(\frac{dS(x)^{(k)}}{dx}(S(x)^{(0)})^{N_q} + N_q\frac{dS(x)^{(0)}}{dx}(S(x)^{(0)})^{N_q-1}S(x)^{(k)}\\
&\quad + \sum_{\substack{0\le j_0,j_1,\ldots,j_{N_q}<k\\ j_0+j_1+\cdots+j_{N_q}=k}}\frac{dS(x)^{(j_0)}}{dx}S(x)^{(j_1)}\cdots S(x)^{(j_{N_q})}\Big)
\end{aligned}$$

and

$$p(S(x)) = p(S(x)^{(0)}) + \sum_{k=1}^{n-1} \big(\frac{dp}{ds}(S(x)^{(0)})S(x)^{(k)} + \sum_{i=0}^{N_p} p_i \sum_{\substack{0\le j_1,\ldots,j_i<k\\ j_1+\cdots+j_i=k}} S(x)^{(j_1)}\cdots S(x)^{(j_i)}\big).$$

By (4.2) for $k = 1, 2, \ldots$, we have a formula

$$\begin{aligned}&\frac{dS(x)^{(k)}}{dx}\\ &= (S(x)^{(0)})^{-N_q}\Big((-N_q\frac{dS(x)^{(0)}}{dx}(S(x)^{(0)})^{N_q-1} + \frac{dp}{ds}(S(x)^{(0)})\Big)S(x)^{(k)}\\ &\quad - \sum_{\substack{0\le j_0,j_1,\ldots,j_{N_q}<k\\ j_0+\cdots+j_{N_q}=k}} \frac{dS(x)^{(j_0)}}{dx}S(x)^{(j_1)}\cdots S(x)^{(j_{N_q})}\\ &\quad + \sum_{i=0}^{N_p} p_i \sum_{\substack{0\le j_1,\ldots,j_i<k\\ j_1+\cdots+j_i=k}} S(x)^{(j_1)}\cdots S(x)^{(j_i)}\Big).\end{aligned} \tag{4.4}$$

We write $S(x)^{(0)} = \sum_{i=L}^{\infty} S^{(0)}_{(i)}x^i$, where $L = \mathrm{ld}(S(x)^{(0)})$.

Suppose that $L = 0$. We shall show that $S(x)^{(k)} \in (\mathrm{End}_{\mathbb{C}} M)[[x]]$ by induction on k. The case $k = 0$ follows from $L = 0$. For $k > 0$, suppose that $\mathrm{ld}(S(x)^{(k)}) < 0$. Since $(S(x)^{(0)})^{-N_q}$ is an element of $\mathbb{C}[[x]]$, the lowest degree of the right-hand side of (4.4) is greater than or equal to $\mathrm{ld}(S(x)^{(k)})$ by the induction assumption. This contradicts that $\mathrm{ld}(dS(x)^{(k)}/dx) = \mathrm{ld}(S(x)^{(k)}) - 1$. It follows from (4.1) that $S(x)$ and $S(x)^{-1}$ are elements in $(\mathrm{End}_{\mathbb{C}} M)[[x]]$ and hence $Y_M(a, x) \in (\mathrm{End}_{\mathbb{C}} M)[[x]]$ for all $a \in \mathbb{C}[s, s^{-1}]$. We conclude that if $L = 0$ then (M, Y_M) is a $\mathbb{C}$-algebra $\mathbb{C}[s, s^{-1}]$-module. This completes the proof of (1).

