

GROTHENDIECK DUALITY UNDER $\text{Spec } \mathbb{Z}$.

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ABSTRACT. We define the derived category of a concrete category in a way which extends the usual definition of the derived category of a ring, and we prove that the bounded-below derived category of $\text{Spec } \mathbb{M}_0$ (an approximation, used by e.g. Connes and Consani, to “Spec of the field with one element”) is the stable homotopy category of connective spectra. We also describe some basic features of Grothendieck duality for the map from $\text{Spec } \mathbb{Z}$ to $\text{Spec } \mathbb{M}_0$, or, what comes to the same thing, the map from $\text{Spec } \mathbb{Z}$ to Spec of the sphere spectrum; these basic features include a computation of the homology of the dualizing complex $f^!(S)$ of abelian groups associated to the sphere spectrum.

1. INTRODUCTION.

It has been known, at least since the time of Waldhausen coining the phrase “brave new ring,” that essentially all of commutative algebra embeds into stable homotopy theory via the Eilenberg-MacLane functor H ; an excellent reference for this idea, which also makes clear the kind of technical obstacles involved in making it precise and proving it, is [10]. From this point of view, E_∞ -ring spectra are a generalization of commutative rings, and one wants to be able to knit together E_∞ -ring spectra to form some geometric objects, in the same way that commutative rings are knitted together to form schemes; such geometric objects would bring together the methods and results of algebraic geometry with the methods and results of stable homotopy theory. Two multi-volume works (Toen and Vezzosi’s HAG series [12] and [13], and the DAG volumes by Lurie) have already been written, establishing the basic definitions and results to allow us to work with these “derived schemes.”

One way of viewing the situation is that all of algebraic geometry takes place, as is well-known, over the base scheme $\text{Spec } \mathbb{Z}$, but stable homotopy theory takes place over the base scheme “ $\text{Spec } S$,” which we have to put in quote marks, since it is not actually a scheme and not actually Spec of a commutative ring in the classical sense; here S is the sphere spectrum, the unit object of the tensor product (smash product) on the category of spectra. In this respect, stable homotopy theory is a kind of algebraic geometry which happens “under $\text{Spec } \mathbb{Z}$.” Actually, it is not the only kind of algebraic geometry “under $\text{Spec } \mathbb{Z}$,” as the various candidates for what should be considered “schemes over $\text{Spec } \mathbb{F}_1$ ” also sit under $\text{Spec } \mathbb{Z}$; see [11] for a useful perspective on this (although it should be remarked that the version of $\text{Spec } \mathbb{F}_1$ in that paper does not have all the properties that non-commutative geometers want to have in a good construction of $\text{Spec } \mathbb{F}_1$, and there are other candidates, e.g. Connes’ and Consani’s \mathbb{F}_1 -schemes from [2], or Borger’s schemes built from λ -rings, which are preferable to those in [11] for various reasons). One approximation to the category of \mathbb{F}_1 -schemes is the category of \mathbb{M}_0 -schemes, which sits under $\text{Spec } \mathbb{Z}$, and which we will be interested in, in this note.

Jack Morava has advocated the study of the morphism $\text{Spec } \mathbb{Z} \rightarrow \text{Spec } S$, and in particular, he has encouraged the development and application of the methods and results from

Hartshorne’s [4] to this morphism. From e.g. [4] one knows that, given a scheme X of finite type over a field k , one has the scheme morphism $X \xrightarrow{f} \text{Spec } k$, and one has a kind of duality functor on the derived category of quasicoherent \mathcal{O}_X -modules, represented by a chain complex $f^!k[0]$; here $k[0]$ is the chain complex of k -modules consisting of k in degree 0 and the trivial module in all other degrees, and $f^!$ is a derived right adjoint to Rf_* , the total right derived functor of the push-forward of modules along f . Clearly, the functor $\text{hom}_{\text{Mod}(k)}(-, k)$ is a duality functor, in the most straightforward sense, on the category of finite-dimensional k -vector spaces; so $f^!k[0]$ is “pulling back” the representing object k for this duality on k -modules to a representing object for a (derived) duality on (chain complexes of) \mathcal{O}_X -modules.

Now we want to do the same thing, but for the morphism $\text{Spec } \mathbb{Z} \xrightarrow{f} \text{Spec } S$. One knows that the sphere spectrum S represents the Spanier-Whitehead duality functor, and one would like to know what kind of “dualizing complex” of abelian groups one gets when one applies $f^!$ to the sphere spectrum, where $f^!$ is a derived right adjoint for Rf_* . In this note, we accomplish at least a large part of that task: we make precise (in two different, non-equivalent ways, each of which has some desirable properties) what we mean when we speak of “the morphism $\text{Spec } \mathbb{Z} \xrightarrow{f} \text{Spec } S$,” we describe the functors f_* , f^* , and $f^!$ associated to f , and we compute the homology groups of the dualizing complex $f^!S$, as well as some interesting properties of this dualizing complex. All of this essentially follows as consequences of results proved by others.

There is another interpretation of our results which is worth considering. One useful approximation to the category of \mathbb{F}_1 -schemes is the category of \mathbb{M}_0 -schemes (see e.g. [2] for definitions and basic properties of \mathbb{M}_0 -schemes, as well as their role in the construction of \mathbb{F}_1 -schemes): these objects resemble schemes but, by design, are given Zariski-locally by commutative monoids with zero elements, rather than by commutative rings. As we show in this note, the derived category of \mathbb{M}_0 is equivalent (as a triangulated category) to the stable homotopy category of spectra; and the description we obtain of $f^!S$, including its homology $H_*(f^!S)$, is also a description of $g^!\mathbb{M}_0[0]$ and its homology $H_*(g^!\mathbb{M}_0[0])$, where f is the morphism $\text{Spec } \mathbb{Z} \xrightarrow{f} \text{Spec } S$ and g is the morphism $\text{Spec } \mathbb{Z} \xrightarrow{g} \text{Spec } \mathbb{M}_0$.

