MODULAR CATEGORIES, INTEGRALITY AND EGYPTIAN FRACTIONS

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ABSTRACT. It is a well-known result of Etingof, Nikshych and Ostrik that there are finitely many inequivalent integral modular categories of any fixed rank n. This follows from a double-exponential bound on the maximal denominator in an Egyptian fraction representation of 1. A naïve computer search approach to the classification of rank n integral modular categories using this bound quickly overwhelms the computer's memory (for $n \geq 7$). We use a modified strategy: find general conditions on modular categories that imply integrality and study the classification problem in these limited settings. The first such condition is that the order of the twist matrix is $2, 3, 4$ or 6 and we obtain a fairly complete description of these classes of modular categories. The second condition is that the unit object is the only simple non-self-dual object, which is equivalent to odd-dimensionality. In this case we obtain a (linear) improvement on the bounds and employ number-theoretic techniques to obtain a classification for rank at most 11 for odd-dimensional modular categories.

1. INTRODUCTION

The problem of classifying low-rank modular categories has its roots in the classification problem for rational conformal field theories going back to the 1980s (predating the definition of modular category [24]). Currently, the most complete results are in [22] where unitary modular categories of rank at most 4 are classified. More generally, Ostrik classified ribbon fusion categories of rank at most 3 [20] and fusion categories of rank at most 2 [19].

By a generalized form of Ocneanu rigidity [7, Prop. 2.31] one may classify modular categories of a given rank up to finite ambiguity by classifying their Grothedieck semirings. In [13] such an approach yielded a classification (up to Gr.-semirings) of modular categories of rank at most 5 with the property that some object is not isomorphic to its dual object.

Wang has conjectured that there are only finitely many inequivalent modular categories of each rank (see [22, Conjecture 6.1]), and the results mentioned above bear this out for rank at most 4 and for rank at most 5 in case some object is non-self-dual. The most general class of modular categories for which Wang's conjecture has been verified is for *weakly integral* categories, that is, categories $\mathcal{C}$

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with $\mathrm{FPdim}(\mathcal{C}) \in \mathbb{N}$ [7, Prop. 8.38]. The proof relies upon a classical result of Landau [14] that the diophantine equation:

$$(1.1) \qquad\qquad 1 = \sum_{i=1}^{n} \frac{1}{x_i}$$

has finitely many solutions with $x_i \in \mathbb{N}$. Such solutions are *Egyptian fraction* representations of 1 which are of independent interest in combinatorial number theory (see [12, Section D11] and [18, Seq. A002966]). Moreover the number of solutions to (1.1) is at least exponential in n since $\frac{1}{x} = \frac{1}{x+1} + \frac{1}{x^2+x}$ so that each (non-constant) n-term Egyptian fraction representation of 1 leads to at least 2 $(n+1)$-term representation. In [13] a computational approach to classifying *integral* modular categories (that is, with $\mathrm{FPdim}(X) \in \mathbb{N}$ for all objects X) of rank n is suggested using two facts:

(1) [6, Lemma 1.2]: $\sqrt{\frac{\mathrm{FPdim}(\mathcal{C})}{\mathrm{FPdim}(X_i)}} \in \mathbb{N}$ for all simple X_i

(2) [3]: $\dim(\mathcal{C}) \leq u_n$ where u_n is inductively defined by $u_1 := 1$ and $u_k := u_{k-1}(u_{k-1} + 1)$.

As u_n is double exponential in n, a direct search for solutions by computer quickly becomes infeasible.

We attempt to circumvent this computational obstacle by finding interesting conditions on modular categories that imply integrality, and then use these conditions to simplify the classification problem for these classes of categories. Specifically, we have:

Theorem. *Suppose $\mathcal{C}$ is a modular category such that either:*

(a) *the twist matrix T satisfies $T^N = I$ for $N \in \{2, 3, 4, 6\}$ or*

(b) *the only simple object in $\mathcal{C}$ satisfying $X \cong X^*$ is the unit object $\mathbf{1}$*

then $\mathcal{C}$ is integral.