Suppose that $L > 0$. In (4.3), the term with the lowest degree of the left-hand side is $L(S^{(0)}_{(L)})^{N_q+1}x^{L(N_q+1)-1}$ and the term with the lowest degree of the right-hand side is $p_{L_p}(S^{(0)}_{(L)})^{L_p}x^{LL_p}$. Comparing these terms, we have $L(L_p - N_q - 1) = -1$ and hence $L = 1$ and $L_p = N_q$. We have $L_p = N_q = 0$ since $p(s)$ and s^{N_q} are coprime. Comparing coefficients of these terms with the lowest degree in (4.3), we have $D = p(s)d/ds, S^{(0)}_{(1)} = \alpha = p(0) \neq 0$, and $S^{(0)}_{(0)} = 0$. For all positive integers n, we shall show the uniqueness of n-dimensional indecomposable vertex algebra $(\mathbb{C}[s, s^{-1}], D)$-module which satisfies the conditions in (2). Setting $N_q = 0$ in (4.4), the same argument as in the case of $L = 0$ shows that $S(x)^{(k)} \in (\mathrm{End}_{\mathbb{C}} M)[[x]]$ for all $k = 0, 1, \ldots, n-1$. For all positive integers m, comparing the coefficients of x^m in (4.3), we have

$$(m+1)S^{(0)}_{(m+1)} = \sum_{i=0}^{N_p} p_i \sum_{\substack{0 \leq j_1, \ldots, j_i \leq m \\ j_0 + j_1 + \cdots + j_i = m}} S^{(0)}_{(j_1)} \cdots S^{(0)}_{(j_i)}. \tag{4.5}$$

It follows by induction on m that every $S^{(0)}_{(m)}$ is uniquely determined by $S^{(0)}_{(1)}$. By (4.4) for all $m > 0$, $S^{(k)}_{(m)}$ is a polynomial in $\{S^{(k)}_{(j)} \mid 0 \leq j \leq m-1\} \cup \{S^{(i)}_{(j)} \mid 0 \leq i \leq k-1, j \geq 0\}$. Since $S^{(i)}_{(0)} = 0$ for all $i = 2, \ldots, n-1$, it follows by induction on k and m that every $S^{(k)}_{(m)}$ is a polynomial in $S^{(1)}_{(0)}$ and hence is uniquely determined by $S^{(1)}_{(0)}$. Since $S^{(1)}_{(0)}$ is the nilpotent part of $S_{(0)}$ and M is indecomposable, $S^{(1)}_{(0)}$ conjugates to J_n. Thus, we have shown that the uniqueness of n-dimensional indecomposable vertex algebra $(\mathbb{C}[s, s^{-1}], D)$-module which satisfies the conditions in (2).

Conversely, suppose that $\alpha = p(0)$. Set $S^{(0)}_{(1)} = \alpha$ and $S^{(0)}_{(i)} = 0$ for all non-positive integers i. By (4.5) we can inductively define $S^{(0)}_{(m)}$ for $m = 2, 3, \ldots$. The obtained $S(x)^{(0)} = \sum_{i=1}^{\infty} S^{(0)}_{(i)} x^i \in \mathbb{C}[[x]]$ satisfies $\mathrm{ld}(S(x)^{(0)}) = 1$, $\mathrm{lc}(S(x)^{(0)}) = \alpha$, and (4.3). Set $S^{(1)}_{(0)} = J_n$, $S^{(k)}_{(0)} = 0$ for all $k = 2, \ldots, n-1$, and $S^{(k)}_{(i)} = 0$ for all $k = 1, \ldots, n-1$ and all negative integers i. After (4.5), we have seen that every $S^{(k)}_{(m)}$ is a polynomial in $S^{(1)}_{(0)}$ if it exists. By the same argument, we can inductively define $S^{(k)}_{(m)} \in \mathrm{End}_{\mathbb{C}} M$ for $k = 1, 2, \ldots, n-1$ and $m = 1, 2, \ldots$. By the argument to get (4.4) and (4.5) above, it is easy to see that the obtained $S(x) = \sum_{k=0}^{n-1} S(x)^{(k)} \in (\mathrm{End}_{\mathbb{C}} M)[[x]]$ satisfies (4.2). Since all coefficients of $S(x)$ are polynomials in $S^{(1)}_{(0)} = J_n$, we see that $S_{(i)}S_{(j)} = S_{(j)}S_{(i)}$ for all $i, j \in \mathbb{Z}$. Thus, we have an n-dimensional vertex algebra $(\mathbb{C}[s, s^{-1}], D)$-module M with $\mathrm{ld}(S(x)^{(0)}) = 1$ and with $\mathrm{lc}(S(x)^{(0)}) = \alpha$. This completes the proof of (2).

