SMOOTH LIE GROUPS ACTIONS ARE PARAMETRIZED DIFFEOLOGICAL SUBGROUPS

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ABSTRACT. We show that every effective smooth action of a Lie group G on a manifold M is a diffeomorphism from G onto its image in $\mathrm{Diff}(M)$, where the image is equipped with the subset diffeology of the functional diffeology.

INTRODUCTION

The group $\mathrm{Diff}(M)$ of diffeomorphisms of a manifold M, and subgroups that preserve certain geometric structures (for example, the group of symplectomorphisms of a symplectic manifold), are important objects of research. Such groups are often considered heuristically as infinite dimensional Lie groups, whose "finite dimensional Lie subgroups" are taken to be the images of smooth finite dimensional Lie group actions.

There are several rigorous approaches to the structure of $\mathrm{Diff}(M)$. When M is compact, $\mathrm{Diff}(M)$ is a Fréchet Lie group [Omo74]. An approach through Frölicher structures is given in [KrMi97].

In this paper we take a simpler rigorous approach: the diffeological approach. Roughly, a diffeology on a set X declares which maps from $\mathbf{R}^n$ to X are smooth, for all $n \in \mathbf{N}$. See Section 1 for relevant definitions. If M is a manifold, there is a natural diffeology on $\mathrm{Diff}(M)$, with respect to which the notion "Lie subgroup of $\mathrm{Diff}(M)$" is unambiguous: it is a subgroup that, equipped with the subset diffeology, is a manifold. The usual notion of an effective smooth action of a Lie group G on a manifold M is the same thing as a group monomorphism $G \to \mathrm{Diff}(M)$ that is smooth in the diffeological sense. *A priori* it is not obvious that in this situation the manifold structure on G is induced from the ambient diffeology of $\mathrm{Diff}(M)$, i.e., that the smooth monomorphism is necessarily a diffeomorphism with its image. The purpose of this paper is to prove this fact. Thus, a Lie subgroup of $\mathrm{Diff}(M)$ really is the same thing as the image in $\mathrm{Diff}(M)$ of a smooth Lie group action on M. We give a formal statement in diffeological terms:

THEOREM — *Let* M *be a manifold. Equip* $\mathrm{Diff}(M)$ *with the functional diffeology. Then every smooth monomorphism from a Lie group to* $\mathrm{Diff}(M)$ *is an induction.*

We now rephrase this statement in more common terms. Let $\rho\colon G \to \mathrm{Diff}(M)$ be an effective smooth action of a Lie group G on a manifold M. That is, ρ is a one-to-one group homomorphism, and the map $(g, m) \mapsto \rho(g)(m)$ from $G \times M$ to M is smooth. Let D be an open subset of $\mathbf{R}^k$ for some k. Let $r \mapsto f_r$ be a map from D to the image of G in $\mathrm{Diff}(M)$. Let $r \mapsto g_r$ be the map from D to G such that $\rho(g_r) = f_r$ for all $r \in D$. Assume that the map $(r, m) \mapsto f_r(m)$, from $D \times M$ to M, is smooth. Then the map $r \mapsto g_r$, from D to G, is smooth.

For example, this theorem immediately implies that the ordinary smooth structure of the linear group $\mathrm{GL}(n, \mathbf{R})$ is induced by the functional diffeology of $\mathrm{Diff}(\mathbf{R}^n)$. This fact was proved in [Igl85] but not as a consequence of a more general theorem. More generally, most of the classical Lie groups – the orthogonal groups, the unitary groups, the symplectic groups etc. – are Lie subgroups (in the diffeological sense) of groups of diffeomorphisms of vector spaces.

Sentences such as "What are, modulo conjugacy, the maximal subtori of a group of symplectomorphisms?", as considered in [KKP07], have an intrinsic diffeological meaning. The above theorem shows that, fortunately, this meaning coincides with the usual interpretation of such sentences.

In Section 2, we prove the above theorem. In Section 3, we give a simple sufficient condition for which the induction is an embedding; this condition holds for the classical Lie groups that are listed above.

Throughout this paper, all manifolds, and in particular all Lie groups, are assumed to be Hausdorff, second countable, and finite dimensional.

1. THE DIFFEOLOGICAL FRAMEWORK

We review here definitions and constructions in diffeology that we will need. More details can be found in [PIZ05-10].

Let X be a set. A *parametrization* on X is a map from an open subset of $\mathbf{R}^n$, for some $n \in \mathbf{N}$, to X. A *diffeology* on X is a set of parametrizations, whose elements are called *plots*, that satisfies three axioms: the *covering axiom* – constant maps are plots; the *locality axiom* – being a plot is a local condition: given $f\colon U \to X$, if every point in U has a neighbourhood V such that $f|_V$ is a plot, then f is a plot; and the

smooth compatibility axiom – precomposition of a plot with a smooth map (in the usual sense) is a plot. A *diffeological space* is a set equipped with a diffeology.

