A RECURSION IDENTITY FOR FORMAL ITERATED LOGARITHMS AND ITERATED EXPONENTIALS

THOMAS J. ROBINSON

Abstract. We prove a recursive identity involving formal iterated logarithms and formal iterated exponentials. These iterated logarithms and exponentials appear in a natural extension of the logarithmic formal calculus used in the study of logarithmic intertwining operators and logarithmic tensor category theory for modules for a vertex operator algebra. This extension has a variety of interesting arithmetic properties. We develop one such result here, the aforementioned recursive identity. We have applied this identity elsewhere to certain formal series expansions related to a general formal Taylor theorem and these series expansions in turn yield a sequence of combinatorial identities which have as special cases certain classical combinatorial identities involving (separately) the Stirling numbers of the first and second kinds.

1. Introduction

In [M] and [HLZ] logarithmic formal calculus was used to set up certain structure for the treatment of logarithmic intertwining operators and ultimately logarithmic tensor category theory for modules for a vertex operator algebra. One particular foundational step in [HLZ] involved two expansions of certain formal series which yielded a classical combinatorial identity involving Stirling numbers of the first kind (see (3.17) in [HLZ]), which was used to solve a problem posed in [Lu] (see Remark 3.8 in [HLZ]). These series expansions were worked out during the course of a proof of a very general logarithmic formal Taylor theorem (see Theorem 3.6 in [HLZ]). A detailed treatment of an efficient algebraic method to obtain formal Taylor theorems for much more general kinds of "formal functions" was given in [R1]. This method was demonstrated on a space involving formal versions of *iterated* logarithmic and exponential variables, extending the setting used in [HLZ]. The method of proof bypasses any series expansions. The series expansions generalizing the one appearing in the proof of Theorem 3.6 in [HLZ] were carried out in [R2] and they yielded, among other identities, both the identity involving Stirling numbers of the first kind mentioned above and also an analogous identity involving the Stirling numbers of the second kind. It was during the course of that work that the recursive identity which is the subject of this paper was discovered (and applied). As is often the case, the side project turned out to be quite as interesting as the original work. Both the statement and proof of this recursive identity are natural and quite different in methods and philosophy from the original work which suggested them, and we felt that they deserved a separate treatment. The formal calculus of iterated logarithms and exponentials which is our setting has further interesting arithmetic properties of which this recursive identity is just one (e.g. see [R4]).

The purpose of this paper is to give a complete proof of this identity, which we now state (without all the definitions) as a preview: For $n \in \mathbb{Z}$, we have

$$\ell_{n+1}(x+y) = \ell_{n+1}(x) + \log\left(1 + \left(\frac{\ell_n(x+y) - \ell_n(x)}{\ell_n(x)}\right)\right),$$

where $\ell_n(x)$ is a formal analogue of the $(-n)$-th iterated exponential for $n < 0$ and the n-th iterated logarithm for $n > 0$ and $\ell_0(x)$ is a formal analogue of x itself, where for a formal object X,

$$\log(1+X) = \sum_{i \geq 1} \frac{(-1)^{i-1}}{i} X^i,$$

whenever this formal expression is well defined (so that there are two different types of "logarithm," as is also the case in [HLZ]). We note also the following equivalent form of the above identity:

$$\ell_n(x+y) = \ell_n(x) e^{(\ell_{n+1}(x+y) - \ell_{n+1}(x))},$$

where for a formal object X

$$e^X = \sum_{i \geq 0} \frac{X^i}{i!},$$

whenever this is well defined. While the result is heuristically clear, it is certainly not trivial when, for instance, one considers the fragile blend of two different types of logarithm, and the proof is hardly obvious. This work is a slightly updated version of part of [R3].

This recursive identity involves expressions with a formal translation occuring in the arguments of some of the expressions. Analogous identities may be developed where we use more general formal changes of variable. One could develop such identities using parallel arguments to those we employ here. However, there is a nicer way to obtain further identities of this type using a different idea, which is developed in [R4] (see also [R1]), to which we refer the interested reader.

