STABLE COHOMOLOGY OF THE UNIVERSAL PICARD VARIETIES AND THE EXTENDED MAPPING CLASS GROUP

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ABSTRACT. We compute the stable cohomology of the universal Picard stack $\mathsf{Pic}_g \to \mathsf{M}_g$, and also its Picard group. The degree zero Picard stack Pic_g^0 has homotopy type the classifying space of Kawazumi's extended mapping class group $\tilde{\Gamma}_g$, and we explain the relation between our calculations and Kawazumi's generalised Morita–Mumford classes.

1. INTRODUCTION

To each Riemann surface Σ there is an associated *Picard variety*, $\mathrm{Pic}(\Sigma)$. As a set this is defined to be $H^1(\Sigma; \mathcal{O}_\Sigma^\times)$, the first cohomology with coefficients in the sheaf of nowhere zero holomorphic functions, so its points correspond to isomorphism classes of holomorphic line bundle on Σ. The exponential map $H^1(\Sigma; \mathcal{O}_\Sigma) \to H^1(\Sigma; \mathcal{O}_\Sigma^\times)$ from a complex vector space endows $H^1(\Sigma; \mathcal{O}_\Sigma^\times)$ with a topology, and using Hodge theory one sees that with this topology $\mathrm{Pic}(\Sigma)$ is a complex torus of dimension the genus of Σ. In this paper we wish to study the collection of all Picard varieties parametrised by the moduli space M_g of all Riemann surface of genus g, considered as a family $\mathsf{Pic}_g \to \mathsf{M}_g$. As Riemann surfaces can have non-trivial automorphisms, M_g is best considered as a stack, and from now on we will do so. We adopt the convention of writing stacks as Abc and their associated homotopy types as Abc.

Denote by Hol_g the stack over **Top** which classifies families of Riemann surfaces of genus g, $E \to B$, equipped with a fibrewise holomorphic line bundle $L \to E$. Denote by Pic_g the stack over **Top** which classifies families of Riemann surfaces of genus g, $E \to B$, equipped with a section $s : B \to \mathrm{Pic}(E)$ of the associated bundle of Picard varieties. We will define these notions more precisely in Section 2, where we show these are both holomorphic stacks. Each of these stacks splits into path components (i.e. open and closed substacks), Hol_g^k and Pic_g^k respectively, indexed by the fibrewise degree of the (isomorphism class of the) line bundle on the total space. Furthermore, both stacks have a natural map to M_g that just remembers the underlying Riemann surface.

The principal relation between these two objects is that there is a morphism $\mathsf{Hol}_g^k \to \mathsf{Pic}_g^k$ over M_g sending a holomorphic line bundle to its isomorphism class, and in Theorem 2.6 we show this is a $\mathbb{C}^\times$-gerbe. In particular there is a homotopy fibre sequence of the associated homotopy types

$$(1.1) \qquad \mathbb{CP}^\infty \longrightarrow \mathrm{Hol}_g^k \longrightarrow \mathrm{Pic}_g^k.$$

On the other hand we can identify the homotopy type Hol_g, and this turns out to be the *moduli space of Riemann surfaces with maps to* $\mathbb{CP}^\infty$, a particular example of moduli spaces of surfaces with maps to a background space introduced by Cohen and Madsen [7] and denoted $\mathcal{S}_g(\mathbb{CP}^\infty)$. When the background space (in this case $\mathbb{CP}^\infty$) is simply-connected, these spaces are known to satisfy homological stability

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[7, 8, 3, 24] and their stable integral cohomology may be identified with that of a certain infinite loop space [7]. This allows us to compute the rational cohomology of the stacks Hol_g^k in a stable range, and the fibration (1.1) allows us to also compute the rational cohomology of the stacks Pic_g^k in a stable range.

To express these results we must first construct certain cohomology classes $\kappa_{i,j} \in H^{2i+2j}(\mathsf{Hol}_g^k; \mathbb{Z})$. Recall that the ordinary cohomology of a stack X is defined to be the cohomology of its homotopy type X. Equivalently, giving a cohomology class on X is just giving a natural transformation $\mathsf{X}(-) \to H^i(-; \mathbb{Z})$ where we consider the functor $H^i(-; \mathbb{Z})$ as taking values in discrete groupoids (i.e., sets). Thus when X is the moduli space of some geometric objects, a cohomology class on it is precisely a characteristic class of families of such objects. Given $(\pi : E \to B, L \to E) \in \mathsf{Hol}_g(B)$ we define the element

$$\kappa_{i,j}(E, L) := \pi_!(c_1(T^v E)^{i+1} \cdot c_1(L)^j) \in H^{2i+2j}(B; \mathbb{Z}).$$

This is invariant under isomorphisms in $\mathsf{Hol}_g(B)$, and so determines a class $\kappa_{i,j} \in H^{2i+2j}(\mathsf{Hol}_g; \mathbb{Z})$.

The class $\kappa_{i,0}$ can be defined on M_g, and here coincides with the cohomology class usually denoted κ_i. The Madsen–Weiss theorem [20] implies that the homomorphism

$$\mathbb{Q}[\kappa_1, \kappa_2, \ldots] \longrightarrow H^*(\mathsf{M}_g; \mathbb{Q})$$

is an isomorphism in degrees $* \leq \lfloor \frac{2g-2}{3} \rfloor$. We recall for later use that the *Hodge class* $\lambda \in H^2(\mathsf{M}_g; \mathbb{Z})$ may be defined as $c_1(\pi_!^K(T^v))$, the first Chern class of the pushforward in K-theory of the vertical tangent bundle, and it satisfies the relation $12\lambda = \kappa_1$.

1.1. **Stable rational cohomology of the universal Picard varieties.** Our first theorem on the cohomology of these stacks identifies the rational cohomology of Hol_g^k in a range that tends to infinity with g, and identifies the rational cohomology of Pic_g^k as a certain subalgebra of the cohomology of Hol_g^k in this stable range.

Theorem A. *There is an isomorphism*

$$H^*(\mathsf{Hol}_g^k; \mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \ldots] \otimes \mathbb{Q}[\kappa_{i,j} \mid i + j > 0, j > 0, i \geq -1]$$

in degrees $* \leq \lfloor \frac{2g}{3} \rfloor - 1$. *Furthermore,* $H^*(\mathsf{Pic}_g^k; \mathbb{Q})$ *injects into* $H^*(\mathsf{Hol}_g^k; \mathbb{Q})$ *in all degrees, with image*

$$\mathbb{Q}[\kappa_1, \kappa_2, \ldots] \otimes \mathbb{Q}[\kappa_{i,j}, k\kappa_{0,1} + (g-1)\kappa_{-1,2} \mid i + j > 1, j > 0, i \geq -1]$$

in degrees $* \leq \lfloor \frac{2g}{3} \rfloor - 1$.

In fact we obtain integral information, but it is difficult to see precisely what is obtained. We discuss this briefly in §3.

1.2. **Low dimensional integral cohomology.** In low degrees we are also able to make integral cohomological computations, and in particular we may compute the first, second and third integral cohomology of the stacks Hol_g^k and Pic_g^k.

Theorem B. *Suppose* $g \geq 6$. *The groups* $H^1(\mathsf{Hol}_g^k; \mathbb{Z})$ *and* $H^1(\mathsf{Pic}_g^k; \mathbb{Z})$ *are trivial.*

The group $H^2(\mathsf{Hol}_g^k; \mathbb{Z})$ *is free abelian of rank three, with free basis the Hodge class* λ, $\kappa_{-1,2}$ *and* $\zeta := \frac{1}{2}(\kappa_{0,1} - \kappa_{-1,2})$. *The group* $H^2(\mathsf{Pic}_g^k; \mathbb{Z})$ *is free abelian of rank two and injects into* $H^2(\mathsf{Hol}_g^k; \mathbb{Z})$. *A free basis for it may be taken to be* λ *and an element* η *that maps to*

$$\frac{1}{\gcd(2g-2, g+k-1)} (k\kappa_{0,1} + (g-1)\kappa_{-1,2})$$

in $H^2(\mathsf{Hol}_g^k; \mathbb{Z})$.

The group $H^3(\mathsf{Hol}_g^k; \mathbb{Z})$ is zero, and $H^3(\mathsf{Pic}_g^k; \mathbb{Z})$ is $\mathbb{Z}/\gcd(2g-2, 1-g-k)$ generated by the Dixmier–Douady class of the gerbe $\mathsf{Hol}_g^k \to \mathsf{Pic}_g^k$.

1.3. Line bundles and the analytic Néron–Severi group.

We now turn our attention to the study of line bundles on Pic_g^k and Hol_g^k. For a holomorphic stack there are several notions of Picard group, and we will study two, the group Pic_{top} of complex line bundles, and the group Pic_{hol} of holomorphic line bundles. For any holomorphic stack X there are homomorphisms

$$\mathrm{Pic}_{hol}(\mathsf{X}) \longrightarrow \mathrm{Pic}_{top}(\mathsf{X}) \xrightarrow{c_1} H^2(\mathsf{X}; \mathbb{Z})$$

and in general neither are isomorphisms. Recall there is a canonical subgroup $\mathrm{Pic}_{hol}^0(\mathsf{X}) \subset \mathrm{Pic}_{hol}(\mathsf{X})$ of the topologically trivial line bundles, and the *Néron–Severi group* is defined to be

$$\mathcal{NS}(\mathsf{X}) := \mathrm{Pic}_{hol}(\mathsf{X})/\mathrm{Pic}_{hol}^0(\mathsf{X})$$

(this can be viewed as the group of topological line bundles that admit a holomorphic structure).

Theorem C. *If $g \geq 6$, the maps*

$$\mathcal{NS}(\mathsf{Hol}_g^k) \longrightarrow \mathrm{Pic}_{top}(\mathsf{Hol}_g^k) \xrightarrow{c_1} H^2(\mathsf{Hol}_g^k; \mathbb{Z})$$

and

$$\mathcal{NS}(\mathsf{Pic}_g^k) \longrightarrow \mathrm{Pic}_{top}(\mathsf{Pic}_g^k) \xrightarrow{c_1} H^2(\mathsf{Pic}_g^k; \mathbb{Z})$$

are isomorphisms.

With Theorem B this implies a theorem of Kouvidakis [18, Theorem 1] on the image of the map

$$\mathcal{NS}(\mathsf{Pic}_g^k) \to \mathcal{NS}(\mathrm{Pic}^k(\Sigma_g)),$$

as it shows that this map takes values $\frac{\chi(\Sigma_g)}{\gcd(\chi(\Sigma_g), g+k-1)}\,\theta$, where $\theta \in \mathcal{NS}(\mathrm{Pic}^k(\Sigma_g))$ is the class of the theta-divisor. In fact, Kouvidakis' result is slightly different: firstly it takes place in the algebraic context, and secondly it treats Pic_g^k restricted to the subspace of M_g of automorphism-free curves, although he points out that the same methods work over the entire stack.

1.4. Relation to algebraic geometry.

In the algebraic setting there exists a smooth algebraic stack $\mathcal{P}ic_g^k$ over the moduli stack of curves $\mathcal{M}_g$, which is an algebraic analogue of our $\mathsf{Hol}_g^k \to \mathsf{M}_g$. There is a natural copy of the multiplicative group $\mathbb{G}_m$ inside the endomorphisms of every object of $\mathcal{P}ic_g^k$, hence it may be rigidified to a stack $\mathcal{P}_g^k$ which is an algebraic analogue of our Pic_g^k. This new stack is smooth and Deligne–Mumford, and the quotient map is representable. We have not proved, but it is likely to be true, that the associated analytic stacks to $\mathcal{P}ic_g^k$ and $\mathcal{P}_g^k$ are Hol_g^k and Pic_g^k respectively. In any case, there is certainly an analytic map $(\mathcal{P}_g^k)^{an} \to \mathsf{Pic}_g^k$, and so we may consider the composition

$$(1.2) \qquad \mathrm{Pic}(\mathcal{P}_g^k) \longrightarrow \mathrm{Pic}_{hol}((\mathcal{P}_g^k)^{an}) \longrightarrow \mathrm{Pic}_{hol}(\mathsf{Pic}_g^k) \longrightarrow \mathcal{NS}(\mathsf{Pic}_g^k),$$

and similarly for $\mathcal{P}ic_g^k$.