Our description of $H_*(f^!S)$ bears a curious relation to the ideal class group from number theory. Let K/\mathbb{Q} be a finite field extension; then the ideal class group $\text{Cl}(K)$ of K admits the following description as a double quotient:

$$\text{Cl}(K) \cong G(\mathbb{A}_K^\infty) \backslash G(\mathbb{A}_K) / G(K),$$

where G is the multiplicative group scheme, \mathbb{A}_K is the adèle ring of K , and \mathbb{A}_K^∞ is the subring of \mathbb{A}_K consisting of the infinite adeles (i.e., the restricted direct product of the completions of K at Archimedean places). Let MO_K be the Moore spectrum of the ring of integers \mathcal{O}_K of K ; the sphere spectrum S is the special case $S \simeq MO_{\mathbb{Q}}$. We prove that one has the isomorphism

$$H_{-1}(f^!(MO_K)) \cong G(\mathbb{A}_K^\infty) \backslash G(\mathbb{A}_K) / G(K),$$

where now G is the additive, rather than multiplicative, group scheme. This isomorphism is natural in the choice of number field K . In this sense, the homology of the “dualizing complex” $f^!(MO_K)$ is a kind of additive analogue of the ideal class group of K . We also show that the homology groups of $f^!(MO_K)$ are trivial in all degrees $\neq -1$.

Everything we do in this note follows from theorems proved by others; we organize those results in a way that provides an answer to the questions asked by J. Morava, and,

we think, sheds a little bit more light on the relations between algebraic geometry, stable homotopy theory, and geometry over \mathbb{F}_1 .

2. \mathbb{M}_0 AND SPECTRA.

Proposition 2.1. *There exists a functor $\text{Rings} \xrightarrow{\beta^*} \text{Monoids}_0$, from the category of commutative rings to the category of commutative monoids equipped with a zero element; this functor β^* simply forgets the addition operation on the ring. This functor also has a left adjoint β , which sends a commutative monoid M with zero element $\mathbf{0}$ to the monoid ring $\mathbb{Z}[M]/(\mathbf{0})$. Let $p\text{Sets}$ be the closed symmetric monoidal category whose objects are pointed sets and whose monoidal product is given by the smash product*

$$X \wedge Y \cong (X \times Y)/(X \vee Y),$$

the Cartesian product of X and Y with the one-point union of X and Y collapsed to the basepoint. Let Ab be the closed symmetric monoidal category whose objects are abelian groups and whose monoidal product is given by the tensor product (over \mathbb{Z}). Let $\text{Ab} \xrightarrow{\alpha^*} p\text{Sets}$ be the forgetful functor and let $p\text{Sets} \xrightarrow{\alpha} \text{Ab}$ be its left adjoint, which sends a pointed set $(S, *)$ to the free abelian group $(\oplus_{s \in S} \mathbb{Z}) / (* = 0)$. Then, on applying Comm (i.e., passing to the categories of commutative monoid objects), we get the commutative diagram of categories and functors

$$\begin{array}{ccccc} \text{Comm}(p\text{Sets}) & \xrightarrow{\text{Comm}(\alpha)} & \text{Comm}(\text{Ab}) & \xrightarrow{\text{Comm}(\alpha^*)} & \text{Comm}(p\text{Sets}) \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ \text{Monoids}_0 & \xrightarrow{\beta} & \text{Rings} & \xrightarrow{\beta^*} & \text{Monoids}_0 \end{array}$$

where the functors marked \simeq are equivalences of categories.

Proof. Let M be a commutative monoid with zero, with multiplication map $M \times M \xrightarrow{\nabla} M$. Then the multiplication map factors uniquely through the smash product:

$$M \times M \xrightarrow{\quad} (M \times M)/(M \vee M) \xrightarrow{\quad} M,$$

∇

since $0 \bullet m = m \bullet 0 = 0$ for any $m \in M$. So to any commutative monoid with zero we associate a commutative monoid in $p\text{Sets}$; this construction is natural in M . Given a commutative monoid $N \wedge N \xrightarrow{\nabla} N$ in $p\text{Sets}$, one has the structure of a commutative monoid with zero on N , given by letting $n \bullet n'$ be the image of (n, n') under the composite

$$N \times N \longrightarrow N \wedge N \xrightarrow{\nabla} N.$$

This gives a monoid operation on N that only fails to be defined on pairs of the form $(0, n)$ and $(n, 0)$; for all such pairs we set $n \bullet 0 = 0 \bullet n = 0$. Hence the leftmost vertical map in diagram 2.1 is an equivalence of categories. That the central vertical map in diagram 2.1 is an equivalence of categories is well-known.

That $\text{Comm}(\alpha^*)$ is naturally isomorphic to the functor β^* follows immediately from inspection—they are both forgetful functors. If M is a commutative monoid with zero, then the underlying abelian group of $\beta(M)$ is $\alpha(M)$; so $\text{Comm}(\alpha^*)$ is naturally isomorphic to β^* . \square

Lemma 2.2. (Simplicial \mathbb{M}_0 -modules “are” pointed topological spaces.) *Let $ps\ Sets$ be the category of pointed simplicial sets (i.e., simplicial sets X equipped with a choice of point in $X[0]$), and let $sp\ Sets$ be the category of simplicial pointed sets (i.e., simplicial objects in the category of pointed sets). Then the forgetful functor*

$$sp\ Sets \rightarrow ps\ Sets$$

is an equivalence of categories.

Proof. A quasi-inverse for the forgetful functor is the functor sending a pointed simplicial set X with point $x \in X[0]$ to the simplicial pointed set whose basepoint in $X[n]$ is $(\sigma_0 \circ \cdots \circ \sigma_0)(x)$, the n -fold composite of the zeroth degeneracy maps applied to the basepoint in $X[0]$. All that needs to be checked is that this actually defines a simplicial pointed set, i.e., that the face and degeneracy maps respect the basepoints. Let x_n denote the basepoint $(\sigma_0 \circ \cdots \circ \sigma_0)(x) \in X[n]$, and suppose that $\delta_j x_i = x_{i-1}$ for all j and all $i < n$. Then

$$\begin{aligned} \delta_0 x_n &= (\delta_0 \circ \sigma_0)(x_{n-1}) \\ &= x_{n-1} \\ &= (\delta_1 \circ \sigma_0)(x_{n-1}) \\ &= \delta_1 x_n, \text{ and} \\ \delta_j x_n &= (\delta_j \circ \sigma_0)(x_{n-1}) \\ &= (\sigma_0 \circ \delta_{j-1})(x_{n-1}) \\ &= \sigma_0(x_{n-2}) \\ &= x_{n-1}, \end{aligned}$$