2. Formal iterated logarithms and exponentials

Let $\ell_n(x)$ be formal commuting variables for $n \in \mathbb{Z}$. We consider the algebra with an underlying vector space basis consisting of all elements of the form

$$\prod_{i \in \mathbb{Z}} \ell_i(x)^{r_i},$$

where $r_i \in \mathbb{C}$ for all $i \in \mathbb{Z}$, and all but finitely many of the exponents $r_i = 0$. The multiplication is the obvious one (when multiplying two monomials simply add the corresponding exponents and linearly extend). We call this algebra

$$\mathbb{C}\{[\ell]\}.$$

We let $\frac{d}{dx}$ be the unique derivation on $\mathbb{C}\{[\ell]\}$ satisfying

$$\frac{d}{dx}\ell_{-n}(x)^r = r\ell_{-n}(x)^{r-1}\prod_{i=-1}^{-n}\ell_i(x),$$

$$\frac{d}{dx}\ell_n(x)^r = r\ell_n(x)^{r-1}\prod_{i=0}^{n-1}\ell_i(x)^{-1},$$

$$\text{and} \qquad \frac{d}{dx}\ell_0(x)^r = r\ell_0(x)^{r-1},$$

for $n > 0$ and $r \in \mathbb{C}$.

Remark 2.1. Secretly, $\ell_n(x)$ is the $(-n)$-th iterated exponential for $n < 0$ and the n-th iterated logarithm for $n > 0$ and $\ell_0(x)$ is x itself.

Remark 2.2. To see that this does indeed uniquely define a derivation, we note that $\frac{d}{dx}$ must coincide with the unique linear map satisfying

$$\frac{d}{dx}\prod_{i\in\mathbb{Z}}\ell_i(x)^{r_i} = \sum_{j\in\mathbb{Z}}\frac{d}{dx}\ell_j(x)^{r_j}\prod_{i\neq j\in\mathbb{Z}}\ell_i(x)^{r_i},$$

on a basis of $\mathbb{C}\{[\ell]\}$. This establishes uniqueness. We need to check that this linear map is indeed a derivation. It is routine and we leave it to the reader to verify that it is enough to check that

$$\frac{d}{dx}(ab) = \left(\frac{d}{dx}a\right)b + a\left(\frac{d}{dx}b\right)$$

for basis elements a and b. Another routine calculation reduces the case to where $a = \ell_i(x)^r$ and $b = \ell_i(x)^s$ for $r, s \in \mathbb{C}$. Checking this case is trivial once one notes that

$$\frac{d}{dx}\ell_j(x)^r = r\ell_j(x)^{r-1}\frac{d}{dx}\ell_j(x).$$

If we let x and y be independent formal variables, then the formal exponentiated derivation $e^{y\frac{d}{dx}}$, defined by the expansion, $\sum_{k\geq 0} y^k\left(\frac{d}{dx}\right)^k/k!$, acts on a (complex) polynomial $p(x)$ as a formal translation in y. That is, as the reader may easily verify, we have

$$e^{y\frac{d}{dx}}p(x) = p(x+y). \tag{2.1}$$

This motivates the following definition (as in [R1]).

Definition 2.1. *Let*

$$\ell_n(x+y) = e^{y\frac{d}{dx}}\ell_n(x) \qquad \textit{for} \quad n\in\mathbb{Z}.$$

We establish a recursive identity for $\ell_n(x+y)$ in terms of $\ell_{n-1}(x+y)$ (and inversely in terms of $\ell_{n+1}(x+y)$) for all $n \in \mathbb{Z}$.

Our approach is based on the following identity:

$$\lim_{r\to 0}\left(\left(\frac{d}{dx}\right)^m \frac{(\ell_n(x))^r}{r}\right) = \left(\frac{d}{dx}\right)^m \ell_{n+1}(x) \quad (m \geq 1). \tag{2.2}$$

But we shall need to define just what we mean by taking a limit in this context in order for the above expression to make precise sense. We first define a new space.

Definition 2.2. We let $F(\mathbb{Z}_+, \ell)$ be the complex vector space of functions from the positive integers into $\mathbb{C}\{[\ell]\}$.

We may define a "lifting" of $\frac{d}{dx}$ on $F(\mathbb{Z}_+, \ell)$.

Definition 2.3. For f and $g \in F(\mathbb{Z}_+, \ell)$, we say that $g = \frac{d}{dx}f$ when $g(r) = \frac{d}{dx}f(r)$ for all $r \geq 0$.