In [22], Melo and Viviani use algebro-geometric methods to study the Picard groups of $\mathcal{P}_g^k$ and $\mathcal{P}ic_g^k$, and also of their compactifications. Comparing with [22, Theorem 4.2 and Corollary 4.4], we see — by computation of both sides — that the composition (1.2) and its analogue for $\mathcal{P}ic_g^k$ are both isomorphisms.

1.5. **Stable cohomology of the extended mapping class group.** The mapping class group of an oriented surface $\Sigma_{g,r}$ of genus g with r boundary components and s marked points is

$$\Gamma_{g,r}^s := \pi_0(\mathrm{Diff}^+(\Sigma_{g,r}), \partial \cup \{x_1, \ldots, x_s\}),$$

the group of isotopy classes of diffeomorphisms which fix the boundary and the marked points pointwise. This group acts on the first homology group of the surface $H_1(\Sigma_{g,r}; \mathbb{Z})$, and following Kawazumi [15] we define the *extended mapping class group* as the semi-direct product

$$\widetilde{\Gamma}_{g,r}^s := H_1(\Sigma_{g,r}; \mathbb{Z}) \rtimes \Gamma_{g,r}^s.$$

Poincaré duality provides an isomorphism $H_1(\Sigma_{g,r}; \mathbb{Z}) \cong H^1(\Sigma_{g,r}, \partial; \mathbb{Z})$ of $\Gamma_{g,r}^s$-modules, and we will denote this module by H, its rationalisation by $H_{\mathbb{Q}}$ and its dual module by H^*. It is worth remarking that if $r = 0$ or 1, there is a natural isomorphism of $\Gamma_{g,r}$-modules $H \cong H^*$ given by Poincaré duality, but there is no such isomorphism for any $r > 1$. As long as $r \leq 1$, we will use the modules H and H^* interchangeably.

The extended mapping class group is related to the discussion so far as there is a homotopy equivalence

$$(1.3) \qquad\qquad B\widetilde{\Gamma}_g \simeq \mathrm{Pic}_g^0,$$

and so the cohomological calculations of Theorem A also give us the stable rational cohomology of the extended mapping class groups of closed surfaces. Using similar methods we are able to study the extended mapping class groups of surfaces with boundary. The Leray–Hochschild–Serre spectral sequence for the extension defining $\widetilde{\Gamma}_{g,r}$ has the form

$$(1.4) \qquad\qquad E_2^{p,q} = H^p(\Gamma_{g,r}; \wedge^q H^*) \Longrightarrow H^{p+q}(\widetilde{\Gamma}_{g,r}; \mathbb{Z})$$

and so the cohomology of the extended mapping class group is closely related to the cohomology of the ordinary mapping class group with coefficients in exterior powers of the module H^*. We call the resulting filtration on $H^*(\widetilde{\Gamma}_{g,r}; \mathbb{Z})$ the *natural filtration*.

It has been known for some time [14] that the groups $H^*(\Gamma_{g,r}; H)$ exhibit homological stability as long as the boundary remains non-empty. In low degrees Morita has found [23] isomorphisms

$$(1.5) \qquad\qquad H^1(\Gamma_{g,1}; H) \cong \mathbb{Z} \qquad H^1(\Gamma_g; H) = 0$$

and

$$(1.6) \qquad\qquad H^2(\Gamma_g; H) \cong \mathbb{Z}/\chi(\Sigma_g),$$

which imply that the stabilisation map $\Gamma_{g,1} \to \Gamma_g$ does not exhibit homological stability with coefficients in the module H. The spectral sequence (1.4) implies that the extended mapping class group also cannot have homological stability for closing the last boundary.

We show that for $g \geq 6$ the group $H^3(\widetilde{\Gamma}_g; \mathbb{Z})$ is precisely $\mathbb{Z}/\chi(\Sigma_g)$, which arises in the spectral sequence (1.4) as the group $H^2(\Gamma_g; H)$. Furthermore, we show that $H^3(\widetilde{\Gamma}_{g,r}; \mathbb{Z})$ is trivial for all $r > 0$. The failure of stability for the extended mapping class group is down entirely to the failure of stability of $H^2(\Gamma_g; H)$, in the following precise sense.

Theorem D. *Let $x \in H^3(\widetilde{\Gamma}_g; \mathbb{Z}) \cong \mathbb{Z}/\chi(\Sigma_g)$ be a generator. Then the composition*

$$B\widetilde{\Gamma}_{g,1} \longrightarrow B\widetilde{\Gamma}_g \xrightarrow{x} K(\mathbb{Z}, 3)$$

is a homology fibration in degrees $* \leq \lfloor \frac{2g}{3} \rfloor - 1$.

The cohomology of $\widetilde{\Gamma}_{g,1}$ has been studied in depth by Kawazumi [15, 16, 17], and he has defined certain classes $m_{i,j} \in H^{2i+j-2}(\Gamma_{g,1}, \wedge^j H)$ which are permanent cycles in the spectral sequence (1.4), and detect classes $\widetilde{m}_{i,j} \in H^{2i+2j-2}(\widetilde{\Gamma}_{g,1}; \mathbb{Z})$ he has also defined. These are defined using explicit group cocycles which depend crucially on the surface having boundary.

For surfaces with boundary we show that there is an equivalence

$$B\widetilde{\Gamma}_{g,r} \simeq \mathcal{S}_{g,r}(\mathbb{CP}^\infty)_0$$

and the characteristic classes $\kappa_{i,j}$ may be defined on this space. When $r = 1$ we show that under this equivalence Kawazumi's classes $\widetilde{m}_{i+1,j}$ correspond to our $\kappa_{i,j}$, and hence that $\kappa_{i,j}$ has natural filtration precisely j. This implies the following description of the rational cohomology of the extended mapping class group and of the rational cohomology of the mapping class group with coefficients in exterior powers of the homology representation, which generalises Kawazumi's results for a single boundary component.

Theorem E. *For $r > 0$ there is an isomorphism of graded $\mathbb{Q}[\kappa_1, \kappa_2, \ldots]$-algebras*

$$H^*(\widetilde{\Gamma}_{g,r}; \mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \ldots] \otimes \mathbb{Q}[\kappa_{i,j} \mid i + j > 0, j > 0, i \geq -1]$$

in degrees $* \leq \lfloor \frac{2g}{3} \rfloor - 1$, where $\kappa_{i,j}$ has degree $2(i+j)$. In this range of degrees the spectral sequence for the natural filtration collapses and the associated graded is

$$\bigoplus_{p,q} H^p(\Gamma_{g,r}; \wedge^q H^*_{\mathbb{Q}}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \ldots] \otimes \mathbb{Q}[x_{i,j} \mid i + j > 0, j > 0, i \geq -1]$$

where κ_i has bidegree $(p,q) = (2i, 0)$ and $x_{i,j}$ has bidegree $(p,q) = (2i+j, j)$.

The above theorem is proved by Kawazumi in [17] in the case $r = 1$, and follows for $r \geq 1$ by homological stability with twisted coefficients, e.g. from the work of Boldsen [3]. However, we are also able to obtain a description of the stable rational cohomology of the extended mapping class group of a closed surface. The following description follows from Theorem D and a careful analysis of the cohomological behaviour of the map $B\widetilde{\Gamma}_{g,1} \to B\widetilde{\Gamma}_g$ with respect to the natural filtrations on both groups.

Theorem F. *There is an isomorphism of graded $\mathbb{Q}[\kappa_1, \kappa_2, \ldots]$-algebras*

$$H^*(\widetilde{\Gamma}_g; \mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \ldots] \otimes \mathbb{Q}[\widetilde{\kappa}_{i,j} \mid i + j > 0, j > 0, i \geq -1, (i,j) \neq (0,1)]$$

in degrees $* \leq \lfloor \frac{2g}{3} \rfloor - 1$, where $\widetilde{\kappa}_{i,j}$ is a class of degree $2(i+j)$ that maps to $\kappa_{i,j}$ in the cohomology of $\widetilde{\Gamma}_{g,1}$. The spectral sequence for the natural filtration collapses and the associated graded in the stable range is

$$\bigoplus_{p,q} H^p(\Gamma_g; \wedge^q H^*_{\mathbb{Q}}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \ldots] \otimes \mathbb{Q}[x_{i,j} \mid i + j > 0, j > 0, i \geq -1, (i,j) \neq (0,1)]$$

where κ_i has bidegree $(p,q) = (2i, 0)$ and $x_{i,j}$ has bidegree $(p,q) = (2i+j, j)$.

Looijenga [19] has studied the stable rational cohomology of the mapping class group with coefficients in any symplectic representation, and his results of course give the same identification of $H^*(\Gamma_g; \wedge^q H_{\mathbb{Q}})$ as an abstract group, but his interpretation of the generators as a $\mathbb{Q}[\kappa_1, \kappa_2, \ldots]$-algebra is different.

2. Geometry of the universal Picard varieties

In order to make precise our claims regarding the stacks Hol_g and Pic_g, we must make precise our definitions of them, and establish several geometrical properties.

2.1. The stack of curves with holomorphic line bundles.

Definition 2.1. Let Cx_g denote the following stack, defined on the site **Top**. An object of $\mathsf{Cx}_g(Y)$ consists of an oriented smooth surface bundle $\pi : E \to Y$ with genus g fibres, and a complex line bundle $L \to E$. An isomorphism $(E, L) \to (E', L')$ is an isomorphism of surface bundles $f : E \to E'$ and an isomorphism $h : L \to f^*L'$ of complex line bundles. We denote by $\mathsf{Cx}_g^k \subset \mathsf{Cx}_g$ the open and closed substack consisting of those bundles having fibrewise degree k.

Definition 2.2. Let Hol_g denote the following stack, defined on the site **Top**. An object of $\mathsf{Hol}_g(Y)$ consists of a family of genus g Riemann surfaces $\pi : E \to Y$ and a fibrewise holomorphic line bundle $L \to E$. An isomorphism $(E, L) \to (E', L')$ is an isomorphism of families $f : E \to E'$ and an isomorphism $h : L \to f^*L'$ of holomorphic line bundles. We denote by $\mathsf{Hol}_g^k \subset \mathsf{Hol}_g$ the open and closed substack consisting of those bundles having fibrewise degree k.

Given a family $E \to B$ of Riemann surfaces of genus g, denote by $\mathsf{Hol}(E)$ the fibre product $B \times_{\mathsf{M}_g} \mathsf{Hol}_g$ and we apply a similar convention to the stack Cx. There are morphisms

(2.1)
$$
\begin{array}{ccc}
\mathsf{Hol}_g \longrightarrow \mathsf{Cx}_g \longleftarrow \mathcal{S}_g(\mathbb{CP}^\infty) \\
\downarrow \qquad\qquad \downarrow \qquad\qquad \swarrow \\
\mathsf{M}_g \longrightarrow *\!\sslash\!\mathrm{Diff}^+(\Sigma_g)
\end{array}
$$

where the left horizontal map forgets the holomorphic structure and the right horizontal one is given by taking the induced bundle. The vertical morphisms forget the line bundle data and the bottom horizontal morphism forgets the complex structure. It is clear that the diagram commutes (up to 2-isomorphism).

Theorem 2.3. *The stacks Hol_g and Cx_g are topological stacks and the horizontal maps in the diagram (2.1) are homotopy equivalences (i.e. universal weak equivalences).*

Proof. Since by definition
$$
\mathcal{S}_g(\mathbb{CP}^\infty) \simeq E\mathrm{Diff}^+(\Sigma_g) \times_{\mathrm{Diff}^+(\Sigma_g)} \mathrm{map}(\Sigma_g; \mathbb{CP}^\infty);
$$
we can rewrite the map $\mathcal{S}_g(\mathbb{CP}^\infty) \to \mathsf{Cx}_g$ as the composition

(2.2) $E\mathrm{Diff}^+(\Sigma_g) \times_{\mathrm{Diff}^+(\Sigma_g)} \mathrm{map}(\Sigma_g; \mathbb{CP}^\infty) \to \mathrm{map}(\Sigma_g; \mathbb{CP}^\infty) \sslash \mathrm{Diff}(\Sigma_g) \to \mathsf{Cx}_g.$

The first map is a universal weak equivalence by general stack-theoretic principles, namely [11, Proposition 2.5]. To analyze the second map, consider a space X and a map $X \to \mathsf{Cx}_g$, representing $L \to E \xrightarrow{\pi} X$, where $\pi : E \to X$ is a surface bundle and $L \to E$ a complex line bundle.