for any $j > 1$, by the simplicial identities and our hypotheses; by induction, the face maps δ_i respect the basepoints $\{x_n\}$. Now suppose that $\sigma_j x_i = x_{i-1}$ for all j and all $i < n$. Then

$$\begin{aligned} \sigma_j x_n &= \sigma_j \sigma_0 x_{n-1} \\ &= \sigma_0 \sigma_{j-1} x_{n-1} \\ &= \sigma_0 \sigma_0 \sigma_{j-2} x_{n-2} \\ &= \dots \\ &= (\sigma_0 \dots \sigma_0)(x_{n-j}) \\ &= x_{n+1} \end{aligned}$$

for any $j > 0$, by the simplicial identities and our hypotheses; so the degeneracy maps σ_i respect the basepoints $\{x_n\}$. \square

Now we will begin to speak of derived categories; in this section, we adopt the convention that all our complexes are chain complexes rather than cochain complexes, i.e., the differentials are of the form $C_i \rightarrow C_{i-1}$; they lower degree, rather than raising degree. By a “bounded-below” chain complex we mean a chain complex C such that there exists an integer M such that $C_m \cong 0$ for all $m < M$.

We would like to say something meaningful about the derived category $D(\mathbb{M}_0)$ of complexes of \mathbb{M}_0 -modules. We can’t speak of the category of chain complexes of \mathbb{M}_0 -modules, since \mathbb{M}_0 -modules are simply pointed sets, and the category $p\ Sets$ is not an abelian category; but by the Dold-Kan theorem, whenever \mathcal{C} is an abelian category, we have an Quillen equivalence between chain complexes, concentrated in nonnegative degrees, of objects in \mathcal{C} , with the projective model structure, and simplicial objects in \mathcal{C} with model structure coming from the standard model structure on simplicial sets. So we want a description of

the operation that takes a commutative ring R to the derived category $D(R)$ which does not use the fact that the category of R -modules is abelian, but perhaps uses the model structure on simplicial R -modules instead.

Here is how we arrive at such a description, which is largely inspired by Lurie's treatment of derived categories in [7] (although this description does not use Lurie's ∞ -categorical technology): let \mathcal{C} be any concrete category (i.e., a category equipped with a forgetful functor to the category of sets), not necessarily an abelian category. Then we have a model structure on $s\mathcal{C}$, the category of simplicial objects in \mathcal{C} , given by letting a morphism in $s\mathcal{C}$ be a fibration if the underlying map of simplicial sets is a fibration, and likewise for weak equivalences and cofibrations. Then let $S^+(s\mathcal{C})$ be the category of bounded-below spectrum objects in $s\mathcal{C}$: this is the category of functors $\mathbb{Z} \times \mathbb{Z} \xrightarrow{F} s\mathcal{C}$ satisfying the following conditions:

- (1) F preserves finite homotopy limits,
- (2) for any object $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ with $m \neq n$, the object $F(m, n)$ is weakly equivalent to a final object of $s\mathcal{C}$,
- (3) for any object $(m, m) \in \mathbb{Z} \times \mathbb{Z}$, the object $F(m, m)$ is a fibrant object of $s\mathcal{C}$, and
- (4) there exists some integer M such that $F(m, m)$ is weakly equivalent to a final object of $s\mathcal{C}$ for every $m < M$.

Here by \mathbb{Z} we mean the integers with their usual partial ordering, considered as a category. The practical significance of this definition is that any such functor F specifies a homotopy pullback square

$$\begin{array}{ccc} F(m, m) & \longrightarrow & \text{pt.} \\ \downarrow & & \downarrow \\ \text{pt.} & \longrightarrow & F(m+1, m+1) \end{array}$$

for each $m \in \mathbb{Z}$, i.e., a choice of weak equivalence $F(m, m) \simeq \Omega F(m+1, m+1)$, and $F(m, m)$ is trivial for $m \ll 0$; when \mathcal{C} is the category of pointed sets, after passage to the homotopy category, this definition recovers the typical homotopy category of bounded-below Ω -spectra. On the other hand, when \mathcal{C} is the category of R -modules, where R is some commutative ring, then the model category $s\mathcal{C}$ is (by the Dold-Kan theorem) equivalent to the category of chain complexes of R -modules concentrated in nonnegative degrees, with the projective model structure; fibrant objects in this model category are those in which every R -module in the complex is a projective R -module, while weak equivalences are quasiisomorphisms, so the category $S^+(s\mathcal{C})$ of bounded-below spectra in R -modules is, after passage to the homotopy category, the usual bounded-below derived category of R -modules; that is, $\text{Ho } S^+(s\text{Mod}(R)) \simeq D^+(R)$. Hence, if \mathcal{C} is a concrete category, we regard $\text{Ho } S^+(s\mathcal{C})$ as the bounded-below derived category of \mathcal{C} , and we may write $D^+(\mathcal{C})$ for it. If \mathcal{C} is a category of modules over some ring- or monoid-like object M then we may also write $D^+(M)$ for $D^+(\mathcal{C})$, a useful abuse of notation.

Proposition 2.3. *The bounded-below derived category $D^+(\mathbb{M}_0)$ of \mathbb{M}_0 -modules is equivalent to the stable homotopy category of bounded-below spectra.*

Proof. The category of \mathbb{M}_0 -modules is precisely the category $p\text{Sets}$ of pointed sets. The category $sp\text{Sets}$ of simplicial pointed sets is equivalent to the category $ps\text{Sets}$ of pointed simplicial sets, by Lemma 2.2, and this equivalence is easily seen to preserve the model structures; so $D^+(\mathbb{M}_0) \simeq \text{Ho } S^+(ps\text{Sets})$, the homotopy category of bounded-below spectrum objects in pointed Kan complexes, which is equivalent to the homotopy category of bounded-below Ω -spectra. \square