Statement (a) is proved below in Theorem 3.1 and was inspired by Davydov who posed the question for $N = 2$ to the second author. Statement (b) is [13, Theorem 2.2] and follows from a Galois theory argument.

We are interested in classifying categories with one of these two properties (a) or (b). We obtain a fairly explicit description of modular categories with property (a) in Theorem 3.2. Modular categories with property (b) of rank at most 11 are shown to be *pointed* in Theorem 4.5, that is, $\mathrm{FPdim}(X_i) = 1$ for each simple object X_i. While property (b) may seem to be a rare condition at first glance one can show (see Prop. 4.3) that it is equivalent to the condition that $\mathrm{FPdim}(\mathcal{C})$ is odd.

The remainder of the paper is structured as follows: in Section 2 we collect together some notation and useful facts about modular categories. We address the classification problem for modular categories having property (a) or (b) in

Sections 3 and 4 respectively, and give some perspectives and futher directions in Section 5.

2. Notation

A modular category $\mathcal{C}$ is a non-degenerate braided, balanced fusion category (see [1] or [21] for the complete axiomatic definition). In this section we establish notation and describe some of the algebraic data and relations coming from the axioms of modular categories.

We shall typically adopt the notation and normalizations of [17] for the data of a modular category. The fusion coefficients are $N_{i,j}^k := \dim \operatorname{Hom}(X_i \otimes X_j, X_k)$ where $\mathbf{1} = X_0, \ldots, X_{n-1}$ are the (isomorphism classes of) simple objects the number of which (n) is called the *rank* of $\mathcal{C}$. The diagonal twist matrix $T_{ij} := \delta_{ij}\theta_i$ has finite order and the S-matrix is normalized so that $S_{00} = 1$. We will denote by d_i the dimension of the simple object X_i, i.e. $d_i = S_{i0} = \dim(X_i)$. Defining the *Gauss sums* by $p_+ = \sum_i \theta_i(d_i)^2$ and $p_- = \overline{p_+}$ we have $(ST)^3 = p_+ S^2$, $S^2 = p_+ p_- C$ where $C_{ij} = \delta_{ij^*}$ is the (involutive) charge conjugation matrix, which commutes with T. In particular (S, T) give rise to a (projective) representation of the modular group $SL(2, \mathbb{Z})$. We define the fusion matrices $(N_i)_{jk} := N_{ik}^j$ and denote by $\operatorname{FPdim}(X_i)$ the largest eigenvalue of N_i, i.e. the *FP-dimension* of X_i. The global dimension of $\mathcal{C}$ is $\dim(\mathcal{C}) = \sum_i (d_i)^2$ and the global FP-dimension is $\operatorname{FPdim}(\mathcal{C}) := \sum_i \operatorname{FPdim}(X_i)^2$. If $\operatorname{FPdim}(\mathcal{C}) = \dim(\mathcal{C})$ then $\mathcal{C}$ is said to be *pseudo-unitary*. The *Verlinde formula* relates the fusion coefficients to the S-matrix entries:

$$(2.1) \qquad N_{ij}^k = \sum_r \frac{S_{ir} S_{jr} \overline{S_{kr}}}{\dim(\mathcal{C}) S_{0r}}.$$

A pair of matrices (S, T) satisfying the above relations such that the right-hand-side of (2.1) is a non-negative integer for all i, j, k is called a *modular datum* ([9]). A modular category $\mathcal{C}$ with corresponding S and T matrices is called a *categorification* of (S, T).

A category $\mathcal{C}$ is called *integral* if $\operatorname{FPdim}(X_i) \in \mathbb{N}$ for all simple objects X_i. All categories encountered in this work will be integral and hence pseudo-unitary so that we may assume that $\operatorname{FPdim}(X_i) = d_i$ and $\operatorname{FPdim}(\mathcal{C}) = \dim(\mathcal{C})$ by [7, Prop. 8.23, 8.24].