Let M be a manifold. There exists on M a natural diffeology: the plots consist of those parametrizations that are smooth in the usual sense. On the group $\mathrm{Diff}(M)$ of diffeomorphisms of M there also exists a natural diffeology, called the *functional diffeology*: the plots consist of those parametrizations $U \to \mathrm{Diff}(M)$, $r \mapsto f_r$, for which the map $(r, m) \mapsto f_r(m)$ from $U \times M$ to M is smooth.

The *discrete diffeology* on a set X is the diffeology for which only the locally constant parametrizations are plots. A space with the discrete diffeology is called *discrete*.

A *smooth map* between two diffeological spaces is a map whose precomposition with every plot of the source is a plot of the target. Diffeological spaces and their smooth maps form a category. The isomorphisms of this category are called *diffeomorphisms*. Thus, a diffeomorphism is a bijective map that is smooth and whose inverse is smooth.

On any subset A of a diffeological space X, there exists a natural diffeology, called the *subset diffeology*. Its plots are the plots of X with values in A. If this diffeology is discrete, A is called a *discrete subspace*. In particular, any countable subset of a manifold inherits the discrete diffeology, for example $\mathbf{Q} \subset \mathbf{R}$.

An *induction* is an injective smooth map between diffeological spaces that is a diffeomorphism onto its image, where the image is equipped with the subset diffeology. For manifolds, every induction is an immersion, and every immersion is locally an induction, but an immersion need not be an induction, even if it is injective. For example, the parametrization $t \mapsto (\sin(t), \sin(2t))$ from $(-\pi, +\pi)$ to $\mathbf{R}^2$, of the lemniscate (figure eight), is an injective immersion but it is not an induction.

A diffeology on X determines a topology on X, called the *D-topology*, in which a subset of X is open if and only if its preimage by every plot is open. Such a subset is called *D-open*. If a subset of X is D-open, then its subset topology coincides with the D-topology of its subset diffeology. The quotient $\mathbf{R}/(\mathbf{Z} + \alpha\mathbf{Z})$, for α irrational, is an example of a highly nontrivial[1] diffeological space whose D-topology is trivial.

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[1] $\mathbf{R}/(\mathbf{Z} + \beta\mathbf{Z})$ is diffeomorphic to $\mathbf{R}/(\mathbf{Z} + \alpha\mathbf{Z})$ if and only if there exists $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2, \mathbf{Z})$ such that $\beta = \frac{a\alpha + b}{c\alpha + d}$ [DonIgl85].

On a manifold, the D-topology coincides with the classical topology. A *manifold* can be redefined as a diffeological space in which every point is contained in a D-open set that is diffeomorphic to an open subset of some $\mathbf{R}^n$. A map of manifolds is smooth in the usual sense if and only if it is smooth in the diffeological sense. A *submanifold* is just a subspace which is a manifold.

The D-topology distinguishes between inductions: an induction that is also a homeomorphism with its image, when the image is taken with the subset topology induced from the D-topology of the ambient space, is called an *embedding*. Not every induction is an embedding. For example, the irrational line $\mathbf{R} \to \mathbf{R}^2/\mathbf{Z}^2$ given by $x \mapsto [x, \alpha x]$ for α irrational is an induction but is not an embedding.

The quotient of a diffeological space X by any equivalence relation $\mathcal{R}$ has a natural diffeology, called the *quotient diffeology*. Its plots are those parametrizations that can be locally lifted to plots of X along the projection $\pi\colon X \to X/\mathcal{R}$. Thus, $f\colon U \to X/\mathcal{R}$ is a plot if and only if for every point $u \in U$ there exists a neighbourhood V of u in U and a plot $\tilde{f}\colon V \to X$ such that $f|_V = \pi \circ \tilde{f}$. A *strict map* $f\colon X \to Y$ is a smooth map that identifies the quotient X/f with the image $f(X)$, as diffeological spaces [Don84].

A *diffeological group* G is a group equipped with a diffeology such that the inversion and the multiplication are smooth. If M is a manifold then $\mathrm{Diff}(M)$, with its functional diffeology, is a diffeological group. A Lie group is the same thing as a diffeological group that is also a manifold. A *Lie subgroup* of $\mathrm{Diff}(M)$ is a subgroup that, with the subset diffeology, is a Lie group.

A *smooth action* of a diffeological group G on manifold M is a smooth homomorphism $\rho\colon G \to \mathrm{Diff}(M)$, where $\mathrm{Diff}(M)$ is equipped with the functional diffeology. When G is a Lie group, this is equivalent to the usual definition of a smooth G action on M: the map $(g, m) \mapsto \rho(g)(m)$ is required to be smooth. The action is *effective* if the homomorphism ρ is injective. The tautological action of $\mathrm{Diff}(M)$ on M is smooth and effective.