Look at the auxiliary space S of pairs (x, f), where $x \in X$ and $f : L|_{\pi^{-1}(x)} \to \pi^{-1}(x) \times \mathbb{C}^\infty$ is a bundle monomorphism. The topology on S is the unique one such that the projection $S \to X$ is a locally trivial fibre bundle and such that the preimage of x has the compact-open topology. The fibre over x is the space of bundle maps $L|_{\pi^{-1}(x)} \to \pi^{-1}(x) \times \mathbb{C}^\infty$, which is contractible. It is a routine verification to identify the fibre product $X \times_{\mathsf{Cx}_g} \mathrm{map}(\Sigma_g; \mathbb{CP}^\infty) \sslash \mathrm{Diff}(\Sigma_g)$ with S. Thus the second map in (2.2) has local sections and it is a universal weak equivalence. This finishes the proof that $\mathcal{S}_g(\mathbb{CP}^\infty) \to \mathsf{Cx}_g$ is both an atlas and a universal weak equivalence.

Next we turn to the map $\mathsf{Hol}_g \to \mathsf{Cx}_g$ which we factor into three maps

$$\mathsf{Hol}_g \xrightarrow{\phi_1} \widetilde{\mathsf{Hol}}_g \xrightarrow{\phi_2} \mathsf{qHol}_g \xrightarrow{\phi_3} \mathsf{Cx}_g.$$

The stack qHol_g parametrizes families of Riemann surfaces together with complex line bundles, and the map ϕ_3 is the forgetful map. It is a universal weak equivalence because the diagram

$$
\begin{array}{ccc}
\mathsf{qHol}_g & \longrightarrow & \mathsf{Cx}_g \\
\downarrow & & \downarrow \\
\mathsf{M}_g & \longrightarrow & *\!/\!/\mathrm{Diff}^+(\Sigma_g)
\end{array}
$$

is a fibre square and the bottom map is a universal weak equivalence by Teichmüller thory.

Now we define the stack $\widetilde{\mathsf{Hol}}_g$. A map $X \to \widetilde{\mathsf{Hol}}_g$ is an element $L \to E \to X$ of $\mathsf{qHol}_g(X)$, together with a family of fibrewise differential operators $D : \Gamma(E; L) \to \Gamma(E; \Lambda_v^{0,1} \otimes L)$ such that

$$(2.3) \qquad\qquad D(fs) = \overline{\partial}f \otimes s + fD(s)$$

for each $s \in \Gamma(E; L)$ and $f \in C^\infty(E)$, where differentiation is understood to be in the fibrewise sense. The map ϕ_2 forgets the differential operator. The condition (2.3) is convex, which implies that each smooth line bundle admits such an operator and that the space of these operators is convex. Therefore ϕ_2 is a universal weak equivalence.

The map ϕ_1 associates to each holomorphic line bundle the Cauchy–Riemann operator on that line bundle. We claim that ϕ_1 is an isomorphism of stacks. This amounts to showing that a holomorphic structure on a line bundle is determined by its Cauchy–Riemann operator (which is a tautology) and that any family of operators satisfying (2.3) induces the structure of a holomorphic line bundle on L, i.e. there exist locally nonzero solutions of $Ds = 0$. The argument we give is due to Atiyah and Bott [1, page 555] (they consider the case of higher-dimensional vector bundles, which is more complicated, but for a fixed Riemann surface, which is easier).

Let $(L \to E \xrightarrow{p} X, D)$ be an element of $\widetilde{\mathsf{Hol}}_g(X)$. For $x \in X$ and $y \in p^{-1}(x)$, we can pick a neighborhood U of x and a map $\alpha : U \times \mathbb{D} \to E$ over X which is an open embedding, fibrewise holomorphic and satisfies $\alpha(x, 0) = y$; furthermore we require L to be trivial over $U \times \mathbb{D}$. We wish to find a section s of L over $\alpha(U \times \mathbb{D})$ that is nowhere zero and satisfies $Ds = 0$ over $U \times \mathbb{D}_{\frac{1}{2}}$. To this end, pick a fibrewise smooth section s_0 of L over $\alpha(U \times \mathbb{D})$ and look for a function f that satisfies $D(e^f s_0) = 0$. So we have to solve the PDE

$$0 = e^{-f}D(e^f s_0) = \overline{\partial}f \otimes s_0 + Ds_0.$$

Write $Ds_0 = -\beta \otimes s_0$ for a $(0, 1)$-form β; this reduces the problem to the equation

$$\overline{\partial}(f) = \beta.$$

Since all that matters is a local section on $U \times \mathbb{D}_{\frac{1}{2}}$, we can multiply β with a cut-off function and thus assume that β has compact support. Now we pick a fibrewise holomorphic embedding $U \times \mathbb{D} \to U \times \mathbb{CP}^1$ and a bundle map from the trivial bundle on $U \times \mathbb{D}$ into the tautological line bundle on $U \times \mathbb{CP}^1$. By Riemann–Roch, the Cauchy–Riemann operator on the tautological line bundle (it has degree -1) is invertible. Hence its inverse is continuous as well. Therefore we can find a continuous solution of $\overline{\partial}f = \beta$ over $U \times \mathbb{D}$. The arguments given so far amount to the construction of an inverse map to ϕ_1 and thus the proof is complete. $\square$

2.2. **The universal Picard stack.** Let $\pi : E \to B$ be a family of Riemann surfaces. We define the associated Picard bundle $\mathrm{Pic}(E) \to B$ as follows. Let $\mathcal{O}$ denote the sheaf of continuous, fibrewise holomorphic functions on E, and $\mathcal{O}^\times$ the subsheaf of nowhere zero functions. The exponential sequence of sheaves $\mathbb{Z} \to \mathcal{O} \to \mathcal{O}^\times$ gives an exact sequence of sheaves on B,

$$(2.4) \qquad 0 \longrightarrow R^1\pi_*\mathbb{Z} \longrightarrow R^1\pi_*\mathcal{O} \longrightarrow R^1\pi_*\mathcal{O}^\times \longrightarrow R^2\pi_*\mathbb{Z} \longrightarrow 0.$$

These are all sheaves of sections of certain bundles of groups on B,

$$(2.5) \qquad 0 \longrightarrow [H^1(F_b;\mathbb{Z})] \longrightarrow [H^1(F_b;\mathcal{O})] \longrightarrow \mathrm{Pic}(E) \longrightarrow \mathbb{Z} \times B \longrightarrow 0$$

where $[H^1(F_b;\mathcal{F})]$ is the bundle of fibrewise first cohomologies with coefficients in the sheaf $\mathcal{F}$, and $\mathrm{Pic}(E)$ is a bundle of abelian groups isomorphic to $\mathbb{Z} \times \mathbb{T}^{2g}$. The group $\pi_1(B)$ acts trivially on the set of path components, and we denote by $\mathrm{Pic}^k(E)$ the degree k component.

Definition 2.4. Let Pic_g denote the following stack, defined on the site **Top**. An object of $\mathsf{Pic}_g(Y)$ is a family of Riemann surfaces $\pi : E \to B$ and a section $s : B \to \mathrm{Pic}(E)$. An isomorphism $(E,s) \to (E',s')$ is an isomorphism of families $f : E \to E'$ such that $f^*(s') = s$. We denote by $\mathsf{Pic}_g^k \subset \mathsf{Pic}_g$ the open and closed substack consisting of those pairs (E,s) where s takes values in $\mathrm{Pic}^k(E) \subset \mathrm{Pic}(E)$.

Lemma 2.5. Pic_g *is a holomorphic Deligne–Mumford stack. In particular, it is a local quotient stack and a topological stack.*

Proof. An atlas for this stack is given as follows: Let $\pi : \mathcal{E}_g \to \mathcal{T}_g$ be the universal family of Riemann surfaces on Teichmüller space. The total space of $\mathrm{Pic}(\mathcal{E}_g)$ is a complex manifold of dimension $4g - 3$. There is a map $\mathrm{Pic}(\mathcal{E}_g) \to \mathsf{Pic}_g$ which is an atlas. This argument also show that Pic_g is a holomorphic stack. $\square$

2.3. **The gerbe of holomorphic maps.** There is a morphism $\phi_g^k : \mathsf{Hol}_g^k \to \mathsf{Pic}_g^k$ sending a fibrewise holomorphic line bundle to its isomorphism class.

Theorem 2.6. *The map ϕ_g^k is a gerbe with band $\mathbb{C}^\times$.*

Corollary 2.7. Hol_g^k *is a holomorphic stack and a local quotient stack.*

Proof. The holomorphicity is clear; and the statement about local quotients is immediate from [13, Corollary 6.3]. $\square$

Probably Theorem 2.6 is a special case of a statement that is well-known among algebraic geometers who are also fond of stacks. The proof below applies only to line bundles on curves of genus $g \geq 2$ and it begins with a trivial observation:

Lemma 2.8. *Tensor multiplication with the cotangent bundle induces a commutative diagram*

$$\begin{array}{ccc} \mathsf{Hol}_g^k & \longrightarrow & \mathsf{Hol}_g^{k+2g-2} \\ \downarrow{\scriptstyle \phi_g^k} & & \downarrow{\scriptstyle \phi_g^{k+2g-2}} \\ \mathsf{Pic}_g^k & \longrightarrow & \mathsf{Pic}_g^{k+2g-2} \end{array}$$

whose horizontal arrows are isomorphisms.

Proof of Theorem 2.6. Lemma 2.8 shows that it is enough to prove Theorem 2.6 for large values of k. Let us assume that the degree $k > 2g - 2$. We have to show the following three axioms [6, §V.2]:

(i) The map $\phi_g^k : \mathsf{Hol}_g^k \to \mathsf{Pic}_g^k$ has local sections.

(ii) Given any $X \to \mathsf{Pic}^k_g$ and two lifts $X \to \mathsf{Hol}^k_g$; then there is a cover $Y \to X$, over which both lifts become isomorphic.

(iii) The group of automorphisms of $X \to \mathsf{Hol}^k_g$ over Pic^k_g is isomorphic to $\mathcal{C}(X; \mathbb{C}^\times)$.

The first property is the most difficult to show. It says that any family of iso-morphism classes of line bundle can — locally — be lifted to an actual family of line bundles. The second property expresses that two such lifts are locally isomorphic. More specifically: two holomorphic line bundles on a family $E \to X$ of Riemann sur-face which are pointwise (in X) isomorphic are locally isomorphic. The third prop-erty is obvious, because the automorphism group of a holomorphic line bundle on a compact Riemann surface is $\mathbb{C}^\times$. For the proof of the first property, we use a clas-sical construction from the geometry of curves. Let $E \to B$ be a family of Riemann surfaces. The fibrewise k-fold symmetric product is denoted by $\mathrm{Sym}^k(E/B) \to B$. It is well-known that this is a fibre bundle with smooth complex manifolds as fi-bres. Recall the classical divisor–line-bundle correspondence [12, p. 129 ff]. This construction yields a fibre-preserving map $\eta : \mathrm{Sym}^k(E/B) \to \mathrm{Pic}^k(E/B)$. As long as $k \geq g$, this map is surjective (because any line bundle of degree k then has a nontrivial holomorphic section). Clearly η is proper and it is a classical result that the fibres are isomorphic to $\mathbb{CP}^{k-g}$. More precisely, Mattuck [21] has shown that for an individual Riemann surface S, the map $\mathrm{Sym}^k(S) \to \mathrm{Pic}^k(S)$ is a projective bundle (the structural group is $\mathbb{P}\,\mathrm{Gl}_{k+1-g}(\mathbb{C})$) as long as $k > 2g - 2$. It follows that $\eta : \mathrm{Sym}^k(E/B) \to \mathrm{Pic}^k(E/B)$ is a proper submersion and hence a locally trivial fibre bundle by Ehresmann's fibration lemma. In particular, η has local sections.