The entries of N_i, S and T satisfy further relations:

$$(2.2) \qquad p_+ p_- = \dim(\mathcal{C})$$

and

$$(2.3) \qquad \theta_i \theta_j S_{ij} = \sum_k N_{i^*j}^k \theta_k d_k.$$

Since dim is a character of the Grothendieck semiring of $\mathcal{C}$ we have:

$$(2.4) \qquad \sum_k N_{ij}^k d_k = d_i d_j.$$

The (second) *FS-indicator* is defined to be

$$\nu_k = \frac{1}{\dim(\mathcal{C})} \sum_{i,j} N_{i,j}^k d_i d_j \left(\frac{\theta_i}{\theta_j}\right)^2$$

and satisfies:

$$(2.5) \qquad \nu_k = \begin{cases} 0 & X_k \not\cong X_k^* \\ \pm 1 & X_k \cong X_k^* \end{cases}.$$

More generally the n-th FS-indiactor $\nu_k^{(n)}$ may be defined for simple objects in modular categories in an analogous way ([16, Theorem 7.5]). The quantity $\mathrm{FSexp}(\mathcal{C})$ is the smallest integer $m > 0$ such that $\nu_k^{(m)} = d_k$ for all simple k. For modular categories $\mathrm{FSexp}(\mathcal{C})$ coincides with the order of the T matrix ([16, Theorem 7.7]).

3. Low-order Twist Matrices

In this section we study modular categories with twist matrix of order $2, 3, 4$ or 6 of arbitrary rank. Our first result is:

Theorem 3.1. *Suppose that T is the twist matrix of a modular category $\mathcal{C}$ such that $T^N = I$ for $N \in \{2, 3, 4, 6\}$. Then $\mathcal{C}$ is integral.*

Proof. First we observe that by [17, Prop. 5.7] the entries of the S-matrix for $\mathcal{C}$ must lie in $\mathbb{Q}(\theta_1, \dots, \theta_{n-1}) = \mathbb{Q}(\zeta_N)$ where ζ_N is a primitive Nth root of unity. Since s_{ij} are algebraic integers and $\varphi(N) \leq 2$ (Euler's totient φ), if $s_{ij} \in \mathbb{R}$ then $s_{ij} \in \mathbb{Z}$. In particular some column of the S-matrix must be an integer multiple of the vector of FP-dimensions (since $\mathcal{C}$ is modular) and so $\mathrm{FPdim}(X_i)$ are rational integers. $\qquad \square$

We now characterize modular categories with $T^N = I$ for $N \in \{2, 3, 4, 6\}$ in the following:

Theorem 3.2. *Suppose $\mathcal{C}$ is a modular category such that $T^N = I$. Then:*

    (a) *If $N = 2$ then $\mathcal{C}$ is braided tensor equivalent to a subcategory of $\mathrm{Rep}(D^\omega G)$ where G is an abelian 2-group of exponent 2 and $\dim(\mathcal{C}) = 2^{2s}$ (in particular, $\mathcal{C}$ is pointed),*

(b) *If $N = 3$ then $\mathcal{C}$ is braided tensor equivalent to a subcategory of $\operatorname{Rep}(D^\omega G)$ where G is a 3-group of exponent 3, and*

(c) *If $N = 4$ then $\mathcal{C}$ is braided tensor equivalent to a subcategory of $\operatorname{Rep}(D^\omega G)$ where G is a 2-group of exponent 2 or 4.*

(d) *If $N = 6$ then $\mathcal{C}$ is solvable and hence weakly integral.*

Proof. If $T^2 = I$ then each X_k is self-dual since the entries of the S-matrix are real. Computing the FS-indicator we have:

$$\nu_k = \frac{1}{\dim \mathcal{C}} \sum_{i,j} N_{ij}^k d_i d_j \left(\frac{\theta_i}{\theta_j}\right)^2 = \pm 1$$

for all k. But $\left(\frac{\theta_i}{\theta_j}\right)^2 = 1$ by assumption and $\sum_i N_{ij}^k d_i = d_k d_j$ (as N_{ij}^k is totally symmetric in the self-dual case) so we may simplify:

$$\nu_k = \frac{1}{\dim \mathcal{C}} \sum_j d_k d_j^2 = d_k.$$

But $\nu_k = \pm 1$ and $d_k > 0$ so this implies $d_k = 1$ for all simple objects X_k. We must then have $X_k^{\otimes 2} \cong \mathbf{1}$ as well. Thus $\mathcal{C}$ is pointed and has the same fusion rules as the group $\mathbb{Z}_2^m$ where 2^m is both the rank and global dimension of $\mathcal{C}$. Now since $p_+ \in \mathbb{R}$ we have $p_+ = p_- \in \mathbb{Z}$ so that by (2.2) we must have $\dim(\mathcal{C}) = 2^m = (p_+)^2$ and hence $m = 2s$. Since $\mathcal{C}$ is braided tensor equivalent to a subcategory of $Z(\mathcal{C}) \cong \operatorname{Rep}(D^\omega G)$ we must have G an elementary abelian 2-group.

For (b) and (c) with $N = 3$ (resp. 4) we use [16, Theorem 8.4] to conclude that $\dim(\mathcal{C}) = p^t$ where $p = 3$ (resp. 2). In particular $\mathcal{C}$ is braided and nilpotent and hence group theoretical by [4, Theorem 6.10]. Thus we conclude that $\mathcal{C}$ is braided tensor equivalent to a subcategory of $Z(\mathcal{C}) \cong \operatorname{Rep}(D^\omega G)$ where G is a p-group and since $\operatorname{FSexp}(\mathcal{C}) = \operatorname{FSexp}(Z(\mathcal{C})) = 3$ (resp. 4) by [16, Corollary 7.8] G must have exponent 3 (resp. 2 or 4 as the exponent of G must divide that of $\operatorname{Rep}(D^\omega G)$).

For (d) with $N = 6$ [16, Theorem 8.4] implies that $\dim(\mathcal{C}) = 2^a 3^b$ with $a, b > 0$ and so by [8, Theorem 1.6] $\mathcal{C}$ is solvable, and in particular weakly group-theoretical. $\qquad\square$

Remark 3.3.

- In principle, Theorem 3.2(a),(b),(c) can be used to completely classify modular categories with twist matrix of orders $2, 3$ and 4.
- Theorem 3.2(d) is sharp in the sense that we cannot conclude $\mathcal{C}$ is group theoretical as [10, Example 5.3(a)] shows that non-group-theoretical modular categories with $T^6 = I$ exist.
- In [2, Prop. 5.1] an infinite family of modular data (S, T) depending on an integer $k \geq 0$ is described with $S_{ij} \in \mathbb{Z}$ for all i, j and $T^2 = I$. Theorem 3.2(a) shows that this family is not categorifiable for $k \geq 1$.

4. Integrality and Egyptian Fractions

In this section we classify integral modular categories of rank at most 6 and maximally non-self-dual modular categories of rank at most 11.

If $\mathcal{C}$ is a (pseudo-unitary) integral modular category with dimensions of simple objects $d_1, d_2, \ldots, d_n = \dim(\mathbf{1}) = 1$ then by [6, Lemma 1.2] the numbers $x_i = \dim(\mathcal{C})/(d_i)^2$ are *integers* and so satisfy eqn. (1.1). In [13, Lemma 2.3] it is shown that after relabelling if necessary so that $x_1 \leq x_2 \leq \cdots \leq x_n = \dim(\mathcal{C})$ then:

$$k \leq x_k \leq u_k(n-i+1) \quad \text{forall} \quad 1 \leq k \leq n,$$

where $u_1 = 1$, $u_k = u_{k-1}(u_{k-1}+1)$. These bounds rely upon two classical results in [14] and [3]. Notice also that $x_n/x_i = (d_i)^2$ is a perfect square for each i. The *trivial solution* $x_1 = \cdots = x_n = n$ corresponds to a *pointed* modular category of rank n as then $d_i = 1$ for all i. Our basic algorithm for finding $x_1 \leq \cdots \leq x_n$ satisfying these conditions is as follows:

Algorithm 4.1.

