ON INTERSECTION OF TWO EMBEDDED SPHERES IN 3-SPACE [1]

A. Rukhovich

Abstract. We find necessary and sufficient conditions on sequences $x = x_1, x_2, \dots, x_n$, $y = y_1, y_2, ..., y_n$ of positive integers, for existence of embeddings $f, g : S^2 \to \mathbb{R}^3$ such that $S^2 - g^{-1}(S^2)$ is the union of spheres with $x_1, x_2, \dots, x_n$ holes and $S^2 - f^{-1}(S^2)$ is the union of a sphere with $y_1, y_2, \dots, y_n$ holes.

We prove the following result, cf. [N, T].

Theorem 1. *Let n be a positive integer and $x = x_1, x_2, \dots, x_n$, $y = y_1, y_2, ...y_n$ be sequences of positive integers. There exist embeddings $f, g : S^2 \to \mathbb{R}^3$ such that*

• *$S^2 - g^{-1}(S^2)$ is the union of a sphere with x_1 holes, a sphere with x_2 holes, ..., a sphere with x_n holes;*

• *$S^2 - f^{-1}(S^2)$ is the union of a sphere with y_1 holes, a sphere with y_2 holes, ..., a sphere with y_n holes;*

if and only if $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i = 2n - 2$.

The 'if' part is essentially known and is essentially proved in [N] (we present elementary proofs, one of them using the Jordan Curve Theorem and another proof by T. Nowik). The 'only if' part is presumably new.

The Lando Conjecture. *Let A be a disjoint union of circles in S^2. Analogously let B be a disjoint union of circles in S^2. Then there exist embeddings $f, g : S^2 \to \mathbb{R}^3$ such that $S^2 - f^{-1}(S^2) = A$ and $S^2 - g^{-1}(S^2) = B$.*

Remark. The following numbered analog of the Lando Conjecture is false. *Let $A_1, A_2, ..., A_k$ be disjoint circles in S^2. Analogously, let $B_1, B_2, ..., B_k$ be disjoint circles in S^2. There exist embeddings $f, g : S^2 \to \mathbb{R}^3$ such that $f(B_s) = g(A_s)$ for each $s \in \{1, 2, ..., k\}$ and $f(S^2), g(S^2)$ have no other intersection points.* A counterexample is obtained for $k = 3$. Let A_1 be a circle of one radian southern latitude, A_2 — an equator of the sphere (i.e., zero radian northern latitude), A_3 — a circle of one radian northern latitude. Let $B_1 = A_2$, $B_2 = A_1$, $B_3 = A_3$.

Proof of the 'if' part. Let us construct a graph G. The vertices of G are the connected components of $S^2 - g^{-1}(S^2)$. The vertices and are connected by an edge if the closures of the corresponding components intersect. Denote by n the number of the vertices. The number of the edges is equal to the number of the circles in $S^2 - g^{-1}(S^2)$. This number is $\sum_{i=1}^n x_i/2$. It is obvious that G is connected. By the Jordan Curve Theorem, G is split by any vertex. So G is a tree. Hence the number of edges is $n - 1 = \sum_{i=1}^n x_i/2$. The 'if' part follows from this. QED

Proof of the 'if' part suggested by T. Nowik. By induction on the number of circles. The statement is true for one circle (there are only 2 disks on each sphere hence $n = 2$). Each additional circle splits one component into two, and adds two boundary circles. QED

Proof of the 'only if' part. Let a be the number of those x_i's that are greater than 1. Define b analogously with x_i replaced by y_i. Draw $a + 1$ circles on the sphere S^2 so that these circles split S^2 into 2 disks and a annuli (an annulus is a disk with one hole). We call *main circles* those $a - 1$ circles that do not bound a disk. We may assume that $x_i = 1$ for all $i > a$. For each i from 1 to a draw $x_i - 2$ non-intersecting disks in the i-th annulus from the top. Main circles and the bounding circles of all these disks split S^2 into n surfaces, which are spheres with $x_1, x_2, ...x_n$ holes.

Then number all these $n - 1$ circles:

• for each i from 1 to a, the i-th main circle from the top we denote $A_{x_1+x_2+...+x_i}$;

• circles in the i-th annulus from the top we denote

$$A_{x_1+x_2+...+x_i+1}, \quad A_{x_1+x_2+...+x_i+2}, \quad \dots, \quad A_{x_1+x_2+...+x_i+x_{i+1}-1}.$$

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Similarly we draw circles corresponding to $y_1, \dots, y_n$ and denote them $B_1, B_2, \dots, B_n$ as above.

It suffices to prove that there exist embeddings

$$f, g : S^2 \to \mathbb{R}^3 \quad \text{such that} \quad S^2 \cap g^{-1}(S^2) = \sqcup_{i=1}^{n-1} A_i \quad \text{and} \quad S^2 \cap f^{-1}(S^2) = \sqcup_{i=1}^{n-1} B_i.$$

We prove this by induction on n. The induction base is $n = 2$. Take embeddings f, g such that $f(S^2) \cap g(S^2)$ is one circle. They are as required.