We also want to describe the effects of the functors $\text{Ab} \xrightarrow{\alpha^*} p\text{Sets}$ and $p\text{Sets} \xrightarrow{\alpha} \text{Ab}$ on the level of derived categories. These two functors jointly describe a morphism

$$\text{Spec } \mathbb{Z} \xrightarrow{g} \text{Spec } \mathbb{M}_0$$

such that $g_* = \alpha^*$ and $g^* = \alpha$; we want to understand the morphisms $D^+(\mathbb{Z}) \xrightarrow{g_*} D^+(\mathbb{M}_0)$ and $D^+(\mathbb{M}_0) \xrightarrow{g^*} D^+(\mathbb{Z})$ induced on the derived categories. Taking a simplicial pointed set, taking the free simplicial abelian group it generates, and identifying the subgroups generated by the basepoints to zero, this is (after using the Quillen equivalence of simplicial pointed sets with pointed CW complexes as well as the Dold-Kan Quillen equivalence of simplicial abelian groups with chain complexes of abelian groups concentrated in non-negative degrees, and passing to the homotopy category) equivalent to taking the pointed simplicial chain complex of a pointed simplicial complex; so, identifying $D^+(\mathbb{M}_0)$ with the bounded-below stable homotopy category, g_* sends a spectrum to (the quasi-equivalence class of) its singular chain complex. Similarly, given a chain complex of abelian groups C^\bullet concentrated in nonnegative degrees, using the Dold-Kan theorem to produce a simplicial abelian group from it, and forgetting the group structure to get a simplicial pointed set, this process yields the generalized Eilenberg-MacLane spectrum HC^* ; this describes g^* . As a result, we have

Proposition 2.4. *The derived adjoint functors*

$$\begin{aligned} D^+(\mathbb{Z}) &\xrightarrow{g_*} D^+(\mathbb{M}_0), \\ D^+(\mathbb{Z}) &\xleftarrow{g^*} D^+(\mathbb{M}_0) \end{aligned}$$

fit into the commutative diagram of triangulated categories

$$\begin{array}{ccccc} D^+(\mathbb{Z}) & \xrightarrow{g_*} & D^+(\mathbb{M}_0) & \xrightarrow{g^*} & D^+(\mathbb{Z}) \\ \downarrow \text{id} & & \downarrow \simeq & & \downarrow \text{id} \\ D^+(\mathbb{Z}) & \xrightarrow{H} & \text{Ho}(\text{Mod}(S)^+) & \xrightarrow{\text{sing}} & D^+(\mathbb{Z}), \end{array}$$

where $\text{Ho}(\text{Mod}(S)^+)$ is the stable homotopy category of bounded-below spectra, H is the Eilenberg-MacLane functor, and sing is the singular chain complex functor.

The derived adjunction of g^* and g_* ,

$$\mathbb{E}xt_{\mathbb{Z}}^0(\text{sing}(X)_\bullet, C_\bullet) \cong [X, HC_\bullet],$$

where $\mathbb{E}xt^0$ is the zeroth hyper-Ext functor, has as special cases the representability of cohomology with coefficients in abelian group A by the Eilenberg-MacLane spectrum HA (this happens when $C_\bullet \simeq A[0]$), and the fact that the homotopy groups of the generalized Eilenberg-MacLane spectrum of a complex C_\bullet recover the homology groups of C_\bullet , that is, $\pi_*(HC^\bullet) \cong H_*(C_\bullet)$ (this happens when X is homotopy equivalent to a sphere, since in that case the spectral sequence

$$E_2^{*,*} \cong \text{Ext}_{\mathbb{Z}}^*(H_*(S^n; \mathbb{Z}), H_*C_\bullet) \Rightarrow \mathbb{E}xt_{\mathbb{Z}}^*(\text{sing}(S^n)_\bullet, C_\bullet)$$

collapses on to a single line, $\text{Ext}_{\mathbb{Z}}^0(H_n(S^n; \mathbb{Z}), H_*(C_\bullet)) \cong H_*(C_\bullet)$, at E_2).

It is worth making some remarks about whether or not $\text{Spec } \mathbb{Z} \xrightarrow{g} \text{Spec } \mathbb{M}_0$ is an effective descent morphism. Given a pointed set S , the abelian group αS naturally has the structure map $\alpha S \rightarrow \alpha\alpha^*\alpha S$ of a coalgebra for the comonad $\alpha\alpha^*$. The question is whether the resulting functor from pointed sets to $\alpha\alpha^*$ -coalgebras is an equivalence of categories; if

so, then g is said to be an “effective descent morphism” (for this same story told in the classical setting of schemes, see [1]). By the weak Beck’s theorem (see [8]), one knows that g is an effective descent morphism if and only if α preserves and reflects equalizers; it is a very elementary exercise to verify that it indeed does. So g is an effective descent morphism. However, it is a much deeper matter to ask whether the derived adjunction g_*, g^* defines an effective descent morphism; naively, this would be asking whether one has an equivalence of stable homotopy categories between bounded-below spectra and g_*g^* -coalgebras; or, in other words, can one recover a bounded-below spectrum X from the coaction map $X \wedge H\mathbb{Z} \rightarrow X \wedge H\mathbb{Z} \wedge H\mathbb{Z}$. The answer is yes for connective X , since this is enough information for one to write down the rest of the $H\mathbb{Z}$ -Adams resolution of X , whose totalization recovers X . So the question of whether g_* and g^* define a derived effective descent morphism, this is really the question of what the $H\mathbb{Z}$ -Adams spectral sequence converges to, i.e., identifying the $H\mathbb{Z}$ -nilpotent completion functor. In general, questions about effective descent for morphisms of derived categories are really questions about E -nilpotent completion and generalized Adams spectral sequences. See [5] for more on these matters.