(1) form the set of pairs $S_2 := \{(j, j \cdot i^2) : 2 \leq j \leq n, 1 \leq i \leq \sqrt{u_n/j}\}$. These are the possible pairs (x_1, x_n).
(2) having determined S_k for some $2 \leq k \leq n-2$ let S_{k+1} be the set of $(k+1)$-tuples $(j_1, \ldots, j_{k-1}, J, j_k)$ such that
    (a) $(j_1, \ldots, j_k) \in S_k$,
    (b) $\max(j_{k-1}, k) \leq J \leq \min(j_k, u_k(n-k+1))$,
    (c) $\sqrt{j_k/J} \in \mathbb{Z}$ and
    (d) $\frac{1}{J} + \sum_{i=1}^{k} \frac{1}{x_i} \leq 1$.
(3) the solution set S_n is the set of n-tuples $(j_1, \ldots, j_{n-2}, J, j_{n-1})$ where
    (a) $(j_1, \ldots, j_{n-1}) \in S_{n-1}$,
    (b) $\max(j_{n-2}, n-1) \leq J \leq \min(j_{n-1}, 2u_{n-1})$,
    (c) $\sqrt{j_{n-1}/J} \in \mathbb{Z}$ and
    (d) $\frac{1}{J} + \sum_{i=1}^{n-1} \frac{1}{x_i} = 1$.

For $n = 5$ and $n = 6$ the solutions are: $\{(5,5,5,5,5), (2,8,8,8,8)\}$ and

$$\{(6,6,6,6,6,6), (3,3,12,12,12,12), (4,4,4,9,9,36)\}$$

respectively. These computations were carried out in a few hours on a desktop computer using Maple, and we have:

Theorem 4.2. *Any integral modular category of rank at most* 6 *is pointed.*

Proof. For rank at most 5 this result is known, see: [22] and [13]. For rank 6 we need only show that no modular category may have simple dimensions $(1,1,1,1,2,2)$ or $(1,2,2,3,3,3)$ corresponding to the non-pointed solutions above. The first case is eliminated by [15, Prop. 4.11]: integral modular categories $\mathcal{C}$ with

FPdim$(\mathcal{C}) = 2^2 3$ must be pointed. On the other hand, a fusion category of dimension 36 must be solvable and contain a non-trivial invertible object by [8, Prop. 8.2] which eliminates the second solution. $\qquad\square$

We remark that there is a non-pointed integral modular category of rank 8, namely Rep$(D(S_3))$, the representation category of the double of the symmetric group S_3. We expect that this is the smallest possible rank for non-pointed integral modular categories. Unfortunately for $n = 7$ one has $u_7 \approx 10^{13}$ and attempts to implement Algorithm 4.1 quickly overwhelms a computer's memory. This motivates passing to a restricted class of modular categories that enjoy integrality.

With a view towards improving the upper bound u_n above, suppose that $\mathcal{C}$ is a maximally non-self dual (MNSD) modular category, i.e. each non-trivial simple object X_i satisfies $X_i \not\cong X_i^*$. Then $\mathcal{C}$ is integral with rank $n = 2k + 1$ by [13, Theorem 2.2]. Since $d_i = \dim(X_i) = \dim(X_i^*)$ we may label the dimensions of simple objects: $d_1, d_1 \ldots, d_k, d_k, d_{k+1} = \dim(\mathbf{1}) = 1$ so that FPdim$(\mathcal{C}) = 1 + 2\sum_{i=1}^{k} d_i^2$ is odd. After reordering the $k + 1$ integers $x_i := \dim(\mathcal{C})/d_i^2$ so that $x_1 \le x_2 \le \cdots \le x_k \le x_{k+1} = \dim(\mathcal{C})$ they satisfy the special case of eqn. (1.1):

$$(4.1) \qquad\qquad \frac{1}{x_{k+1}} + \sum_{i=1}^{k} \frac{2}{x_k} = 1$$

The maximal non-self-dual condition may seem somewhat exceptional, but it follows from [13, Theorem 2.2], [16, Corollary 8.2(ii)] and [5, Corollary 2.33] that:

Proposition 4.3. *Let $\mathcal{C}$ be a modular category of rank n. The the following are equivalent:*