Of course, $\frac{d}{dx}f$ may not exist for all $f \in F(\mathbb{Z}_+, \ell)$. We shall actually be interested in a subspace of $F(\mathbb{Z}_+, \ell)$, which we call $P(\mathbb{Z}_+, \ell)$ on which $\frac{d}{dx}f$ does always exist.

Definition 2.4. We let $P(\mathbb{Z}_+, \ell) \subset F(\mathbb{Z}_+, \ell)$ be the space of functions $f(r)$, from the nonzero natural numbers into $\mathbb{C}\{[\ell]\}$, which may be represented in the form

$$f(r) = \sum_{j\geq 0} q_j(r) \prod_{i\in\mathbb{Z}} \ell_i(x)^{p_{i,j}(r)},$$

where $q_j(r), p_{i,j}(r)$ are complex polynomials in r for all $j \geq 0$, $i \in \mathbb{Z}$ and where we further require that for all $j \geq 0$ there exists some $I_j \geq 0$ such that $p_{i,j}(r) = 0$ when $|i| \geq I_j$ and finally that there exists $J \geq 0$ such that $q_j(r)$ is the zero polynomial for $j \geq J$. We call such a representation a *formal polynomial form* of the function. The function is given by the obvious substitution procedure for r in the formal polynomial form.

Definition 2.5. For $f(r) \in P(\mathbb{Z}_+, \ell)$ we say a formal polynomial form representation,

$$f(r) = \sum_{j\geq 0} q_j(r) \prod_{i\in\mathbb{Z}} \ell_i(x)^{p_{i,j}(r)},$$

is in *reduced formal polynomial form* or *reduced form,* when for all $j \neq k$, $j, k \geq 0$ there is some $i \in \mathbb{Z}$ such that

$$p_{i,j}(r) \neq p_{i,k}(r).$$

Then we get

Proposition 2.1. *If $f(r) \in P(\mathbb{Z}_+, \ell)$, then it is uniquely expressible in reduced formal polynomial form.*

Proof. Let us say that

$$M(r) = q(r) \prod_{i \in \mathbb{Z}} \ell_i(x)^{p_i(r)}$$

is a monomial summand in one reduced formal polynomial form of $f(r)$. Then consider any other reduced formal polynomial form expression for $f(r)$. Since two formally unequal complex polynomials can only agree for a finite number of substitution values, it is not difficult to see that there must be a monomial summand in the second reduced polynomial form of the form

$$N(r) = \bar{q}(r) \prod_{i \in \mathbb{Z}} \ell_i(x)^{p_i(r)}.$$

But since our forms are reduced, then in fact $N(r)$ is the only monomial summand of this form in the second representation, and therefore $q(r) = \bar{q}(r)$. The result now obviously follows by induction. $\square$

It is now easy to define what is meant by $\lim_{r \to 0} f(r)$ when $f(r) \in P(\mathbb{Z}_+, \ell)$. One simply expresses $f(r)$ in its unique reduced formal polynomial expansion and substitutes 0 for r to yield a well-defined element of $\mathbb{C}\{[\ell]\}$. Before we return to the identity which we want we should note that $P(\mathbb{Z}_+, \ell)$ is obviously closed under $\frac{d}{dx}$. It is also necessary to prove one lemma.

Lemma 2.1. *For any $A_r(x) \in P(\mathbb{Z}_+, \ell)$ we have that*

$$\lim_{r \to 0} \frac{d}{dx} A_r(x) = \frac{d}{dx} \lim_{r \to 0} A_r(x).$$

Proof. Since $\frac{d}{dx}$ is linear we only have to consider the case where $A_r(x)$ is a monomial. For convenience we call $\lim_{r \to 0} A_r(x) = A_0(x)$. Let $A_r(x) = B_r(x) C_r(x)$. then

$$\begin{aligned} \lim_{r \to 0} \frac{d}{dx} A_r(x) &= \lim_{r \to 0} \frac{d}{dx} (B_r(x) C_r(x)) \\ &= (\lim_{r \to 0} \frac{d}{dx} B_r(x)) C_0(x) + B_0(x) \lim_{r \to 0} \frac{d}{dx} (C_r(x)) \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dx} \lim_{r \to 0} A_r(x) &= \frac{d}{dx} \lim_{r \to 0} (B_r(x) C_r(x)) \\ &= (\frac{d}{dx} B_0(x)) C_0(x)) + B_0(x) \frac{d}{dx} C_0(x)), \end{aligned}$$