Let us prove the induction step. Suppose the Theorem is proved for $1, 2, \dots, n-1$. Let us prove it for $n > 2$. Since $n > 2$, we have $x_1 > 1$ and $y_1 > 1$. Without loss of generality, assume that $x_1 \geq y_1$. Hence $\sum_{i=2}^{n-y_1+2} x_i = (\sum_{i=1}^{n} x_i) - y_1 - (y_1 - 2) = 2(n - y_1 + 1) - 2$. Then by the induction hypothesis, there are embeddings $f', g' : S^2 \to \mathbb{R}^3$ such that the Theorem holds for the sequences $x_2, x_3, ..., x_{n-x_1+2}$ and $y_2, y_3, ..., y_{n-y_1+2}$.

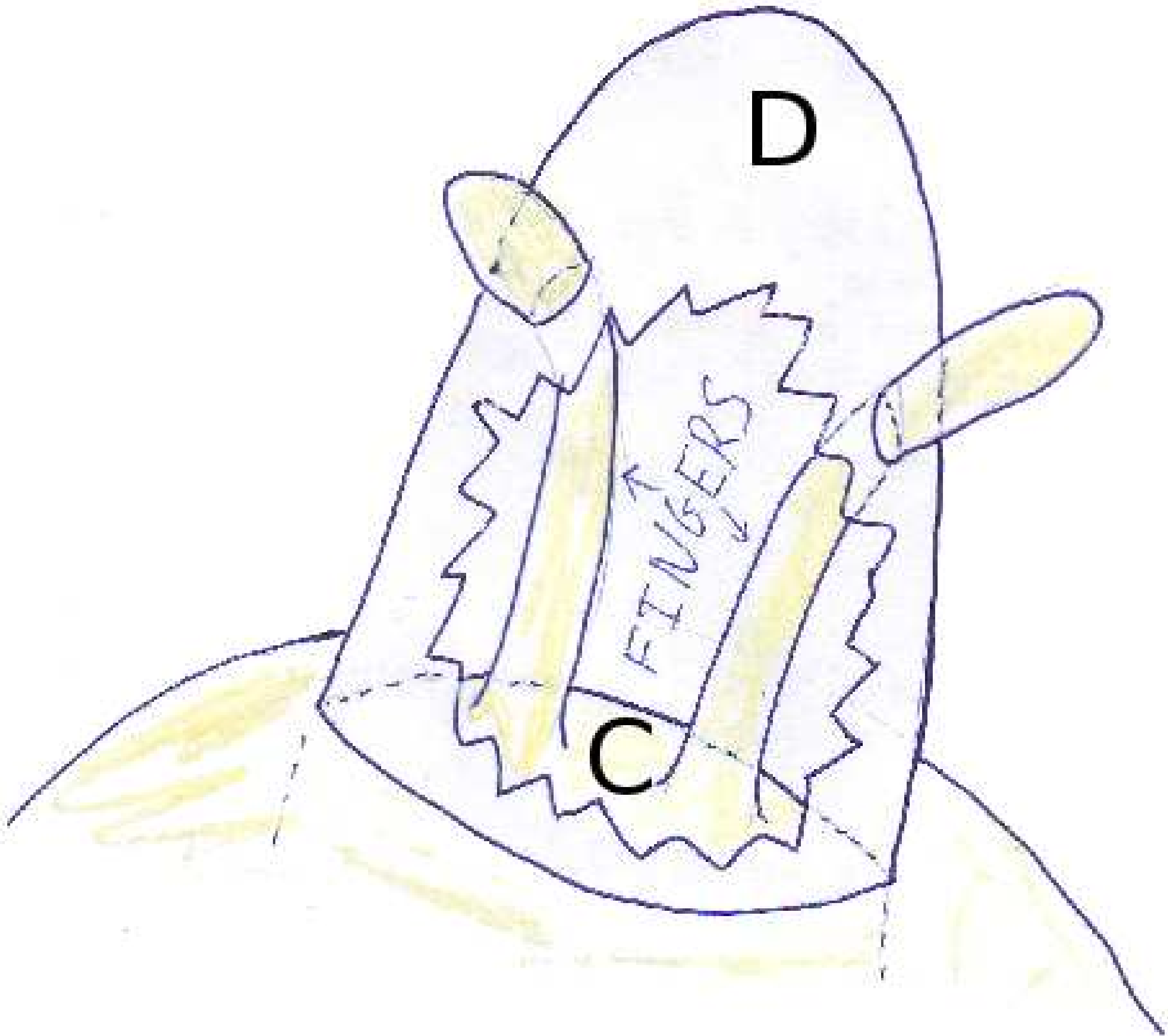

Figure 1

Denote by D the connected component of $S^2 - B_1$ that is a disk not containing other circles B_i. If $x_1 = y_1$, then denote by C the connected component of $S^2 - A_1$ that is a disk not containing other circles A_i. If $x_1 > y_1$, then denote by C the connected component of $S^2 - f^{-1}(S^2)$ that is bounded by the circles $A_1, A_2, ..., A_{x_1-y_1}$. We modify the embeddings f', g' by joining C and D by y_1 fingers, see Figure 1. Denote the new embeddings by f, g. We have added y_1 circles both to the first family and to the second family of circles. Number the new y_1 circles by $1, 2, ..., y_1$ (in both families). Each of the other circles will increase its previously assigned number by y_1. So the embeddings f, g are as required.

The induction step is proved. QED

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[H] T. Hirasa, Dissecting the torus by immersions, preprint.
[N] T. Nowik, Dissecting the 2-sphere by immersions, http://arxiv.org/abs/math/0612796