2. Smooth Lie group actions on manifolds

In the following paragraphs, we use the following notation. G is a Lie group, M is a manifold, and $\rho\colon G \to \mathrm{Diff}(M)$ is a smooth action of G on M. We denote the stabilizer of m in G by G_m. We denote $\rho(g)(m)$

by $g \cdot m$, and we denote the G-orbit of m by $G \cdot m$. For every point m of M, we denote by $m_\sharp \colon G \to M$ the *orbit map* $m_\sharp(g) := g \cdot m$.

The orbit map $m_\sharp$ descends to a smooth map $\overline{m}_\sharp \colon G/G_m \to M$. Thus, $m_\sharp = \overline{m}_\sharp \circ \pi$, where $\pi \colon G \to G/G_m$ is the projection. Moreover, because the stablizer G_m of m is a closed subgroup of G, this stabilizer G_m is an embedded submanifold of G, the quotient G/G_m is a manifold, and the projection $\pi \colon G \to G/G_m$ is a submersion. Moreover, π is a (locally trivial) principal bundle with G_m as structure group acting on G by $G_m \times G \ni (h, g) \mapsto gh^{-1}$; see for example [KobNom63].

1. THE ORBIT MAP.   *Let M be a manifold, let G be a Lie group, and let $\rho \colon G \to \mathrm{Diff}(M)$ be a smooth action of G on M. Let m be a point of M. Then the orbit map $m_\sharp \colon G \to M$ is a strict map.*

NOTE 1. This statement means that the map $\overline{m}_\sharp$ is a diffeomorphism from G/G_m to $G \cdot m$, where the former is equipped with the quotient diffeology of G and the latter is equipped with the subset diffeology induced from M. In other words, $\overline{m}_\sharp$ is an induction, which implies in particular that the orbit $G \cdot m$ is a submanifold.

NOTE 2. The induction $\overline{m}_\sharp \colon G/G_m \to M$ is not necessarily an embedding. For example, consider the group $G = \mathbf{R}$, the manifold $M = \mathbf{R}^2/\mathbf{Z}^2 \times \mathbf{R}$, and the action $\rho(t)([x, y], \alpha) = ([x + t, y + \alpha t], \alpha)$. Let $m = ([x, y], \alpha)$. If α is rational, then $m_\sharp$ is an embedding, and if α is irrational, then $m_\sharp$ is an induction but not an embedding.

Proof. Because the quotient map $\pi \colon G \to G/G_m$ admits smooth local sections, the map $\overline{m}_\sharp \colon G/G_m \to M$ is a smooth injection. Let σ be a smooth section of π, let $\mathcal{O}$ be its image in G, and assume that $\mathbf{1}_G \in \mathcal{O}$. Then $\mathcal{O}$ is transverse to G_m at the identity $\mathbf{1}_G$.

The differential of the orbit map $m_\sharp \colon g \mapsto g \cdot m$, computed at the identity $\mathbf{1}_G$, is the map $\xi \mapsto \xi_M(m)$, where ξ is an element of the Lie algebra $\mathcal{G}$ of G and ξ_M denotes the fundamental vector field on M associated to ξ. The kernel of this map is equal to the Lie algebra $\mathcal{G}_m$ of G_m:

$$\mathcal{G}_m = T_{\mathbf{1}_G} G_m = \{\xi \in \mathcal{G} \mid \xi_M(m) = 0\}.$$

The image of this map is the subspace $\mathcal{G} \cdot m = \{\xi_M(m) \mid \xi \in \mathcal{G}\}$ of $T_m M$. Let D be a disc in M through m that is transverse at m to this subspace.

Consider the map

$$\psi \colon \mathcal{O} \times D \to M \quad , \quad (a, x) \mapsto a \cdot x.$$

By the implicit function theorem, after possibly shrinking $\mathcal{O}$ and D, the image $U_m := \psi(\mathcal{O} \times D)$ is open in M, and ψ is a diffeomorphism of $\mathcal{O} \times D$ with U_m.

For every $x \in D$, consider the set

$$A_x := \{g \in G \mid g \cdot m \in \psi(\mathcal{O} \times \{x\})\}.$$

Note that A_x is nonempty if and only if $x \in G \cdot m \cap D$.

Let us now show that

$$G \cdot m \cap D \quad \text{is countable.} \qquad (\clubsuit)$$

For this, we will first show that

($\diamondsuit$) for every $x \in D$, the set A_x is an open subset of G.
($\heartsuit$) for every $x, y \in D$, if $x \neq y$ then $A_x \cap A_y = \varnothing$.