(1) *$\mathcal{C}$ is maximally non-self-dual*
(2) *FPdim$(\mathcal{C})$ is odd*
(3) *$\mathcal{C}$ is (monoidally) equivalent to Rep(A) where A is an odd-dimensional, semisimple quasi-Hopf algebra.*

The following generalization of [13, Lemma 2.3] gives a linear improvement of the upper bounds in Algorithm 4.1 for a restricted class of integral modular categories and includes the MNSD setting as the special case $\ell = 2$:

Lemma 4.4. *Suppose that $\mathcal{C}$ is an integral modular category of rank n and there exist k (weakly decreasing) integers $p_1 \ge p_2 \ge \cdots \ge p_k$ such that the non-trivial (isomorphisms classes of) simple objects can be partitioned into k sets $P_1, \ldots, P_k$ such that $X \in P_i$ has $\dim(X) = p_i$ and $|P_i| = \ell$ for all $1 \le i \le k$. Then:*

(i) *The numbers $x_i := \frac{\dim \mathcal{C}}{p_i^2}$ form a weakly increasing sequence of integers such that $\sum_{i=1}^{k} \frac{\ell}{x_i} + \frac{1}{x_{k+1}} = 1$, and*
(ii) *$\ell i \le x_i \le (n + \ell - i\ell)\frac{A_i}{\ell}$ for $i \le k$ and $(k+1) \le x_{k+1} \le \frac{A_{k+1}}{\ell}$ where $A_1 = \ell$ and $A_i = A_{i-1}(A_{i-1} + 1)$.*

Proof. The proof of (i) proceeds exactly as in [13, Lemma 2.3] so we focus on (ii). First define $x'_i = x_i$ for $1 \le i \le k$ and $x'_{k+1} = x_{k+1}/\ell$ so that:

$$(4.2) \qquad \sum_{i=1}^{k+1} \frac{1}{x'_i} = \frac{1}{\ell}$$

and $x'_1 \le \cdots \le x'_{k+1} = \ell x_{k+1}$. Observing that $n = k\ell + 1$ we finds that (ii) becomes $\ell i \le x'_i \le (n + \ell - i\ell)\frac{A_i}{\ell}$. Since the x'_i are weakly increasing the lower bounds are clear: for if $\ell i > x'_i$ for some i then $\ell i > x'_i \ge x'_{i-1} \ge \cdots \ge x'_1$ so $1/\ell = i\frac{1}{\ell i} > \sum_{j=1}^i \frac{1}{x'_j}$, a contradiction. For the upper bounds follow the same strategy as in [13]: we use Takenouchi's [23] bound on the largest denominator x'_{k+1} in eqn. (4.2) and Landau's [14] estimate. In particular, if

$$\sum_{i=1}^{k} \frac{1}{y_i} = r$$

with $y_1 \le y_2 \le \cdots \le y_k$ then Landau's result says $y_i \le (k-i+1)/r_{i-1}$ where $r_0 = r$ and $r_i = r_{i-1} - 1/y_i$. Takenouchi's result says that the maximum denominator y_i of a solution to $\frac{1}{\ell} = \sum_{i=1}^k \frac{1}{y_i}$ is A_k where $A_1 = \ell$ and $A_i := A_{i-1}(A_{i-1} + 1)$. So in particular $1/r_{i-1} \le A_i$ for all i and the result follows.

$\square$

Lemma 4.4 provides a linear improvement on the bounds in [13, Lemma 2.3]: one has roughly $\mathrm{rank}(\mathcal{C})/\ell$ variables as opposed to $\mathrm{rank}(\mathcal{C})$. Taking the case $\ell = 2$ one can modify Algorithm 4.1 to search for possible MNSD modular categories in the obvious way. As the bounds are still double exponential the best result we can achieve with our (modified) algorithm is:

Theorem 4.5. *All maximally non-self dual modular categories of rank at most* 11 *are pointed.*