which means that we only need consider the case where $A_r(x) = p(r)\ell_i(x)^{q(r)}$ where $i \in \mathbb{Z}$. Now we get:

$$\begin{aligned}
\lim_{r\to 0} \frac{d}{dx} A_r(x) &= \lim_{r\to 0} q(r)p(r)\ell_i(x)^{q(r)-1}\frac{d}{dx}\ell_i(x) \\
&= q(0)p(0)\ell_i(x)^{q(0)-1}\frac{d}{dx}\ell_i(x) \\
&= \frac{d}{dx}p(0)\ell_i(x)^{q(0)} \\
&= \frac{d}{dx}\lim_{r\to 0} A_r(x).
\end{aligned}$$

□

We now prove the desired identity (2.2).

Lemma 2.2. *For $m \geq 1$ and $n \in \mathbb{Z}$*

$$\lim_{r\to 0}\left(\left(\frac{d}{dx}\right)^m \frac{(\ell_n(x))^r}{r}\right) = \left(\frac{d}{dx}\right)^m \ell_{n+1}(x).$$

Proof. First note that $\frac{d}{dx}\frac{(\ell_n(x))^r}{r} \in P(\mathbb{Z}_+, \ell)$. Then we may calculate to get:

$$\begin{aligned}
&\lim_{r\to 0}\left(\left(\frac{d}{dx}\right)^m \frac{(\ell_n(x))^r}{r}\right) = \left(\frac{d}{dx}\right)^{m-1} \lim_{r\to 0}\left(\frac{d}{dx}\frac{(\ell_n(x))^r}{r}\right) \\
&= \left(\frac{d}{dx}\right)^{m-1} \lim_{r\to 0}\left(\ell_n(x)^{r-1}\frac{d}{dx}\ell_n(x)\right) \\
&= \left(\frac{d}{dx}\right)^{m-1} \left(\ell_n(x)^{-1}\frac{d}{dx}\ell_n(x)\right).
\end{aligned}$$

And now we proceed in the two separate cases $n \geq 0$ and $n \leq -1$. First, when $n \geq 0$ we have,

$$\begin{aligned}
\lim_{r\to 0}\left(\left(\frac{d}{dx}\right)^m \frac{(\ell_n(x))^r}{r}\right) &= \left(\frac{d}{dx}\right)^{m-1}\left(\ell_n(x)^{-1}\prod_{i=0}^{n-1}(\ell_i(x))^{-1}\right) \\
&= \left(\frac{d}{dx}\right)^{m-1}\prod_{i=0}^{n}(\ell_i(x))^{-1} \\
&= \left(\frac{d}{dx}\right)^{m}\ell_{n+1}(x).
\end{aligned}$$

And second, when $n \leq -1$ we have,

$$\lim_{r\to 0}\left(\left(\frac{d}{dx}\right)^m \frac{(\ell_n(x))^r}{r}\right) = \left(\frac{d}{dx}\right)^{m-1}\left(\ell_n(x)^{-1}\prod_{i=-1}^{n}\ell_i(x)\right)$$
$$= \left(\frac{d}{dx}\right)^{m-1}\left(\prod_{i=-1}^{n+1}\ell_i(x)\right)$$
$$= \left(\frac{d}{dx}\right)^{m}\ell_{n+1}(x).$$

$\square$

With some care, we now see that for $n \in \mathbb{Z}$

$$\lim_{r\to 0}\left(e^{y\frac{d}{dx}}\left(\frac{(\ell_n(x))^r}{r}\right) - \frac{\ell_n(x)^r}{r}\right) = e^{y\frac{d}{dx}}\ell_{n+1}(x) - \ell_{n+1}(x).$$

One must note that indeed $e^{y\frac{d}{dx}}\left(\frac{(\ell_n(x))^r}{r}\right) - \frac{\ell_n(x)^r}{r} \in P(\mathbb{Z}_+, \ell)$ because the first term of $e^{y\frac{d}{dx}}\left(\frac{(\ell_n(x))^r}{r}\right)$ cancels $\frac{\ell_n(x)^r}{r}$. Next we get for $n \in \mathbb{Z}$

$$\ell_{n+1}(x+y) = \ell_{n+1}(x) + \lim_{r\to 0}\left(\frac{\ell_n(x+y)^r - \ell_n(x)^r}{r}\right).$$