3. TRIANGULATED $\text{Spec } \mathbb{Z}$ AND $\text{Spec } S$.

Following Grothendieck, we work with categories of modules over rings or schemes, rather than the rings or schemes themselves; and furthermore, in this section, we want to explore the situation when we take the bounded-below derived category $D^+(\mathbb{Z})$, and not simply the category of \mathbb{Z} -modules, as the real object of interest when working with $\text{Spec } \mathbb{Z}$, so in this section, when we write $\text{Spec } S$, we mean the homotopy category $\text{Ho}(\text{Mod}(S)^+)$ of connective S -modules (that is, S -modules whose homotopy groups vanish in sufficiently low dimensions), of [3]; when we write $\text{Spec } \mathbb{Z}$, we have in mind the category $D^+(\mathbb{Z})$; and when we speak of a morphism

$$(3.1) \quad \text{Spec } \mathbb{Z} \xrightarrow{f} \text{Spec } S,$$

we mean a pair of functors

$$\begin{aligned} \text{Mod}^+(S) &\xrightarrow{f^*} D^+(\mathbb{Z}), \\ \text{Mod}^+(S) &\xleftarrow{f_*} D^+(\mathbb{Z}), \end{aligned}$$

with f_* fully faithful and with f^* left adjoint to f_* after passage to derived categories:

$$\mathbb{E}xt_{\mathbb{Z}}^*(f^*(X), C_\bullet) \cong [\Sigma^{-*}X, f_*C_\bullet].$$

Here $\mathbb{E}xt$ is the hyper-Ext functor. Clearly we want f_* to be the generalized Eilenberg-MacLane spectrum functor H , and as a consequence f^* has to be the singular chain complex functor. Of course, as a result of Shipley’s theorem (see [10]) identifying the homotopy category of bounded-below chain complexes of abelian groups with the homotopy category of bounded-below $H\mathbb{Z}$ -modules, the functor f^* should be thought of as base change to $H\mathbb{Z}$ —that is, the functor $X \mapsto X \wedge H\mathbb{Z}$.

Now we move on to Grothendieck duality. Given a map of schemes $X \xrightarrow{f} Y$, “Grothendieck duality for f ” is the existence of a functor $D^+(O_Y)_{\text{qcoh}} \xrightarrow{f^!} D^+(O_X)_{\text{qcoh}}$ which is right adjoint to the functor $D^+(O_X)_{\text{qcoh}} \xrightarrow{Rf_*} D^+(O_Y)_{\text{qcoh}}$ induced by f_* on the derived categories:

$$Rf_*R\text{hom}_{O_X}(C_\bullet, f^!D_\bullet) \cong R\text{hom}_{O_Y}(Rf_*C_\bullet, D_\bullet).$$

When $Y \cong \text{Spec } k$ for some field k and f is finite type, the functor $D^+(O_X) \xrightarrow{I} D^+(O_X)^{\text{op}}$ given by

$$IC_{\bullet} = R\text{hom}_{O_X}(C_{\bullet}, f^!(O_Y[0])),$$

(where $O_Y[0]$ is the complex consisting of O_Y in degree zero and the trivial module in all other degrees) satisfies the isomorphism $IIC_{\bullet} \cong C_{\bullet}$ for sufficiently small (“dualizable”) complexes C_{\bullet} ; this is the sense in which Grothendieck duality is really a “duality.”

We wish to describe such a situation for the morphism $\text{Spec } \mathbb{Z} \xrightarrow{f} \text{Spec } S$, i.e., we want a derived right adjoint for f_* . This takes the form of a functor

$$\text{Ho}(\text{Mod}(S)^+) \xrightarrow{f^!} D^+(Z)$$

from the homotopy category of bounded-below spectra to the bounded-below derived category of \mathbb{Z} -modules, such that we have the isomorphism

$$\text{Ext}_{\mathbb{Z}}^*(C_{\bullet}, f^!X) \cong [\Sigma^{-*}HC_{\bullet}, X],$$

natural in the choice of chain complex C_{\bullet} and spectrum X .

That $f^!$ exists can be shown in more than one way: in [3] it is shown that $f^!$ actually exists as a functor from S -modules to $H\mathbb{Z}$ -modules, as a right adjoint to the forgetful functor, and $f^!(X)$ is isomorphic to the function spectrum $F(H\mathbb{Z}, X)$; this fits with the definition of $f^!$ for a finite morphism of schemes, in [4]. Another method for proving the existence of $f^!$ is Neeman’s proof in [9], using the Brown representability theorem for triangulated categories: since the Eilenberg-MacLane spectrum functor $D(\mathbb{Z}) \xrightarrow{H} \text{Ho}(S)$ is a functor defined on a compactly generated triangulated category, taking values in a triangulated category, and which preserves coproducts, the existence of a right adjoint for H is automatic (and indeed, is even defined on the unbounded derived category). Neeman used this argument to produce $f^!$ for a map of schemes f but it works equally well to produce $f^!$ for the morphism $\text{Spec } \mathbb{Z} \xrightarrow{f} \text{Spec } S$. Of course, by the argument of the previous section, the functors f_* and f^* (which we regard as defining a morphism $\text{Spec } \mathbb{Z} \xrightarrow{f} \text{Spec } S$) coincide with the derived adjoint functors g_* and g^* between $D^+(\mathbb{Z})$ and $D^+(\mathbb{M}_0)$; so the Grothendieck duality we describe here for f is equally well a Grothendieck duality for $\text{Spec } \mathbb{Z} \xrightarrow{g} \text{Spec } \mathbb{M}_0$.

Proposition 3.1. *Let f be the morphism*

$$\text{Spec } \mathbb{Z} \xrightarrow{f} \text{Spec } S,$$

with the definition we have given immediately following morphism 3.1. Then the homology groups of the chain complex $f^!(S)$ are concentrated in a single degree. Specifically:

$$\begin{aligned} H_i(f^!(S)) &\cong [\Sigma^i H\mathbb{Z}, S] \\ &\cong \begin{cases} \text{Ext}_{\mathbb{Z}}^1(\mathbb{Q}, \mathbb{Z}) & \text{if } i = -1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The complex $f^!(S)$ has trivial homology in all degrees when pulled back to any closed point of $\text{Spec } \mathbb{Z}$. In other words, if j is the morphism

$$\text{Spec } \mathbb{F}_p \xrightarrow{j} \text{Spec } S,$$

then

$$\begin{aligned} H_i(j^!(S)) &\cong [\Sigma^i H\mathbb{F}_p, S] \\ &\cong 0. \end{aligned}$$

Proof. Recall the theorem of Lin [6]: for A a cyclic abelian group and X a finite cell complex, we have the isomorphism

$$[\Sigma^i HA, X] \cong \text{Ext}_{\mathbb{Z}}^1(A \otimes_{\mathbb{Z}} \mathbb{Q}, H_{i+1}(X; \mathbb{Z})).$$