Next suppose that $L < 0$. Set $\tilde{s} = 1/s$. By (4.1), we have $Y_M(\tilde{s}, x)^{[0]} =$

$(S(x)^{-1})^{[0]} = (S(x)^{[0]})^{-1}$ and

$$D = -\tilde{s}^{N_q+2}p(1/\tilde{s})\frac{d}{d\tilde{s}}.$$

Since $S(x)^{-1}$ is a polynomial in $S(x)$, all coefficients in $S(x)^{-1}$ are commutative. Thus, this case reduces to the case of $L > 0$. □

5 Examples

Throughout this section, D is a non-zero derivation of $\mathbb{C}[s, s^{-1}]$. For a positive integer n, the $\mathbb{C}$-algebra $A = \mathbb{C}[s, s^{-1}][t]/(t^n - s)$ is a Galois extension of $\mathbb{C}[s, s^{-1}]$ (cf. [14, Lemma 5.1 in Chapter 0]). The Galois group of A over $\mathbb{C}[s, s^{-1}]$ is the cyclic group of order n generated by τ with $\tau(t) = \zeta_n t$. Since $t^n - s$ is an irreducible element in the unique factorization domain $\mathbb{C}[s, s^{-1}][t]$, $t^n - s$ is a prime element. Therefore, A is an integral domain and hence is a connected $\mathbb{C}$-algebra. We can extend D to a unique derivation of A, which we denote by the same notation D, by setting $D(t) = s^{-1}tD(s)/n$. It is easy to see that D is invariant under the action of τ.

In Theorem 2, we have classified the finite-dimensional indecomposable $(\mathbb{C}[s, s^{-1}], D)$-modules (M, Y_M) which are not $\mathbb{C}$-algebra $\mathbb{C}[s, s^{-1}]$-modules. In this section, we shall investigate twisted vertex algebra (A, D)-module structures over (M, Y_M). We denote $Y_M(s, x)$ by $S(x)$ and $S(x)^{[0]} = \sum_{i=L}^{\infty} S_{(i)}^{[0]} x^i \in \mathbb{C}((x))$ with $S_{(L)}^{[0]} \neq 0$ as in Section 4. It follows from Theorem 2 that $L = \mathrm{ld}(S(x)^{[0]}) = 1$ or -1.

Proposition 4. *Let (M, Y_M) be a finite-dimensional indecomposable vertex algebra $(\mathbb{C}[s, s^{-1}], D)$-module which is not a $\mathbb{C}$-algebra A-module. Set $L = \mathrm{ld}(S(x)^{[0]})$. Then, for the $\mathbb{C}$-algebra $A = \mathbb{C}[s, s^{-1}][t]/(t^n - s)$, (M, Y_M) has exactly n τ^{-L}-twisted vertex algebra (A, D)-module structure.*

Proof. We use the notation in the proof of Theorem 1 (1). If $\mathrm{ld}(S(x)^{[0]}) = 1$, then every root of the polynomial $Z^n - S(x)^{[0]}$ in $\Omega = \cup_{i=1}^{\infty}\mathbb{C}((x^{1/i}))$ is an element in $x^{1/n}\mathbb{C}((x)) = x^{-(-1/n)}\mathbb{C}((x))$. It follows from the argument in the proof of Theorem 1 (1) that (M, Y_M) has a τ^{-1}-twisted vertex algebra (A, D)-module structure $(M, \tilde{Y}_M)$ with $\tilde{Y}_M(t, x) \in x^{-(-1/n)}(\mathrm{Mat}_{\mathbb{C}} M)((x))$. We conclude by Theorem 1 (3) and (4) that (M, Y_M) has exactly n τ^{-1}-twisted vertex algebra (A, D)-module structures. The same argument tells us that if $\mathrm{ld}(S(x)^{[0]}) = -1$, then (M, Y_M) has exactly n τ-twisted vertex algebra (A, D)-module structures. □

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