Proof. The only non-trivial solution the modified algorithm produces is in rank 11, consisting of 9 simple objects of dimension 1 and two simple objects of dimension 3. To eliminate this possibility we use a result in [11]: any integral modular category $\mathcal{C}$ is faithfully graded by its universal grading group $U(\mathcal{C})$, which is isomorphic to the group of invertible objects in $\mathcal{C}$. Since each component of a faithful grading must have the same FP-dimension we see that this is impossible as $|U(\mathcal{C})| = 9$ and any component $\mathcal{C}_g$ of the grading containing a simple object X with $\mathrm{FPdim}(X) = 3$ must have $\mathrm{FPdim}(\mathcal{C}_g) \ge 9$. $\square$

Remark 4.6. An obvious source of MNSD modular categories are those of the form $\mathrm{Rep}(D^\omega G)$ where $|G|$ is odd (that is, the representation category of the twisted double of a finite group of odd order). The smallest rank example of a non-pointed MNSD modular category we are aware of is the rank 25 category

$\text{Rep}(D(\mathbb{Z}_3 \ltimes \mathbb{Z}_7))$ of dimension 441 where $\mathbb{Z}_3 \ltimes \mathbb{Z}_7$ is a non-abelian semidirect product. One might speculate that any MNSD modular category of rank at most 23 is pointed.

We close with the following related question that we find interesting:

Question 4.7. *Is there an odd dimensional modular category that is non-group-theoretical?*

5. CONCLUSIONS AND FUTURE DIRECTIONS

We have made significant progress towards classifiying modular categories with twist matrices of order $2, 3, 4$ or 6. It is not clear how to extend these techniques to other orders. One might hope that one could generalize [16, Theorem 8.4] to non-integral categories to get some statement about the possible prime divisors of $\text{FPdim}(\mathcal{C})$ in terms of those of the order of the twist matrix. However, the relevant number field would no longer be $\mathbb{Z}$ so that other complications would arise particularly concerning units.

For rank greater than 11 even the improved bounds for MNSD modular categories become too large for our computational techniques. However, if one considers MNSD modular categories with:

(1) rank r
(2) $\text{FPdim}(X_i) \leq M$, for each simple X_i and
(3) $|\{\text{FPdim}(X_i) : 0 \leq i \leq r-1\}| = t$ (X_i simple).

further headway can be made. For example, taking $r \leq 23$, $M = 45$ and $t \leq 6$ we set $k_i = |\{j : \text{FPdim}(X_j) = d_i\}|$ with $d_0 = 1$ we obtain (using [6, Lemma 1.2] and [7, Prop. 8.15]):

$$k_0 \equiv 1 \mod 2$$
$$k_i \equiv 0 \mod 2, (i \neq 0)$$
$$\sum_{i=0}^{t-1} k_i = r$$
$$\sum_{j \neq i} k_j (d_j)^2 \equiv 0 \mod (d_i)^2$$
$$\sum_{j \neq 0} k_j (d_j)^2 \equiv 0 \mod k_0$$

For any fixed $r \leq 23$ and subsets of $\{d : 3 \leq d \leq 45, d \in 2\mathbb{Z} + 1\}$ of size at most 5 we convert these congruences to a system of linear diophantine equations and then use the Smith normal form of the corresponding matrix to solve for the k_i. Applying further classification theorems and *ad hoc* techniques produces the following partial result:

Theorem 5.1. *If $\mathcal{C}$ is a non-pointed maximally non-self dual modular category with $13 \leq rank(\mathcal{C}) \leq 23$ then either $\mathcal{C}$ has:*

(a) $\mathrm{FPdim}(X_i) \geq 47$ *for some simple X_i or*
(b) $|\{\mathrm{FPdim}(X_i) : 0 \leq i \leq m - 1\}| \geq 7$

This is further evidence that perhaps a non-pointed MNSD modular category of smallest rank is indeed the category of rank 25 and dimension 441 described above.

More generally, one can also obtain partial classification results for integral modular categories of small rank by bounding the FP-dimension further (below the double-exponential bound of u_n in Algorithm 4.1). For example, using a streamlined version of Algorithm 4.1 provided by K. Rusek and some classification theorems we can prove:

Theorem 5.2. *If a non-pointed modular category $\mathcal{C}$ has rank 7 then $\mathrm{FPdim}(\mathcal{C}) > 10^5$.*

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