Finally, the map η lifts to a map to the stack $\mathrm{Hol}^k(E/B)$. Note that this is not entirely tautological. Given a divisor D on a Riemann surface S, the classical construction gives an actual line bundle (and not merely an isomorphism class) when one specifies local holomorphic functions on S that define D, i.e. have the same zeroes (with mutliplicity). Observe that such local functions can be picked continuously when the divisor varies continuously. This argument shows that η lifts (locally). To finish the proof of the first gerbe axiom, pick a local section of η and compose it with the lift of η: this is a local section of ϕ.

Having two lifts of the same map to $\mathrm{Pic}(E/B)$ is the same as having two line bundles that are pointwise (in B) isomorphic. We have to show that they are locally isomorphic. This is accomplished by another switch of perspective. Given two holomorphic line bundles L_0, L_1 on $E \to B$, then asking for a fibrewise holomorphic isomorphism is the same as asking for a fibrewise holomorphic (and nowhere zero) section of the degree 0 line bundle $\mathrm{Hom}(L_0; L_1)$. This is a solution of a Cauchy-Riemann equation, i.e. we are looking for local sections in the kernel of a Fredholm family. But the assumption says that the kernel dimension is everywhere positive; on the other hand, the dimension is at most one (since a degree 0 line bundle that has a nonzero section has to be trivial and thus cannot admit further holomorphic sections). Thus the kernel form a line bundle on B. Take a local section; this is a local isomorphism. This finishes the argument that ϕ^k_g is a gerbe with band $\mathbb{C}^\times$. $\quad\square$

We denote by $\mathcal{G}^k_g \in H^3(\mathsf{Pic}^k_g; \mathbb{Z})$ the Dixmier–Douady class of ϕ^k_g. Our next goal is to show that $\mathcal{G}^k_g$ has finite order and to give an upper bound for its order.

Corollary 2.9 (of the proof). *The structural group of the fibre bundle $\mathrm{Sym}^k(E/B) \to \mathrm{Pic}^k(E/B)$ with fibre $\mathbb{CP}^{k-g}$ is $\mathbb{P}\,\mathrm{Gl}_{k-g+1}(\mathbb{C})$. The following diagram (defined for*

$k > 2g - 2$) is cartesian:

$$
\begin{array}{ccc}
\mathsf{Hol}_g^k & \longrightarrow & * /\!\!/ \operatorname{Gl}_{k-g+1}(\mathbb{C}) \\
\downarrow & & \downarrow \\
\mathsf{Pic}_g^k & \longrightarrow & * /\!\!/ \mathbb{P}\operatorname{Gl}_{k-g+1}(\mathbb{C}).
\end{array}
$$

Proof. The pullback of $\operatorname{Sym}^k(E/B) \to \operatorname{Pic}^k(E/B)$ to $\operatorname{Hol}^k(E/B)$ is the projectivization of the vector bundle $V \to \operatorname{Hol}^k(E/B)$ whose fibre at a point (x, L) is the $k - g + 1$-dimensional vector space $H^0(E_x, L_x)$. This proves the second claim once the first is established.

To show the first claim, we have to study the action of the automorphism group of a line bundle $L \to S$ on $H^0(S, L)$. But it is clear that it acts with weight 1. Thus V is a vector bundle of weight 1 and its projectivization descends to $\operatorname{Pic}^k(E/B)$. But this descended bundle is nothing else than $\operatorname{Sym}^k(E/B)$ and the first claim is proven. $\square$

There is a morphism $\mathsf{Pic}_g \to \mathsf{M}_g$ that forgets the fibrewise Picard data, and this gives a locally trivial bundle $\operatorname{Pic}_g^k \to \mathsf{M}_g$ on homotopy types. The zero component $\operatorname{Pic}_g^0 \to \mathsf{M}_g$ is a bundle of abelian groups isomorphic to $\mathbb{T}^{2g}$. It is classified by the usual map $\Gamma_g \to \operatorname{Sp}_{2g}(\mathbb{Z})$. The bundle Pic_g^k is a torsor over Pic_g^0 (or "twisted $\mathbb{T}^{2g}$-principal bundle"). As such, it is classified by an element of $H^1(M_g; \operatorname{Pic}_g^0)$ (the sheaf of continuous sections in the fibre bundle $\operatorname{Pic}_g^0 \to M_g$). There is a tensor product operation on twisted principal bundles over a bundle of abelian groups, corresponding to the addition in $H^1(M_g; \operatorname{Pic}_g^0)$. Moreover, it is rather clear that $\operatorname{Pic}_g^k = (\operatorname{Pic}_g^1)^{\otimes k}$.

Proposition 2.10. *The group $H^1(M_g; \operatorname{Pic}_g^0)$ is isomorphic to $\mathbb{Z}/(2g - 2)$ and Pic_g^1 is a generator.*

In particular, $\mathsf{Pic}_g^k \to \mathsf{M}_g$ admits a section (and hence is a trivial bundle) if and only if $\chi \mid k$. This should be compared with the strong Franchetta conjecture.

Proof. Look at the short exact sequence of sheaves on M_g

$$
0 \to R^1 p_* \mathbb{Z} \to R^1 p_* \mathcal{O} \to \operatorname{Pic}_g^0 \to 0.
$$

Since $R^1 p_* \mathcal{O}$ is fine, we get an exact sequence in cohomology

$$
0 = H^1(M_g, R^1 p_* \mathcal{O}) \to H^1(M_g; \operatorname{Pic}_g^0) \to H^2(M_g; R^1 p_* \mathbb{Z}) \to H^2(M_g; R^1 p_* \mathcal{O}) = 0.
$$

Morita has computed that $H^2(M_g; R^1 p_* \mathbb{Z}) \cong \mathbb{Z}/(2g - 2)$, and another proof may be found in [8].

The element $\operatorname{Pic}_g^k \in H^1(M_g; \operatorname{Pic}_g^0)$ is trivial if and only if there is a global section $s : M_g \to \operatorname{Pic}_g^k$. Since $H^3(M_g; \mathbb{Z}) = 0$, the element $s^* \mathcal{G}_{g,k} = 0$. In other words, the gerbe $\phi_{g,k}$ becomes trivial when pulled back via s. This implies that we can lift s to a section of $\operatorname{Hol}_g^k \to M_g$, which in turn implies the existence of a fibrewise holomorphic line bundle on $M_g^1 \to M_g$ of fibrewise degree k. But the main result of [5] implies that all line bundles on $M_{g,1}$ have fibrewise degree divisible by $2g - 2$. So the triviality of Pic_g^k implies that $k \equiv 0 \pmod{2g - 2}$. Therefore $\operatorname{Pic}_g^1 \in H^1(M_g; \operatorname{Pic}_g^0)$ is a generator. $\square$

2.4. **Homotopy types.** Taking homotopy types of the stacks introduced so far, we obtain a commutative diagram

$$
\begin{array}{ccccc}
\mathbb{CP}^\infty & \longrightarrow & \mathrm{map}^k(\Sigma_g, \mathbb{CP}^\infty) & \longrightarrow & \mathrm{Pic}^k(\Sigma_g) \\
\| & & \downarrow & & \downarrow \\
\mathbb{CP}^\infty & \longrightarrow & \mathrm{Hol}_g^k \simeq \mathcal{S}_g^k(\mathbb{CP}^\infty) & \longrightarrow & \mathrm{Pic}_g^k \\
& & \downarrow & & \downarrow \\
& & \mathrm{M}_g & = & \mathrm{M}_g
\end{array}
$$

where the rows are principal fibrations, the columns are principal fibrations if k is zero and torsors over principal fibrations in general, and the top right hand square is homotopy cartesian. There is a class $\mathcal{G}_g^k \in H^3(\mathrm{Pic}_g^k; \mathbb{Z})$ classifying the middle row, and a class $\mathcal{P}_g^k \in H^2(\mathrm{M}_g; H)$ classifying the right hand column, corresponding to the classes already defined in the cohomology of the stack.

3. Cohomology of $\mathcal{S}_{g,r}(\mathbb{CP}^\infty)$

Recall we have defined the space $\mathcal{S}_{g,r}(\mathbb{CP}^\infty)$ as the homotopy quotient

$$\mathcal{S}_{g,r}(\mathbb{CP}^\infty) := \mathrm{map}_\partial(\Sigma_{g,r}, \mathbb{CP}^\infty) /\!\!/ \mathrm{Diff}_\partial^+(\Sigma_{g,r}).$$

This splits into path components $\mathcal{S}_{g,r}^k(\mathbb{CP}^\infty)$ indexed by the degree k of maps to $\mathbb{CP}^\infty$. The space $\mathcal{S}_{g,r}^k(\mathbb{CP}^\infty)$ classifies oriented smooth surface bundles

$$\Sigma_{g,r} \longrightarrow E \xrightarrow{\ \pi\ } B$$

equipped with a complex line bundle $L \to E$ which has degree k on each fibre. Such bundles have a natural generalisation of the Mumford–Morita–Miller classes κ_i, defined by the pushforward

$$\kappa_{i,j}(E) := \pi_!\big(e(T^v E)^{i+1} \cdot c_1(L)^j\big) \in H^{2(i+j)}(B; \mathbb{Z})$$

where $T^v E$ is the vertical tangent bundle. In particular we can define these classes on $\mathcal{S}_{g,r}(\mathbb{CP}^\infty)$ to obtain $\kappa_{i,j} \in H^{2(i+j)}(\mathcal{S}_{g,r}(\mathbb{CP}^\infty); \mathbb{Z})$. These characteristic classes can be defined in slightly more generality than on surface bundles, as we now describe.

Consider the vector bundle $\gamma_{2,n} \to \mathrm{Gr}_2^+(\mathbb{R}^{n+2})$ given by the complement of the tautological bundle, and write M_n for its Thom space. The stabilisation maps $\mathrm{Gr}_2^+(\mathbb{R}^{n+2}) \to \mathrm{Gr}_2^+(\mathbb{R}^{n+3})$ pull back $\gamma_{2,n+1}$ to $\epsilon^1 \oplus \gamma_{2,n}$ and so give maps of Thom spaces $\Sigma M_n \to M_{n+1}$. Thus the collection $\{M_n\}_{n \geq 0}$ forms a *spectrum* in the sense of stable homotopy theory, which is denoted $\mathbf{MTSO}(2)$. Similarly, we define a spectrum $\mathbf{MTSO}(2) \wedge \mathbb{CP}_+^\infty$ having as its n-th space the smash product $M_n \wedge \mathbb{CP}_+^\infty$.