But we don't want the limit in the expression, so, recalling that r stands for a positive integer, we get:

$$\ell_{n+1}(x+y) = \ell_{n+1}(x) + \lim_{r\to 0}\left(\frac{(\ell_n(x) + (\ell_n(x+y) - \ell_n(x)))^r - \ell_n(x)^r}{r}\right)$$
$$= \ell_{n+1}(x) + \lim_{r\to 0}\sum_{p\geq 1}\frac{r(r-1)\cdots(r-(p-1))}{rp!}\ell_n(x)^{r-p}\left(\ell_n(x+y) - \ell_n(x)\right)^p$$
$$= \ell_{n+1}(x) + \sum_{p\geq 1}\frac{(-1)^{p-1}}{p}\left(\frac{\ell_n(x+y) - \ell_n(x)}{\ell_n(x)}\right)^p$$
$$= \ell_{n+1}(x) + \log\left(1 + \left(\frac{\ell_n(x+y) - \ell_n(x)}{\ell_n(x)}\right)\right),$$

where for a formal object X

$$\log(1+X) = \sum_{i\geq 1}\frac{(-1)^{i-1}}{i}X^i,$$

whenever this formal expression is well defined.

Theorem 2.1. *For $n \in \mathbb{Z}$ we have*

$$\ell_{n+1}(x+y) = \ell_{n+1}(x) + \log\left(1 + \left(\frac{\ell_n(x+y) - \ell_n(x)}{\ell_n(x)}\right)\right). \tag{2.3}$$

□

We note that we may solve for $\ell_n(x+y)$ in (2.3) to get for all $n \in \mathbb{Z}$

$$\ell_n(x+y) = \ell_n(x) e^{(\ell_{n+1}(x+y) - \ell_{n+1}(x))}, \tag{2.4}$$

where for a formal object X

$$e^X = \sum_{i \geq 0} \frac{X^i}{i!},$$

whenever this is well defined. We used that $e^{\log(1+X)} = 1 + X$, which is perhaps checked most easily by calculating that $\frac{d}{dx}\left(e^{\log(1+X)}(1+X)^{-1}\right) = 0$ which gives $e^{\log(1+X)} = c(1+X)$ for some constant, c which is, in turn, solved for by substituting 0 for X or, in other words, checking the constant term.

Remark 2.3. Both (2.3) and (2.4) make sense heuristically, as may be seen easily, when one recalls that $\ell_n(x)$ is "really" an iterated logarithm or exponential.

References

[HLZ] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor product theory for generalized modules for a conformal vertex algebra, arXiv:0710.2687v3 [math.QA]

[Lu] D. Lubell, Problem 10992, Problems and Solutions, *American Mathematical Monthly* **110**, 2003, 155.

[M] A. Milas, Weak modules and logarithmic intertwining operators for vertex operator algebras, in: *Recent Developments in Infinite-Dimensional Lie Algebras and Conformal Field Theory*, ed. S. Berman, P. Fendley, Y.-Z. Huang, K. Misra, and B. Parshall, Contemp. Math., Vol. 297, American Mathematical Society, Providence, RI, 2002, 201–225.

[R1] T. Robinson, New perspectives on exponentiated derivations, the formal Taylor theorem, and Faà di Bruno's formula in: *Proceedings of the Conference on Vertex Operator Algebras, Illinois State University*, (2008) ed. by: Maarten Bergvelt, Gaywalee Yamskulna, and Wenhua Zhao, Contemporary Math. **497**, Amer. Math. Soc., Providence (2009), 185–198; arXiv:0903.3391.

[R2] T. Robinson, On generalizing the formal Taylor theorem and some associated combinatorics, to appear.

[R3] T. Robinson, Formal calculus, umbral calculus, and basic axiomatics of vertex algebras, Ph.D. thesis, Rutgers University, 2009.

[R4] T. Robinson, Intertwining substitutions and the formal Taylor theorem, to appear.

Department of Mathematics, Rutgers University, Piscataway, NJ 08854

E–mail address: thomasro@math.rutgers.edu