Since $f^!$ is, by definition, a derived right adjoint for $f_* = H$, we have the isomorphism

$$\mathbb{E}xt_{\mathbb{Z}}^i(C_{\bullet}, f^!S) \cong [\Sigma^{-i}HC_{\bullet}, S].$$

Let $\mathbb{Z}[0]$ be the complex with \mathbb{Z} in degree 0 and the trivial abelian group in all other degrees. Then the spectral sequence

$$\text{Ext}_{\mathbb{Z}}^s(\mathbb{Z}, H_t f^!S) \Rightarrow \mathbb{E}xt_{\mathbb{Z}}^{s-t}(\mathbb{Z}[0], f^!S)$$

collapses on to the $s = 0$ line, so we get isomorphisms

$$\begin{aligned} H_t f^!S &\cong \text{hom}_{\mathbb{Z}}(\mathbb{Z}, H_t f^!S) \\ &\cong \mathbb{E}xt_{\mathbb{Z}}^{-t}(\mathbb{Z}[0], f^!S) \\ &\cong [\Sigma^t H\mathbb{Z}, S], \end{aligned}$$

which, by Lin's theorem, is $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Q}, \mathbb{Z})$ when $t = -1$, and zero otherwise.

The spectral sequence

$$\text{Ext}_{\mathbb{F}_p}^s(\mathbb{F}_p, H_t j^!S) \Rightarrow \mathbb{E}xt_{\mathbb{F}_p}^{s-t}(\mathbb{F}_p[0], j^!S)$$

also collapses on to the $s = 0$ line, so we get isomorphisms

$$\begin{aligned} H_t j^!S &\cong \text{hom}_{\mathbb{F}_p}(\mathbb{F}_p, H_t j^!S) \\ &\cong \mathbb{E}xt_{\mathbb{F}_p}^{-t}(\mathbb{F}_p[0], j^!S) \\ &\cong [\Sigma^t H\mathbb{F}_p, S], \end{aligned}$$

which is zero for all t , again by Lin's theorem. \square

We now give a description of $H_{-1}(f^!S)$ which is interesting for number-theoretic reasons. If K is a number field and G is an algebraic group, one frequently encounters the groups

$$o \backslash G(\mathbb{A}_K) / G(K),$$

where \mathbb{A}_K is the adèle ring of K (i.e., the restricted direct product of the completions of K at all its places) and o is a compact open subgroup of $G(\mathbb{A}_K^{\infty})$, the evaluation of G at the infinite adèle ring of K . One recovers the complex points of Shimura varieties in this way; in particular, when $G \cong GL_1$, the multiplicative group scheme, and $o = G(\mathbb{A}_K^{\infty})$, then the group $o \backslash G(\mathbb{A}_K) / G(K)$ is isomorphic to the ideal class group of K . We identify the nonvanishing homology group $H_{-1}(f^!S)$ of the Grothendieck dualizing complex arising from the sphere spectrum as an additive analogue of the ideal class group:

Proposition 3.2. *Let \mathcal{NF} be the category of finite extensions of \mathbb{Q} , and let Ab be the category of abelian groups. Then the functor*

$$\begin{aligned} \mathcal{NF} &\rightarrow \text{Ab} \\ K &\mapsto G(\mathbb{A}_K^{\infty}) \backslash G(\mathbb{A}_K) / G(K), \end{aligned}$$

where G is the additive group scheme, \mathbb{A}_K is the adèle ring of K , and \mathbb{A}_K^{∞} is the ring of infinite adèles of K (i.e., the restricted direct product of the completions of the K at its Archimedean places), is naturally isomorphic to the functor

$$\begin{aligned} \mathcal{NF} &\rightarrow \text{Ab} \\ K &\mapsto H_{-1}(f^!(M_{O_K})), \end{aligned}$$

where MO_K is the Moore spectrum of the ring of integers o_K of K .

Proof. From the proof of Prop. 3.1 we know that

$$\begin{aligned} H_{-1}(f^1(MO_K)) &\cong \text{Ext}_{\mathbb{Z}}^1(\mathbb{Q}, H_0(MO_K; \mathbb{Z})) \\ &\cong \text{Ext}_{\mathbb{Z}}^1(\mathbb{Q}, o_K), \end{aligned}$$

and this isomorphism is natural in K ; we need to show that $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Q}, o_K)$ is naturally isomorphic to $G(\mathbb{A}_K^\infty) \backslash G(\mathbb{A}_K) / G(K)$, where G is the additive group scheme. Our method of proof is based on an unpublished proof of J. Michael Boardman's. We have the short exact sequence of abelian groups

$$0 \rightarrow o_K \rightarrow K \rightarrow K/o_K \rightarrow 0$$

and, after applying the functor $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Q}, -)$ to this short exact sequence, we get the short exact sequence

$$0 \rightarrow \text{hom}_{\mathbb{Z}}(\mathbb{Q}, K) \rightarrow \text{hom}_{\mathbb{Z}}(\mathbb{Q}, K/o_K) \rightarrow \text{Ext}_{\mathbb{Z}}^1(\mathbb{Q}, o_K) \rightarrow 0,$$

since $\text{hom}_{\mathbb{Z}}(\mathbb{Q}, o_K) \cong \text{Ext}_{\mathbb{Z}}^1(\mathbb{Q}, K) \cong 0$. Now the torsion group K/o_K decomposes as the direct sum $K/o_K \cong \bigoplus_p (K/o_K)_{(p)}$ of its localizations at the primes of \mathbb{Z} , so we have isomorphisms