The infinite loop space $\Omega^\infty \mathbf{MTSO}(2) \wedge \mathbb{CP}_+^\infty$ associated to this spectrum is defined to be the homotopy colimit (or mapping telescope)

$$\Omega^\infty \mathbf{MTSO}(2) \wedge \mathbb{CP}_+^\infty := \mathrm{hocolim}_{n \to \infty} \Omega^n(M_n \wedge \mathbb{CP}_+^\infty).$$

There is an isomorphism $\pi_0(\Omega^\infty \mathbf{MTSO}(2) \wedge \mathbb{CP}_+^\infty) \cong \mathbb{Z}^2$ given by half the Euler characteristic and the degree of the line bundle, and we write $\Omega_{h,k}^\infty \mathbf{MTSO}(2) \wedge \mathbb{CP}_+^\infty$ to denote the path component corresponding to $(h, k) \in \mathbb{Z}^2$. Via Pontryjagin–Thom theory, the space $\Omega^\infty \mathbf{MTSO}(2) \wedge \mathbb{CP}_+^\infty$ classifies cobordism classes of "formal" surface bundles equipped with a complex line bundle on the total space. Formal surface bundles still have pushforwards and a notion of Euler class of the vertical

tangent bundle (although they do not have a vertical tangent bundle itself) and the classes $\kappa_{i,j}$ can still be defined. More precisely, under the composition

$$\widetilde{H}^{*+2}(BSO(2) \times \mathbb{CP}^\infty) \cong H^*(\mathbf{MTSO}(2) \wedge \mathbb{CP}^\infty_+) \xrightarrow{\sigma_*} H^*(\Omega_0^\infty \mathbf{MTSO}(2) \wedge \mathbb{CP}^\infty_+)$$

of the Thom isomorphism in spectrum cohomology with the cohomology suspension[1], the class $e^{i+1} \otimes c_1^j$ maps to a class we define to be

$$\kappa_{i,j} \in H^{2i+2j}(\Omega_0^\infty \mathbf{MTSO}(2) \wedge \mathbb{CP}^\infty_+; \mathbb{Z}).$$

Cohen and Madsen [7, 8] have defined a comparison map

$$\mathcal{S}_{g,r}^k(\mathbb{CP}^\infty) \longrightarrow \Omega_{1-g,k}^\infty \mathbf{MTSO}(2) \wedge \mathbb{CP}^\infty_+$$

and shown that it is an integral homology isomorphism in a range of degrees (when r is non-zero, or otherwise with rational coefficients). Work of Boldsen [3] or the second author [24] shows that it is a homology isomorphism in degrees $* \leq \lfloor \frac{2g}{3} \rfloor - 1$ when r is non-zero, and work of the second author [24] also shows one may remove this restriction on boundaries. Furthermore, it is simple to see that this comparison map pulls back the classes $\kappa_{i,j}$ on the infinite loop space to the classes of the same name on $\mathcal{S}_{g,r}^k(\mathbb{CP}^\infty)$.

Let us say some brief words about the integral cohomology of Hol_g^k. Theorem 2.3 implies that $H^*(\mathsf{Hol}_g^k; \mathbb{Z}) \cong H^*(\mathcal{S}_g^k(\mathbb{CP}^\infty); \mathbb{Z})$ and the above homology stability results identifies this ring with $H^*(\Omega_{1-g,k}^\infty \mathbf{MTSO}(2) \wedge \mathbb{CP}^\infty_+; \mathbb{Z})$ in degrees $* \leq \lfloor \frac{2g}{3} \rfloor - 1$. This gives a tremendous amount of torsion homology in Hol_g^k, but it is very difficult to extract; for this reason we now study the rational cohomology.

3.1. **Rational cohomology.** The rational cohomology of an infinite loop space has a very restricted structure, and is easily deduced from the rational cohomology of the associated spectrum.

Lemma 3.1. *There is a isomorphism of $\mathbb{Q}[\kappa_1, \kappa_2, \ldots]$-algebras*

$$H^*(\Omega_{1-g,k}^\infty \mathbf{MTSO}(2) \wedge \mathbb{CP}^\infty_+; \mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \ldots] \otimes \mathbb{Q}[\kappa_{i,j} \mid i+j > 0, j > 0, i \geq -1]$$

where $\kappa_{i,j}$ is in degree $2(i+j)$.

Proof. It is well known that the rational cohomology of a 0-connected infinite loop space $\Omega^\infty \mathbf{X}$ is given by the free graded commutative algebra on the vector space $\tau_{*>0} H^*(\mathbf{X}; \mathbb{Q})$ of positive degree elements in spectrum cohomology.

The spectrum $\mathbf{MTSO}(2) \wedge \mathbb{CP}^\infty_+$ splits as a wedge $\mathbf{MTSO}(2) \vee \mathbf{MTSO}(2) \wedge \mathbb{CP}^\infty$. The first factor contributes the classes $\mathbb{Q}[\kappa_1, \kappa_2, \ldots]$ on its infinite loop space. The rational spectrum cohomology of $\mathbf{MTSO}(2) \wedge \mathbb{CP}^\infty$ in positive degrees is the vector space

$$\mathbb{Q}\langle u_{-2} \cdot e^{i+1} \wedge c_1^j \mid i+j > 0, j > 0, i \geq -1 \rangle$$

where $u_{-2} \in H^{-2}(\mathbf{MTSO}(2); \mathbb{Q})$ denotes the Thom class. The element $u_{-2} \cdot e^{i+1} \wedge c_1^j$ under the cohomology suspension gives the element $\kappa_{i,j}$ on the infinite loop space. $\square$

Combining this calculation with a homological stability theorem we obtain the following corollary, which gives a stable description of the rational cohomology of $\mathcal{S}_{g,r}^k(\mathbb{CP}^\infty)$.

Corollary 3.2. *The map of $\mathbb{Q}[\kappa_1, \kappa_2, \ldots]$-algebras*

$$\mathbb{Q}[\kappa_1, \kappa_2, \ldots] \otimes \mathbb{Q}[\kappa_{i,j} \mid i+j > 0, j > 0, i \geq -1] \longrightarrow H^*(\mathcal{S}_{g,r}^k(\mathbb{CP}^\infty); \mathbb{Q})$$

is an isomorphism in degrees $* \leq \lfloor \frac{2g}{3} \rfloor - 1$.

---

[1]For any spectrum $\mathbf{X} = \{X_n\}$, the evaluation maps $\Sigma^n \Omega^n X_n \to X_n$ induce maps on cohomology $H^{*+n}(X_n) \to H^*(\Omega^n X_n)$, which after taking limits over n gives $\sigma^* : H^*(\mathbf{X}) \to H^*(\Omega^\infty \mathbf{X})$.

3.2. **Integral cohomology in degrees ≤ 3.** Throughout this section let us suppose that $g \geq 6$ so that the cohomology of $\mathcal{S}_{g,r}^k(\mathbb{CP}^\infty)$ is stable in degrees $* \leq 3$.

Lemma 3.3. *The cohomology group $H^1(\mathcal{S}_{g,r}^k(\mathbb{CP}^\infty); \mathbb{Z})$ is zero, $H^2(\mathcal{S}_{g,r}^k(\mathbb{CP}^\infty); \mathbb{Z})$ is free abelian of rank three, and $H^3(\mathcal{S}_{g,r}^k(\mathbb{CP}^\infty); \mathbb{Z})$ is zero.*

Proof. By the calculations in rational cohomology in the last section we know the statement is true modulo torsion. Thus we must show there is no torsion in cohomology in degrees $* \leq 3$, or equivalently that there is no torsion in homology in degrees $* \leq 2$. As we are in the stable range we may show this in $H_*(\Omega^\infty \mathbf{MTSO}(2) \wedge \mathbb{CP}_+^\infty; \mathbb{Z})$.

In low degrees, the homotopy groups of $\mathbf{MTSO}(2) \wedge \mathbb{CP}_+^\infty$ may be calculated using the Atiyah–Hirzebruch spectral sequence and the known homotopy groups of $\mathbf{MTSO}(2)$ up to degree 3 (which have for example been computed by the first named author in [10]). The result is displayed below.

i	-2	-1	0	1	2	3
$\pi_i(\mathbf{MTSO}(2))$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}/24$
$\pi_i(\mathbf{MTSO}(2) \wedge \mathbb{CP}_+^\infty)$	$\mathbb{Z}$	0	$\mathbb{Z}^2$	0	$\mathbb{Z}^3$	$-$

Hence we see that $\Omega_0^\infty \mathbf{MTSO}(2) \wedge \mathbb{CP}_+^\infty$ is simply-connected, so in particular H_1 is zero and hence torsion free. Furthermore

$$\pi_2(\mathbf{MTSO}(2) \wedge \mathbb{CP}_+^\infty) \cong H_2(\Omega_0^\infty \mathbf{MTSO}(2) \wedge \mathbb{CP}_+^\infty; \mathbb{Z})$$

by Hurewicz' theorem, and hence we see that the second homology is free abelian of rank three. The third homology must be torsion (by our rational calculations), and hence the third cohomology is zero. $\square$

Next we determine a $\mathbb{Z}$-basis of $H^2(\mathcal{S}_{g,r}^k(\mathbb{CP}^\infty); \mathbb{Z})$, which begins with naming elements. Recall that $H^2(M_g; \mathbb{Z}) \cong \mathbb{Z}$ has a generator λ that satisfies $12\lambda = \kappa_1$. To define the next element, we need a divisibility result, based on the Grothendieck–Riemann–Roch theorem. Let $L \to E \xrightarrow{\pi} B$ be in $\mathsf{Hol}_g^k(B)$ and abbreviate $x = c_1(T^v E)$, $y = c_1(L)$. Consider the Dolbeault operator $\overline{\partial}_{T^{\otimes r} L^{\otimes s}}$ on the tensor product bundle of r copies of the vertical tangent bundle and s copies of L. The Chern character of its index bundle is, by Grothendieck–Riemann–Roch:

$$\mathrm{ch}(\mathrm{Ind}(\overline{\partial}_{T^{\otimes r} L^{\otimes s}})) = \pi_!(\mathrm{Td}(x) e^{rx} e^{sy})$$

and the degree 2 part is

(3.1) $$(6r^2 + 6r + 1)\lambda + \frac{1}{2} s^2(\kappa_{0,1} - \kappa_{-1,2}) + (rs + \frac{1}{2}(s - s^2))\kappa_{0,1}.$$

The first and third summand are integral, hence so is the middle summand. In other words:

Proposition 3.4. *The class $\kappa_{0,1} - \kappa_{-1,2} \in H^2(\mathcal{S}_{g,r}^k(\mathbb{CP}^\infty); \mathbb{Z})$ is divisible by 2.*

Thus we may define

$$\zeta := \tfrac{1}{2}(\kappa_{0,1} - \kappa_{-1,2}) \in H^2(\mathcal{S}_{g,r}^k(\mathbb{CP}^\infty); \mathbb{Z}).$$

Theorem 3.5. *A free basis for $H^2(\mathcal{S}_{g,r}^k(\mathbb{CP}^\infty); \mathbb{Z})$ is $\mathcal{B} = (\lambda, \kappa_{0,1}, \zeta)$.*

As we have stressed before, $H^2(\mathcal{S}_{g,r}^k(\mathbb{CP}^\infty); \mathbb{Z}) \cong H^2(\Omega_{0,0}^\infty \mathbf{MTSO}(2) \wedge \mathbb{CP}_+^\infty; \mathbb{Z})$ as long as $g \geq 6$, which is the range we are studying. It is enough to give a map $H^2(\Omega_{0,0}^\infty \mathbf{MTSO}(2) \wedge \mathbb{CP}_+^\infty; \mathbb{Z}) \to \mathbb{Z}^3$ that maps the tuple $(\lambda, \kappa_{0,1}, \zeta)$ to a basis. To achieve this, we construct three examples of surface bundles equipped with complex line bundles. By the above remark, the genus and the degree of the line bundle are irrelevant for this purpose. Here are the examples:

Example 3.6. For g in the stable range, we consider the universal surface bundle on $B\Gamma_g$, together with the trivial line bundle on it. We know that $H^2(B\Gamma_g; \mathbb{Z}) \cong \mathbb{Z}\langle\lambda\rangle$, and it is clear that $\mathcal{B}$ evaluates to $(1, 0, 0)$.

Example 3.7. Consider the trivial surface bundle $\pi : \mathbb{CP}^\infty \times \Sigma_g \to \mathbb{CP}^\infty$, with complex line bundle given by π^*L. On this surface bundle, we have $\kappa_1 = 0$, hence $\lambda = 0$. Moreover, $\kappa_{0,1}$ is $\chi(\Sigma_g)$ times a generator. Finally, $\kappa_{-1,2}$ is zero. If we put $g = 2$, we obtain that $\mathcal{B}$ is mapped to $(0, 2, 1)$.

Example 3.8. Let H_1 be the Hirzebruch surface (we use the notation of [9]. It is an $\mathbb{S}^2$-bundle over $\mathbb{S}^2$, which is not spin and has $\kappa_1 = 0$ (since the signature of the total space is 0). A basis for $H_2(H_1; \mathbb{Z})$ is given by the fundamental class u of the fibre and the image v of the section $\mathbb{S}^2 \to H_1$ at ∞. Let (x, y) be the Poincaré dual basis to (u, v). Using the intersection matrix given in [9], it is not hard to see that the Euler class of the vertical tangent bundle is $e = 2x + y$. Let $L \to H_1$ be the line bundle with Chern class y. Again using the intersection matrix, we compute $\kappa_{0,1} = -1$ and $\kappa_{-1,2} = -1$. Thus $\mathcal{B}$ is mapped to $(0, -1, 0)$.

Lemma 3.9. *The map $\mathbb{CP}^\infty \to \mathcal{S}_g^k(\mathbb{CP}^\infty)$ on second cohomology sends λ to zero, ζ to $(1 - g - k)$ times the generator and $\kappa_{0,1}$ to $(2 - 2g)$ times the generator.*

Proof. The map $\mathbb{CP}^\infty \to \mathrm{map}^k(\Sigma_g, \mathbb{CP}^\infty)/\!\!/\mathrm{Diff}^+(\Sigma_g) = \mathcal{S}_g^k(\mathbb{CP}^\infty)$ is the action of the constant maps. Thus is classifies the trivial family $\Sigma_g \to \Sigma_g \times \mathbb{CP}^\infty \xrightarrow{\pi} \mathbb{CP}^\infty$ with the line bundle $c_1(L) := k[\Sigma_g]^* \otimes 1 + 1 \otimes x$ on the total space. the Euler class of the vertical tangent bundle is $e(\Sigma_g) = \chi(\Sigma_g)[\Sigma_g]^* \otimes 1$. For this bundle we may calculate

$$
\begin{aligned}
\kappa_1 &= \pi_!(e(\Sigma_g)^2) = 0, \\
\kappa_{0,1} &= \pi_!(e(\Sigma_g) \cdot c_1(L)) = \chi(\Sigma_g) \cdot x, \\
\kappa_{-1,2} &= \pi_!(c_1(L)^2) = 2kx.
\end{aligned}
$$

$\square$

We note another consequence of the equations (3.1):

Corollary 3.10. *Chern classes of holomorphic line bundles generate $H^2(\mathsf{Hol}_g^k; \mathbb{Z})$*

Proof. To realize λ as a first Chern class, put $r = s = 0$ in 3.1. To realize $\lambda + \zeta$, put $(r, s) = (0, 1)$. Finally, $s = r = 1$ yields $13\lambda + \zeta + \kappa_{0,1}$, and we are done. $\square$

4. Stable cohomology of Pic_g^k

We will now employ the diagram of Section 2.4 to compute the rational cohomology of Pic_g^k in a certain stable range. We will also compute the low dimensional integral cohomology, in particular we will show $H^2(\mathsf{Pic}_g^k; \mathbb{Z})$ is free abelian and produce a free basis for it. There is a principal fibration sequence

$$(4.1) \qquad \mathbb{CP}^\infty \longrightarrow \mathcal{S}_g^k(\mathbb{CP}^\infty) \longrightarrow \mathsf{Pic}_g^k$$

classified by an element $\mathcal{G}_g^k \in H^3(\mathsf{Pic}_g^k; \mathbb{Z})$. This is the Dixmier–Douady class of the gerbe of §2.3.

Proposition 4.1. *The group $H^3(\mathsf{Pic}_g^k; \mathbb{Z})$ is cyclic of order $\gcd(2 - 2g, 1 - g - k)$ and the element $\mathcal{G}_g^k$ is a generator. The group $H^1(\mathsf{Pic}_g^k; \mathbb{Z})$ is zero and $H^2(\mathsf{Pic}_g^k; \mathbb{Z})$ is free abelian of rank two. A free basis for $H^2(\mathsf{Pic}_g^k; \mathbb{Z})$ is given by the Hodge class λ and a class η which in $H^2(\mathcal{S}_g^k(\mathbb{CP}^\infty); \mathbb{Z})$ restricts to*

$$\frac{1}{\gcd(\chi(\Sigma_g), g + k - 1)}(k\kappa_{0,1} + (g - 1)\kappa_{-1,2}).$$

Proof. We study the Leray–Serre spectral sequence in cohomology for the principal fibration (4.1), which has E_2 page as shown in Figure 1 and which converges to
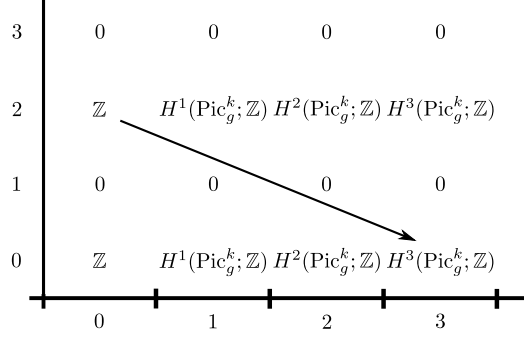

FIGURE 1. E_2 page of the Serre spectral sequence for (4.1).

zero in total degree 3 by Lemma 3.3. From this we deduce that $d^3 : E_2^{0,2} = \mathbb{Z} \to H^3(\mathrm{Pic}_g^k; \mathbb{Z})$ must be onto, so this last group must be cyclic, and a generator of it classifies the fibration. The statements about H^1 and H^2 follow from Lemma 3.3. It remains to determine the order of $H^3(\mathrm{Pic}_g^k; \mathbb{Z})$.

The edge homomorphism

$$H^2(\mathcal{S}_g^k(\mathbb{CP}^\infty)) \longrightarrow H^2(\mathbb{CP}^\infty; \mathbb{Z}) \cong \mathbb{Z}$$

has been calculated in Lemma 3.9 to be

$$\begin{aligned} \lambda &\mapsto 0 \\ \zeta &\mapsto 1 - g - k \\ \kappa_{0,1} &\mapsto 2 - 2g. \end{aligned}$$

Furthermore we have shown λ, $\kappa_{0,1}$ and ζ to be a basis, which implies that $H^3(\mathrm{Pic}_g^k; \mathbb{Z})$ is cyclic of order $\gcd(2 - 2g, 1 - g - k)$. The order of the gerbe can also be computed from Lemma 2.8 and Corollary 2.9. A basis for the kernel of the edge homomorphism is given by the Hodge class λ and the class η in the statement. $\square$

As the fibration (4.1) is classified by a torsion element, its Leray–Serre spectral sequence collapses rationally, and we find that the rational cohomology of Pic_g^k injects into the rational cohomology of $\mathcal{S}_g^k(\mathbb{CP}^\infty)$ and (in the stable range) has image the subalgebra

$$\mathbb{Q}[\kappa_1, \kappa_2, \ldots] \otimes \mathbb{Q}[\kappa_{i,j}, \; k\kappa_{0,1} + (g-1)\kappa_{-1,2} \mid i + j > 1, j > 0, i \geq -1].$$

This establishes Theorem A. Combining the calculations of Theorem 3.5 and Proposition 4.1 establishes Theorem B.

5. PICARD GROUPS AND ANALYTIC NÉRON–SEVERI GROUPS

We first need to discuss some generalities regarding the (holomorphic) Picard group of a holomorphic stack X. Recall that the exponential sequence of sheaves $\mathbb{Z} \to \mathcal{O}_\mathsf{X} \to \mathcal{O}_\mathsf{X}^\times$ on X provides a long exact sequence

$$\longrightarrow H^1(\mathsf{X}; \mathbb{Z}) \longrightarrow H^1(\mathsf{X}; \mathcal{O}_\mathsf{X}) \longrightarrow \mathrm{Pic}_{hol}(\mathsf{X}) \xrightarrow{c_1} H^2(\mathsf{X}; \mathbb{Z}) \longrightarrow H^2(\mathsf{X}; \mathcal{O}_\mathsf{X}) \longrightarrow .$$

On the other hand, a holomorphic stack is in particular a topological stack, so the exponential sequence $\mathbb{Z} \to \mathcal{C}_\mathsf{X} \to \mathcal{C}_\mathsf{X}^\times$ of sheaves of continuous functions provides a long exact sequence

$$\longrightarrow H^1(\mathsf{X}; \mathbb{Z}) \longrightarrow H^1(\mathsf{X}; \mathcal{C}_\mathsf{X}) \longrightarrow \mathrm{Pic}_{top}(\mathsf{X}) \xrightarrow{c_1} H^2(\mathsf{X}; \mathbb{Z}) \longrightarrow H^2(\mathsf{X}; \mathcal{C}_\mathsf{X}) \longrightarrow$$

and the map of sheaves $\mathcal{O}_X^\times \to \mathcal{C}_X^\times$ from the nowhere zero holomorphic functions to the nowhere zero continuous complex-valued functions provides a homomorphism

$$\mathrm{Pic}_{hol}(X) \longrightarrow \mathrm{Pic}_{top}(X).$$

Recall from the introduction that the Néron–Severi group $\mathcal{NS}(X)$ is by definition the image of this homomorphism, i.e. the quotient of $\mathrm{Pic}_{hol}(X)$ by the subgroup of topologically trivial line bundles. We first observe what can be said about the first Chern class map in the topological situation.

Lemma 5.1. *Let* X *be a differentiable local quotient stack. Then* $\mathcal{C}_X$ *is acyclic and so the homomorphism* $c_1 : \mathrm{Pic}_{top}(X) \to H^2(X; \mathbb{Z})$ *is an isomorphism.*

Proof. This follows from a twofold application of the descent spectral sequence. If X and X are stacks, $X \to X$ a representable surjective map and $\mathcal{F}$ a sheaf on X, then there is a spectral sequence

$$E_1^{p,q} = H^q(X_p, \mathcal{F}_p) \Longrightarrow H^{p+q}(X; \mathcal{F}),$$

where $X_p := X \times_X X \times_X \ldots \times_X X$ (p factors) and $\mathcal{F}_p$ is the pullback of $\mathcal{F}$ to X_p. If $X = X /\!\!/ G$ is a global quotient of a manifold by a compact Lie group, then the descent spectral sequence is

$$H^p(G; H^q(X, \mathcal{C}_X)) \Longrightarrow H^{p+q}(X; \mathcal{C}_X);$$

this is zero if $q > 0$ since the sheaf $\mathcal{C}_X$ is fine. Thus the spectral sequence collapses to $H^*(G; \mathcal{C}(X, \mathbb{C}))$, which vanishes in positive degrees as the group G is compact and the coefficient module is a locally convex topological vector space with a continuous G-action. For details of that argument, consult [2, Proposition 6.3].

If X is merely a local quotient stack, there is an open cover by substacks X_i each of which is a global quotient stack. Since fibre products of global quotients are global quotient stacks, the descent spectral sequence for the map $\coprod_i X_i \to X$ has $E_1^{p,q} = 0$ for $q > 0$. The $E^{p,0}$-line is just the Čech complex for the open cover. Because there exist partitions of unity in this situation, see [11, Appendix A], this complex is acyclic. $\square$

This lemma applies to Hol_g^k and Pic_g^k, because both are local quotient stacks. Together with Corollary 3.10, we establish the following result, which is the first part of Theorem C.

Proposition 5.2. *We have a sequence of maps*

$$\mathrm{Pic}_{hol}(\mathsf{Hol}_g^k) \longrightarrow \mathcal{NS}(\mathsf{Hol}_g^k) \longrightarrow \mathrm{Pic}_{top}(\mathsf{Hol}_g^k) \xrightarrow{c_1} H^2(\mathsf{Hol}_g^k; \mathbb{Z})$$

where the last map is an isomorphism, the first is surjective, and the middle map is injective. For $g \geq 6$ the middle map is also an isomorphism.

We have shown in Theorem 3.5 that for $g \geq 6$ the group $H^2(\mathsf{Hol}_g^k; \mathbb{Z})$ is free abelian of rank three, with free basis λ, $\kappa_{0,1}$ and ζ, and so have also determined $\mathcal{NS}(\mathsf{Hol}_g^k)$ and $\mathrm{Pic}_{top}(\mathsf{Hol}_g^k)$ in this range. In order to prove the equivalent result for Pic_g^k we require the following lemma.