$$\begin{aligned} \text{hom}_{\mathbb{Z}}(\mathbb{Q}, K/o_K) &\cong \text{hom}_{\mathbb{Z}}(\mathbb{Q}, \bigoplus_p (K/o_K)_{(p)}) \\ &\subseteq \text{hom}_{\mathbb{Z}}(\mathbb{Q}, \prod_p (K/o_K)_{(p)}) \\ &\cong \prod_p \text{hom}_{\mathbb{Z}}(\mathbb{Q}, (K/o_K)_{(p)}) \\ &\cong \prod_p \text{hom}_{\mathbb{Z}}(\mathbb{Z}_{(p)}, (K/o_K)_{(p)}) \\ &\cong \prod_p \text{hom}_{\mathbb{Z}}\left(\text{colim}\left(\mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \dots\right), (K/o_K)_{(p)}\right) \\ &\cong \prod_p \lim\left(\dots \xrightarrow{p} \text{hom}_{\mathbb{Z}}(\mathbb{Z}, (K/o_K)_{(p)}) \xrightarrow{p} \text{hom}_{\mathbb{Z}}(\mathbb{Z}, (K/o_K)_{(p)})\right) \\ &\cong \prod_p \lim\left(\dots \xrightarrow{p} (K/o_K)_{(p)} \xrightarrow{p} (K/o_K)_{(p)}\right) \\ &\cong \prod_p K_p, \end{aligned}$$

where the isomorphism $\text{hom}_{\mathbb{Z}}(\mathbb{Q}, (K/o_K)_{(p)}) \cong \text{hom}_{\mathbb{Z}}(\mathbb{Z}_{(p)}, (K/o_K)_{(p)})$ holds because $(K/o_K)_{(p)}$ is uniquely m -divisible for every integer m prime to p , so every morphism $\mathbb{Z}_{(p)} \rightarrow (K/o_K)_{(p)}$ extends uniquely to a morphism $\mathbb{Q} \rightarrow (K/o_K)_{(p)}$. Morphisms $\mathbb{Q} \rightarrow (K/o_K)_{(p)}$ which correspond to elements in $(\hat{o}_K)_p \subseteq K_p$ are precisely those morphisms which factor through the quotient map $\mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z}$, so we get Diagram 1, a commutative diagram with exact rows.

Morphisms $\mathbb{Q} \rightarrow \prod_p (K/o_K)_{(p)}$ which factor through the inclusion $\bigoplus_p (K/o_K)_{(p)} \hookrightarrow \prod_p (K/o_K)_{(p)}$ are the morphisms such that the composite of the p th component map $\mathbb{Q} \rightarrow (K/o_K)_{(p)}$ with the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is zero for all but finitely many p , i.e., these are the morphisms $\mathbb{Q} \rightarrow \prod_p (K/o_K)_{(p)}$ such that the p th component map $\mathbb{Q} \rightarrow (K/o_K)_{(p)}$ factors through $\mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z}$ for all but finitely many p ; these are

Diagram 1.

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & \mathrm{hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, (K/\mathcal{O}_K)_{(p)}) & \longrightarrow & \mathrm{hom}_{\mathbb{Z}}(\mathbb{Q}, (K/\mathcal{O}_K)_{(p)}) & \longrightarrow & \mathrm{hom}_{\mathbb{Z}}(\mathbb{Z}, (K/\mathcal{O}_K)_{(p)}) & \longrightarrow & 0 \\
 \downarrow & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \\
 0 & \longrightarrow & \mathrm{hom}_{\mathbb{Z}}(\mathrm{colim}_n \mathbb{Z}/p^n \mathbb{Z}, (K/\mathcal{O}_K)_{(p)}) & \longrightarrow & \mathrm{hom}_{\mathbb{Z}}(\mathrm{colim}_n \mathbb{Z}, (K/\mathcal{O}_K)_{(p)}) & \longrightarrow & (K/\mathcal{O}_K)_{(p)} & \longrightarrow & 0 \\
 \downarrow & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \\
 0 & \longrightarrow & \lim_n \mathrm{hom}_{\mathbb{Z}}(\mathbb{Z}/p^n, (K/\mathcal{O}_K)_{(p)}) & \longrightarrow & \lim \left(\cdots \xrightarrow{p} (K/\mathcal{O}_K)_{(p)} \xrightarrow{p} (K/\mathcal{O}_K)_{(p)} \right) & \longrightarrow & (K/\mathcal{O}_K)_{(p)} & \longrightarrow & 0 \\
 \downarrow & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \\
 0 & \longrightarrow & (\hat{\mathcal{O}}_K)_p & \longrightarrow & K_p & \longrightarrow & (K/\mathcal{O}_K)_{(p)} & \longrightarrow & 0.
 \end{array}$$

the elements of $\mathrm{hom}_{\mathbb{Z}}(\mathbb{Q}, \prod_p (K/o_K)_{(p)}) \cong \prod_p K_p$ which correspond to elements of $\prod_p K_p$ whose p th component is in $(\hat{o}_K)_p \subseteq K_p$ for all but finitely many p . Hence we have the commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & \mathrm{hom}_{\mathbb{Z}}(\mathbb{Q}, K) & \longrightarrow & \mathrm{hom}_{\mathbb{Z}}(\mathbb{Q}, K/o_K) & \longrightarrow & \mathrm{Ext}_{\mathbb{Z}}^1(\mathbb{Q}, o_K) & \longrightarrow & 0 \\
 \downarrow & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \\
 0 & \longrightarrow & K & \longrightarrow & \mathbb{A}_K^{fin} & \longrightarrow & \mathbb{A}_K^{fin}/K & \longrightarrow & 0,
 \end{array}$$

where by \mathbb{A}_K^{fin} we mean the finite adeles of K , i.e., $\mathbb{A}_K^{fin} \cong \mathbb{A}_K/\mathbb{A}_K^\infty$. \square

4. $\mathrm{Spec} \mathbb{Z}$ AND $\mathrm{Spec} S$, AFTER TOËN-VAQUIÉ.

In this section we change gears entirely, and try a different approach to the construction and study of a morphism $\mathrm{Spec} \mathbb{Z} \rightarrow \mathrm{Spec} S$: we will see what happens when we take $\mathrm{Spec} \mathbb{Z}$ to mean the category of abelian groups, and we take $\mathrm{Spec} S$ to be the category of connective S -modules, that is, the category of S -modules X such that $\pi_i(X)$ is trivial for $i < 0$. In this setup, unlike that of the previous section in which we were really concerned with triangulated categories, we do not have a desuspension operation defined on either of the categories under consideration; but there are some ways in which this setup is a desirable one. For one thing, we will find that the base-change functor from $\mathrm{Spec} S$ to $\mathrm{Spec} \mathbb{Z}$ is, in this setup, the zeroth homotopy group functor π_0 , rather than the singular chain complex functor as in the previous section; this is desirable because, in derived algebraic geometry, one would like to think of a derived scheme X^{der} as consisting of an ordinary scheme X together with a homotopy-Cartesian presheaf \mathcal{E} of E_∞ -ring spectra on the local Zariski (or perhaps étale, or syntomic, or...) site of X with the property that $\pi_0(\mathcal{E})$ recovers the structure ring sheaf of that site. From this point of view, we have a commutative square

$$\begin{array}{ccc}
 X & \longrightarrow & X^{\mathrm{der}} \\
 \downarrow & & \downarrow \\
 \mathrm{Spec} \mathbb{Z} & \longrightarrow & \mathrm{Spec} S
 \end{array}$$

which is a pullback square, in the sense that X is obtained by applying π_0 to X^{der} , i.e., X is obtained by base-change along $\mathrm{Spec} \mathbb{Z} \rightarrow \mathrm{Spec} S$ (of course this cannot be a pullback square in a categorical sense unless we actually describe a category in which each of these objects lives; this also is not a pullback square in the “everything is triangulated” setup of the previous section in this note). These observations we make in this note are by no means a replacement for the well-developed theory of HAG, due to Toën and Vezzosi (primarily in [12] and [13]), and DAG, due to Lurie (in a continuing series, currently at six volumes; the first is [7]); for example, our situation (which involves only connective spectra, not periodic spectra) is still weaker than the technology required to construct the derived moduli stack of derived elliptic curves, and the global sections of its “ ∞ -sheafification” (Joyal-Jardine fibrant replacement), (a localization of) the spectrum tmf of topological modular forms.

There is one other sense in which the present setup (in which $\mathrm{Spec} \mathbb{Z}$ is taken as the category of \mathbb{Z} -modules, and not some category of chain complexes of \mathbb{Z} -modules) may be more desirable than one in which we are really concerned with triangulated categories: the framework of Toën and Vaquié for “relative algebraic geometry,” from the paper [11], takes as input a complete, co-complete, closed symmetric monoidal category \mathcal{C} , and produces a category of “ \mathcal{C} -schemes” such that the category \mathcal{C} winds up being precisely the category

of quasicohherent/Cartesian modules over the terminal object in the category of \mathcal{C} -schemes. When applied to the category of abelian groups, with monoidal product the tensor product, Toën-Vaquié's framework produces the classical category of schemes; and when applied to (a category Quillen-equivalent to) the category of S -modules, with monoidal product the smash product, Toën-Vaquié's framework produces a useful category of “derived schemes” or “spectral schemes.” One can also put the category of chain complexes of abelian groups through the machinery of Toën-Vaquié, but then one gets a category of derived schemes over $\text{Spec } H\mathbb{Z}$ (which was essentially the way we treated $\text{Spec } \mathbb{Z}$ in the previous section), not the classical category of schemes.

In most other ways the approach taken in the previous section is preferable to that of this section: the connectivity assumption made on spectra in the previous section is weaker, and more importantly, the adjunction of the previous section really is the “correct” notion of a derived adjunction.

First, we want to know that there really is a morphism

$$\text{Spec } \mathbb{Z} \xrightarrow{i} \text{Spec } S$$

in our present setup; in other words, we want to know that there is a faithful functor

$$\text{Mod}(\mathbb{Z}) \xrightarrow{i_*} \text{Mod}(S)^{\geq 0}$$

where $\text{Mod}(S)^{\geq 0}$ is the category of connective S -modules, and we want a functor

$$\text{Mod}(\mathbb{Z}) \xleftarrow{i^*} \text{Mod}(S)^{\geq 0}$$

which is well-defined on the homotopy category $\text{Ho}(\text{Mod}(S)^{\geq 0})$ and such that we have a homotopy adjunction between them, i.e., an isomorphism

$$\text{hom}_{\text{Mod}(\mathbb{Z})}(i^*X, A) \cong [X, i_*A]$$

natural in the choice of spectrum X and abelian group A . As usual we want i_* to be the Eilenberg-MacLane functor H , since it is the most obvious and natural embedding of the category of abelian groups into the category of spectra. We claim that i_* has homotopy left adjoint π_0 , the zeroth homotopy group functor. Indeed, we have the “trivial” Adams spectral sequence

$$\text{Ext}_{\text{grMod}(\pi_*(S))}^s(\Sigma^t \pi_*(X), A) \Rightarrow [\Sigma^{t-s} X, HA],$$

where $\text{Ext}_{\text{grMod}(\pi_*(S))}$ is the derived functor of hom in the category of graded modules over the graded ring $\pi_*(S)$ of stable homotopy groups of spheres. Since X and HA are both connective, there is no room for differentials to come from or hit the class in bidegree $(0, 0)$, so we have isomorphisms

$$\begin{aligned} [X, HA] &\cong \text{Ext}_{\text{grMod}(\pi_*(S))}^0(\pi_*(X), A) \\ &\cong \text{hom}_{\text{grMod}(\pi_*(S))}(\pi_*(X), A) \\ &\cong \text{hom}_{\text{grMod}(\pi_0(S))}(\pi_0(X), A) \\ &\cong \text{hom}_{\text{Mod}(\mathbb{Z})}(\pi_0(X), A). \end{aligned}$$

So π_0 is homotopy left adjoint to H .

Now we proceed to Grothendieck duality. We want a homotopy right adjoint to $i_* = H$, i.e., we want a functor $i^!$ with an isomorphism

$$\text{hom}_{\mathbb{Z}}(A, i^!X) \cong [HA, X]$$

natural in the choice of connective spectrum X and abelian group A . As a consequence of the results of the previous section, such a functor exists and is simply the zeroth homology of the complex $f^!X$:

$$i^!X \cong H_0 f^!X,$$

and by Lin's theorem, $i^!$ is given on finite cell complexes X by

$$\begin{aligned} i^!X &\cong \operatorname{hom}_{\mathbb{Z}}(\mathbb{Z}, i^!X) \\ &\cong [HZ, X] \\ &\cong \operatorname{Ext}_{\mathbb{Z}}^1(\mathbb{Q}, H_1(X, \mathbb{Z})). \end{aligned}$$

Hence $i^!$ vanishes on finite cell complexes with finite first integral homology group H_1 , e.g. the sphere spectrum.

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