Lemma 5.3. *Let* X *and* Y *be holomorphic local quotient stacks and* $X \to Y$ *a* $\mathbb{C}^\times$-*gerbe. If the comparison map* $\mathrm{Pic}_{hol}(X) \to \mathrm{Pic}_{top}(X)$ *is surjective, then so is* $\mathrm{Pic}_{hol}(Y) \to \mathrm{Pic}_{top}(Y)$.

Proof. Any line bundle (holomorphic or topological) $L \to X$ has an action of $\mathbb{C}^\times$ coming from the gerbe structure. As usual, $z \in \mathbb{C}^\times$ acts by multiplication with z^w for a uniquely determined $w \in \mathbb{Z}$, called the *weight* of L. It is not difficult to see that $w : \mathrm{Pic}_{top}(X) \to \mathbb{Z}$ is a homomorphism and coincides with the edge homomorphism

$H^2(X; \mathbb{Z}) \to H^2(\mathbb{CP}^\infty; \mathbb{Z})$ derived from the Leray–Serre spectral sequence. Now consider the diagram:

$$
\begin{array}{ccccc}
\mathrm{Pic}_{hol}(\mathsf{Y}) & \longrightarrow & \mathrm{Pic}_{hol}(\mathsf{X}) & \overset{w}{\longrightarrow} & \mathbb{Z} \\
\downarrow & & \downarrow & & \| \\
0 \longrightarrow \mathrm{Pic}_{top}(\mathsf{Y}) & \longrightarrow & \mathrm{Pic}_{top}(\mathsf{X}) & \overset{w}{\longrightarrow} & \mathbb{Z}.
\end{array}
$$

The bottom sequence is everywhere exact and the top sequence is exact at $\mathrm{Pic}_{hol}(\mathsf{X})$: a line bundle descends if and only if its weight is zero. A fragmentary version of the 5-lemma holds in this situation and shows that the left vertical map is onto. $\square$

Applying this and Lemma 5.1, we establish the last part of Theorem C.

Proposition 5.4. *We have a sequence of maps*

$$
\mathrm{Pic}_{hol}(\mathsf{Pic}_g^k) \longrightarrow \mathcal{NS}(\mathsf{Pic}_g^k) \longrightarrow \mathrm{Pic}_{top}(\mathsf{Pic}_g^k) \overset{c_1}{\longrightarrow} H^2(\mathsf{Pic}_g^k; \mathbb{Z})
$$

where the last map is an isomorphism, the first is surjective, and the middle map is injective. For $g \geq 6$ the middle map is also an isomorphism.

We have shown in Proposition 4.1 that $H^2(\mathsf{Pic}_g^k; \mathbb{Z})$ is free abelian of rank two, with basis the Hodge class λ and a class we denoted η, and so have also determined $\mathcal{NS}(\mathsf{Pic}_g^k)$ and $\mathrm{Pic}_{top}(\mathsf{Pic}_g^k)$ in this range.

6. Cohomology of the extended mapping class groups

Recall Kawazumi's definition of the extended mapping class group,

$$
\widetilde{\Gamma}_{g,r}^s := H \rtimes \Gamma_{g,r}^s
$$

where $H = H_1(\Sigma_{g,r}; \mathbb{Z})$ is considered as a $\Gamma_{g,r}^s$-module by the usual action of the mapping class group on homology (so H is also isomorphic to $H^1(\Sigma_{g,r}, \partial; \mathbb{Z})$ by Poincaré duality). There is a fibration

$$
(6.1) \qquad\qquad BH \longrightarrow B\widetilde{\Gamma}_{g,r} \longrightarrow B\Gamma_{g,r}
$$

which when $r = 0$ is equivalent to the fibration

$$
\mathrm{Pic}^0(\Sigma_g) \longrightarrow \mathrm{Pic}_g^0 \longrightarrow \mathcal{M}_g.
$$

We require a geometric model for this fibration when $r > 0$ also.

Lemma 6.1. *Let $r > 0$. There is a $\mathrm{Diff}^+(\Sigma_{g,r})$-equivariant homotopy equivalence*

$$
\mathrm{map}_\partial^0(\Sigma_{g,r}, \mathbb{CP}^\infty) \simeq BH_1(\Sigma_{g,r}; \mathbb{Z}).
$$

Proof. Non-equivariantly, the space $\mathrm{map}_\partial^0(\Sigma_{g,r}, \mathbb{CP}^\infty)$ may be seen to be a $K(\pi, 1)$ with fundamental group $H_1(\Sigma_{g,r}; \mathbb{Z}) = H$, and the group $\mathrm{Diff}^+(\Sigma_{g,r})$ acts preserving the basepoint (which we take to be the constant map to the basepoint in $\mathbb{CP}^\infty$). Thus it suffices to show that the group $\mathrm{Diff}^+(\Sigma_{g,r})$ acts in the usual way on the fundamental group, but this is clear. $\square$

Thus for $r > 0$ there is a homotopy equivalence

$$
(6.2) \qquad B\widetilde{\Gamma}_{g,r} \simeq \mathrm{map}_\partial^0(\Sigma_{g,r}, \mathbb{CP}^\infty) /\!\!/ \mathrm{Diff}_\partial^+(\Sigma_{g,r}) = \mathcal{S}_{g,r}^0(\mathbb{CP}^\infty),
$$

and for $r = 0$ a fibration sequence

$$
(6.3) \qquad\qquad \mathbb{CP}^\infty \longrightarrow \mathcal{S}_g^0(\mathbb{CP}^\infty) \longrightarrow B\widetilde{\Gamma}_g.
$$

Corollary 3.2 implies that for $r > 0$ there is a map of $\mathbb{Q}[\kappa_1, \kappa_2, \ldots]$-algebras

$$
\mathbb{Q}[\kappa_1, \kappa_2, \ldots] \otimes \mathbb{Q}[\kappa_{i,j} \mid i + j > 0, j > 0, i \geq -1] \longrightarrow H^*(B\widetilde{\Gamma}_{g,r}; \mathbb{Q})
$$

which is an isomorphism in degrees $* \leq \lfloor \frac{2g}{3} \rfloor - 1$. This establishes Theorem E.

As the fibration (6.3) is classified by a torsion class (by Proposition 4.1), its rational Serre spectral sequence collapses. Thus there is an isomorphism of $\mathbb{Q}[\kappa_1, \kappa_2, \ldots]$-algebras

$$H^*(B\widetilde{\Gamma}_g; \mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \ldots] \otimes \mathbb{Q}[\widetilde{\kappa}_{i,j} \mid i + j > 0, j > 0, i \geq -1, (i,j) \neq (0,1)]$$

in degrees $* \leq \lfloor \frac{2g}{3} \rfloor - 1$, where $\widetilde{\kappa}_{i,j}$ is a class that maps to $\kappa_{i,j}$ in the cohomology of $\mathcal{S}_g^0(\mathbb{CP}^\infty)$. This establishes Theorem F. As $\mathcal{S}_{g,r}^0(\mathbb{CP}^\infty) \to \mathcal{S}_g^0(\mathbb{CP}^\infty)$ is a homology equivalence in degrees $* \leq \lfloor \frac{2g}{3} \rfloor - 1$, the fibration (6.3) may be replaced by a homology fibration in degrees $* \leq \lfloor \frac{2g}{3} \rfloor - 1$

$$\mathbb{CP}^\infty \longrightarrow B\widetilde{\Gamma}_{g,r} \longrightarrow B\widetilde{\Gamma}_g \dashrightarrow K(\mathbb{Z}, 3)$$

classified by $\mathcal{G}_g^0$, which establishes Theorem D.

6.1. **Relation to the work of Kawazumi.** Kawazumi [15] has defined certain cohomology classes

$$\widetilde{m}_{i,j} \in H^{2i+2j-2}(\widetilde{\Gamma}_{g,1}; \mathbb{Z}),$$

in the framework of group cohomology, as pushforwards $\pi_!(e^i \cdot \widetilde{\omega}^j)$ where e is the group level analogue of the Euler class of the vertical tangent bundle, and $\widetilde{\omega} \in H^2(\widetilde{\Gamma}_{g,1}^1, \widetilde{\Gamma}_{g,1} \times \mathbb{Z}; \mathbb{Z})$ is represented by a cocycle he defines manually. He does not phrase it in this way, but it is the Euler class of the relative central extension

$$
\begin{array}{ccccc}
\mathbb{Z} & \longrightarrow & H_1(\Sigma_{g,1}^1) \rtimes \Gamma_{g,1}^1 & \longrightarrow & H_1(\Sigma_{g,1}) \rtimes \Gamma_{g,1}^1 =: \widetilde{\Gamma}_{g,1}^1 \\
\big\uparrow & & \big\uparrow & & \big\uparrow \\
\mathbb{Z} & \longrightarrow & (H_1(\Sigma_{g,1}^1) \rtimes \Gamma_{g,1}) \times \mathbb{Z} & \longrightarrow & \widetilde{\Gamma}_{g,1} \times \mathbb{Z}.
\end{array}
$$

The homomorphism $\rho : \Gamma_{g,1} \times \mathbb{Z} \to \Gamma_{g,1}^1$ is obtained by gluing in an annulus with a Dehn twist, see [15, p. 140]. Here $H_1(\Sigma_{g,1}^1)$ means the homology of the surface with the marked points removed. The top sequence arises from the short exact sequence $0 \to \mathbb{Z} \to H_1(\Sigma_{g,1}^1) \to H_1(\Sigma_{g,1}) \to 0$ of $\Gamma_{g,1}^1$-modules. The bottom sequence is induced from the top one via ρ and it is split via the $\Gamma_{g,1}$-equivariant inclusion $H_1(\Sigma_{g,1}) \to H_1(\Sigma_{g,1}^1)$ obtained by boundary connect sum with a punctured disc. We wish to reproduce his construction topologically. The following lemma is elementary and we do not include a proof.

Lemma 6.2. *The evaluation map*

$$ev : \Sigma_{g,1} \times \mathrm{map}_\partial^0(\Sigma_{g,1}, \mathbb{CP}^\infty) \longrightarrow \mathbb{CP}^\infty,$$

is $\mathrm{Diff}_\partial^+(\Sigma_{g,1})$*-invariant and vanishes on* $\partial\Sigma_{g,1}$*, so induces a cohomology class* $c_1(L) \in H^2(\mathcal{S}_{g,1}^0(\mathbb{CP}^\infty), \partial; \mathbb{Z})$. *Considered as an element of*

$$H^2(\Sigma_{g,1} \times \mathrm{map}_\partial^0(\Sigma_{g,1}, \mathbb{CP}^\infty), \partial; \mathbb{Z}) \cong H^2(\Sigma_{g,1}, \partial; \mathbb{Z}) \oplus H^{\otimes 2}$$

ev is given by the characteristic element $\mu \in H^{\otimes 2}$ *determined by the intersection pairing.*

By Lemma 6.1 there is a $\mathrm{Diff}_\partial^+(\Sigma_{g,1})$-equivariant homotopy equivalence $BH \simeq \mathrm{map}_\partial^0(\Sigma_{g,1}, \mathbb{CP}^\infty)$ and so this lemma produces a class

$$c_1(L) \in H^2((\Sigma_{g,1} \times BH) /\!\!/ \mathrm{Diff}_\partial^+(\Sigma_{g,1}), (\partial\Sigma_{g,1} \times BH) /\!\!/ \mathrm{Diff}_\partial^+(\Sigma_{g,1}); \mathbb{Z}),$$

and we may identify this group with $H^2(\widetilde{\Gamma}_{g,1}^1, \widetilde{\Gamma}_{g,1} \times \mathbb{Z}; \mathbb{Z})$, where Kawazumi's class $\widetilde{\omega}$ lies. Restricting these classes to $(\Sigma_{g,1} \times BH, \partial\Sigma_{g,1} \times BH)$, Kawazumi's cocycle on the group level becomes $\widetilde{\omega}((\gamma, u), (\gamma', u')) = [\gamma'] \cap u$ whereas $c_1(L)$ becomes the intersection pairing $\mu \in H^{\otimes 2}$. These are the same, and so by the map of spectral sequences from

$$H^p(\Gamma_{g,1}; H^q(\Sigma_{g,1}, \partial\Sigma_{g,1})) \Longrightarrow H^{p+q}(\Gamma_{g,1}^1, \Gamma_{g,1} \times \mathbb{Z}; \mathbb{Z})$$

to
$$H^p(\Gamma_{g,1}; H^q(\Sigma_{g,1} \times BH, \partial\Sigma_{g,1} \times BH)) \implies H^{p+q}(\widetilde{\Gamma}^1_{g,1}, \widetilde{\Gamma}_{g,1} \times \mathbb{Z}; \mathbb{Z})$$

we deduce that these classes agree modulo classes coming from $H^2(\Gamma^1_{g,1}, \Gamma_{g,1} \times \mathbb{Z}; \mathbb{Z}) \longrightarrow H^2(\widetilde{\Gamma}^1_{g,1}, \widetilde{\Gamma}_{g,1} \times \mathbb{Z}; \mathbb{Z})$, i.e. modulo the Euler class. However, restricted to $(\Sigma_{g,1}, \partial\Sigma_{g,1})$ both $c_1(L)$ and $\tilde{\omega}$ vanish, and so their difference cannot be a non-zero multiple of the Euler class. Thus $c_1(L) = \tilde{\omega}$, and we have proved:

Proposition 6.3. *In $H^*(B\widetilde{\Gamma}_{g,1}; \mathbb{Z})$ we have an equality $\widetilde{m}_{i+1,j} = \kappa_{i,j}$.*

6.2. **Filtrations.** Kawazumi shows that the class $\widetilde{m}_{i+1,j}$ on the total space of the fibration
$$BH \longrightarrow B\widetilde{\Gamma}_{g,1} \xrightarrow{\pi} B\Gamma_{g,1}$$

is detected in the Leray–Serre spectral sequence for this fibration by a certain class
$$m_{i+1,j} \in H^{2i+j}(\Gamma_{g,1}; \wedge^j H)$$

which he constructs. Equivalently, we may say that $\widetilde{m}_{i+1,j}$ has natural filtration precisely j. In [17, Theorem 3.2] he then shows that rationally the associated graded algebra to the natural filtration is
$$\mathcal{G}r(H^*(\widetilde{\Gamma}_{g,1})) = H^*(\Gamma_{g,1}; \mathbb{Q}) \otimes \mathbb{Q}[m_{i+1,j} \,|\, j > 0].$$

We use Kawazumi's theorem to establish the filtration part of Theorem E.

Corollary 6.4. *For each $r > 0$ and $2(i + j) \leq \lfloor \frac{2g}{3} \rfloor$, the class $\kappa_{i,j}$ is detected by a class $M_{i,j} \in H^{2i+j}(\Gamma_{g,r}; \wedge^j H^*)$ in the Leray–Serre spectral sequence for the fibration*
$$BH \longrightarrow B\widetilde{\Gamma}_{g,r} \xrightarrow{\pi} B\Gamma_{g,r}.$$

Equivalently, $\kappa_{i,j}$ has natural filtration j. Rationally the associated graded algebra is
$$H^*(\Gamma_{g,r}; \mathbb{Q}) \otimes \mathbb{Q}[x_{i,j} \,|\, j > 0].$$

Proof. Let us first treat the case $r = 1$. Proposition 6.3 shows that $\widetilde{m}_{i+1,j} = \kappa_{i,j}$, so $\kappa_{i,j}$ is detected by the class $x_{i,j} := m_{i+1,j} \in H^{2i+j}(\Gamma_{g,1}; \wedge^j H^*)$ and the statement about the associated graded is implied by Kawazumi's theorem. For $r \geq 1$ we now use the homological stability theorem of Boldsen [3] for the mapping class group with twisted coefficients. The coefficient system $\wedge^j H^*$ has degree j and so the map
$$H^{2i+j}(\Gamma_{g,r}; \wedge^j H^*) \longrightarrow H^{2i+j}(\Gamma_{g,1}; \wedge^j H^*)$$

is an isomorphism for $2i + j \leq \lfloor \frac{2g}{3} \rfloor - j$. The rational associated graded algebra for the natural filtration is $H^*(\Gamma_{g,r}; \wedge^* H^*_{\mathbb{Q}})$, and as the classes $\kappa_{i,j}$ are natural for stabilisation maps the result follows. $\square$

It remains to prove the part of Theorem E concerning the associated graded algebra for the natural filtration on $H^*(\widetilde{\Gamma}_g; \mathbb{Z})$. This spectral sequence collapses, as the fibration $BH \to B\widetilde{\Gamma}_g \to B\Gamma_g$ is equivalent to $\mathrm{Pic}^0(\Sigma_g) \longrightarrow \mathsf{Pic}^0_g \longrightarrow \mathsf{M}_g$ which is a Kähler fibration, and so its rational Leray–Serre spectral sequence collapses at E_2 by Deligne's degeneration theorem [25, Theorem 4.15]. The same reasoning shows that the spectral sequence for $BH \to B\widetilde{\Gamma}^1_g \to B\Gamma^1_g$ collapses at E_2. By Theorem D, there is an injection $H^*(\widetilde{\Gamma}_g; \mathbb{Q}) \hookrightarrow H^*(\widetilde{\Gamma}_{g,1}; \mathbb{Q})$ in the stable range. The claim about the associated graded algebra to the natural filtration is a consequence of the following proposition.

Proposition 6.5. *The induced filtration on $H^*(\widetilde{\Gamma}_g; \mathbb{Q}) \hookrightarrow H^*(\widetilde{\Gamma}_{g,1}; \mathbb{Q})$ coincides with the natural filtration.*

Proof. There are natural forgetful maps $B\widetilde{\Gamma}_{g,1} \to B\widetilde{\Gamma}_g^1 \to B\widetilde{\Gamma}_g$ and fibrations

$$
\begin{array}{ccccc}
B\widetilde{\Gamma}_{g,1} \longrightarrow B\widetilde{\Gamma}_g^1 \xrightarrow{\ e\ } \mathbb{CP}^\infty & \quad & \Sigma_g \longrightarrow B\widetilde{\Gamma}_g^1 \longrightarrow B\widetilde{\Gamma}_g \\
\downarrow \qquad\quad \downarrow \qquad\quad \| & & \| \qquad\quad \downarrow \qquad\quad \downarrow \\
B\Gamma_{g,1} \longrightarrow B\Gamma_g^1 \xrightarrow{\ e\ } \mathbb{CP}^\infty & \quad & \Sigma_g \longrightarrow B\Gamma_g^1 \longrightarrow B\Gamma_g.
\end{array}
$$

The class e has natural filtration 0, so the natural filtration on $H^*(\widetilde{\Gamma}_g^1; \mathbb{Q})$ descends to one on $H^*(\widetilde{\Gamma}_g^1; \mathbb{Q})/(e)$. Furthermore, there is a Leray–Serre spectral sequence converging to this filtration,

$$
H^p(\Gamma_g^1; \wedge^q H_\mathbb{Q})/(e) \Longrightarrow H^*(\widetilde{\Gamma}_g^1; \mathbb{Q})/(e).
$$

There is another Leray–Serre spectral sequence

$$
H^p(\mathbb{CP}^\infty; H^q(\Gamma_{g,1}; \wedge^k H_\mathbb{Q})) \Longrightarrow H^{p+q}(\Gamma_g^1; \wedge^k H_\mathbb{Q})
$$

and after quotienting by the ideal (e) it collapses to give an isomorphism of the quotient $H^*(\Gamma_g^1; \wedge^k H_\mathbb{Q})/(e)$ with $H^*(\Gamma_{g,1}; \wedge^k H_\mathbb{Q})$. It then follows that there is an isomorphism of filtered algebras

$$
H^*(\widetilde{\Gamma}_g^1; \mathbb{Q})/(e) \cong H^*(\widetilde{\Gamma}_{g,1}; \mathbb{Q}).
$$

We have maps of filtered algebras $H^*(\widetilde{\Gamma}_g; \mathbb{Q}) \to H^*(\widetilde{\Gamma}_g^1; \mathbb{Q})/(e) \to H^*(\widetilde{\Gamma}_{g,1}; \mathbb{Q})$, the second map is an isomorphism of filtered algebras and the composition is injective in the stable range by Theorem D. Thus the first map is also injective in the stable range, and it is enough to show that the induced filtration on $H^*(\widetilde{\Gamma}_g; \mathbb{Q})$ agrees with the natural one. For this it is enough to show that the filtration induced by the injection $H^*(\widetilde{\Gamma}_g; \mathbb{Q}) \hookrightarrow H^*(\widetilde{\Gamma}_g^1; \mathbb{Q})$ coincides with the natural one.

In Lemma A.1 of the Appendix we show that there are Becker–Gottlieb transfer maps for cohomology with local coefficients, for maps whose homotopy fibre has the stable homotopy type of a finite complex. In particular, the maps

$$
H^p(\Gamma_g; \wedge^q H_\mathbb{Q}) \longrightarrow H^p(\Gamma_g^1; \wedge^q H_\mathbb{Q})
$$

are split injections in all bidegrees. As the spectral sequences associated to the natural filtrations of both $\widetilde{\Gamma}_g$ and $\widetilde{\Gamma}_g^1$ collapse at E_2, this gives a split injection of associated graded algebras, which implies that the induced filtration on $H^*(\widetilde{\Gamma}_g; \mathbb{Q})$ agrees with the natural one. $\qquad\square$

Appendix A. Transfer maps and coefficient systems

The purpose of this appendix is to make an observation about Becker–Gottlieb transfer maps [4] and their existence for (co)homology with local coefficients. This is surely known to experts, but we were unable to locate a reference. Consider a fibration $F \longrightarrow E \xrightarrow{\ \pi\ } B$ and let us suppose that F has the stable homotopy type of a finite complex, so that π admits a Becker–Gottlieb transfer map trf_π.

Lemma A.1. *Let A be a $\mathbb{Z}[\pi_1(B)]$-module, or equivalently a bundle of abelian groups over B. There is a transfer map*

$$
\mathrm{trf}_\pi : H_*(B; A) \longrightarrow H_*(E; \pi^* A)
$$

so that $\pi_* \circ \mathrm{trf}_\pi = \chi(F) \cdot \mathrm{Id}$, and similarly in cohomology.

Proof. We will identify these homology groups as the ordinary homology of related spaces. Recall that a *Moore space* $M(A, n)$ is a pointed space with reduced integral homology given by the group A in degree n and 0 otherwise. These spaces can be taken to be functorial in A. There is a fibration

$$
M(A, n) \longrightarrow X_n \xrightarrow{\ p\ } B
$$

obtained by applying $M(-, n)$ fibrewise to the bundle of abelian groups A over B, which has a section s given by the basepoint of each $M(A, n)$. We have the diagram

$$
\begin{array}{ccc}
M(A, n) & = & M(A, n) \\
\downarrow & & \downarrow \\
\end{array}
$$

$$
\begin{array}{ccccc}
F & \longrightarrow & \pi^* X_n & \xrightarrow{\pi_n} & X_n \\
\| & & {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\
F & \longrightarrow & E & \xrightarrow{\pi} & B.
\end{array}
$$

The Leray–Serre spectral sequence for (X_n, B) is then $H_p(B; H_q(M(A, n), *; \mathbb{Z}))) \Rightarrow H_{p+q}(X_n, B)$. This is concentrated along the line $q = n$, so collapses and hence

$$H_*(B; A) \cong H_{*+n}(X_n, B; \mathbb{Z}).$$

Similarly $H_*(E, \pi^* A) \cong H_{*+n}(\pi^* X_n, E; \mathbb{Z})$. The ordinary Becker–Gottlieb transfer gives a stable map of pairs

$$\mathrm{trf}_{(\pi_n, \pi)} : (X_n, B) \to (\pi^* X_n, E)$$

where we have used that $(E \to B) = s^*(\pi^* X_n \to X_n)$ so we have the formula $s' \circ \mathrm{trf}_\pi = \mathrm{trf}_{\pi'} \circ s$. Applying integral homology and using the isomorphisms above, this gives the required transfer map. The effect of composing with π_* follows from the usual formula for the transfer. □

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