Let $g \in G$. Suppose that $g \cdot m = a \cdot x$ for $a \in \mathcal{O}$. The subset $\mathcal{O} \cdot G_x \cdot a^{-1} \cdot g$ of G is a neighbourhood of g in G, and for every g' in this subset, $g' \cdot m \in \psi(\mathcal{O} \times \{x\})$. This shows ($\diamondsuit$).

Now suppose that g is in both A_x and A_y. Then both x and y are equal to $\mathrm{pr}_D \circ \psi^{-1}(g \cdot m)$, where $\psi^{-1} \colon U_m \to \mathcal{O} \times D$ is the inverse of ψ and where $\mathrm{pr}_D \colon \mathcal{O} \times D \to D$ is the projection to the second coordinate. So x and y are equal to each other. This shows ($\heartsuit$).

Thus, $\{A_x\}_{x \in D}$ is a family of disjoint open subsets of G. Because G is second countable, at most countably many elements of this family are nonempty. But A_x is nonempty exactly if $x \in G \cdot m \cap D$. This shows ($\clubsuit$).

Let us now check that the map $\overline{m}_\natural \colon G/G_m \to M$ is an induction (diffeomorphism with its image). Fix an arbitrary parametrization $\overline{\gamma} \colon V \to G/G_m$, and assume that the composition $\overline{m}_\natural \circ \overline{\gamma}$ is smooth. We need to show that $\overline{\gamma}$ itself is smooth.

Let r_0 be a point in V and let $g \in G$ be such that $\overline{\gamma}(r_0)$ is equal to the coset gG_m in G/G_m. We need to find a neighbourhood of r_0 in V in which $\overline{\gamma}$ is smooth. Shrinking V, precomposing $\overline{\gamma}$ by translation by r_0, and composing $\overline{\gamma}$ by left multiplication by the element g^{-1} of G, we reduce the problem to the following special case.

We assume that V is a small open ball centred at the origin 0, that $\overline{\gamma}(0) = \pi(\mathbf{1}_G)$, and that $\overline{\gamma}(V) \subset \pi(\mathcal{O})$. The map $\gamma := \sigma \circ \overline{\gamma} \colon V \to \mathcal{O}$ is then well defined, and it is a lift of $\overline{\gamma}$, that is, $\pi \circ \gamma = \overline{\gamma}$. The map $v \mapsto \gamma(v) \cdot m$, being equal to the composition $\overline{m}_\natural \circ \overline{\gamma}$, is smooth by assumption; also, it takes values in the subset $U_m(= \psi(\mathcal{O} \times D))$ of M.

By the definition of ψ, we can write
$$\psi^{-1}(\gamma(v) \cdot m) = (\alpha(v), \mu(v)),$$
where α is a parametrization of G that takes values in $\mathcal{O}$, where μ is a parametrization of M that takes values in D, and where
$$\alpha(v) \cdot \mu(v) = \gamma(v) \cdot m$$
for all $v \in V$. Rewriting the last equality as $\mu(v) = (\alpha(v)^{-1}\gamma(v)) \cdot m$, we see that μ also takes values in $G \cdot m$. So μ takes values in the set $G \cdot m \cap D$, which is countable by ($\clubsuit$).

Because $v \mapsto \psi^{-1}(\gamma(v) \cdot m)$, being the composition of two smooth maps, is smooth, the parametrizations α and μ are also smooth. Being a smooth map on an open ball that takes values in a countable set, the map μ is constant. So $\mu(v) = m$ for all v, and $\alpha(v) \cdot m = \gamma(v) \cdot m$ for all v. This, in turns, implies that $\pi(\alpha(v)) = \pi(\gamma(v))$ for all v. Because $\alpha(v)$ and $\gamma(v)$ both belong to $\mathcal{O}$, composing with σ gives $\gamma(v) = \alpha(v)$. But we already know that α is smooth. So γ is also smooth, and, hence, so is $\overline{\gamma}$. $\qquad\square$

2. WHEN G IS COUNTABLE. *Let M be a manifold, G a countable discrete group, and $\rho \colon G \to \mathrm{Diff}(M)$ a smooth action. Then the image of ρ is a discrete subset of $\mathrm{Diff}(M)$. In particular, if the action is effective, then ρ is an induction.*

NOTE 1. Because we assume manifolds to be second countable, a countable discrete group is the same thing as a zero dimensional Lie group.

NOTE 2. The image of a discrete diffeological group into $\mathrm{Diff}(M)$ by a smooth monomorphism has no reason to be discrete *a priori*. For example, a Lie group G injects into $\mathrm{Diff}(G)$ by left multiplication. If G is equipped with the discrete diffeology, then the injection $G \to \mathrm{Diff}(G)$ is smooth, but it is not an induction if $\dim G > 0$. We rule out this example by assuming that G is countable.

Proof. Let $r \mapsto f_r$ be a smooth parametrization in $\mathrm{Diff}(M)$ with values in $\rho(G)$, and let r_0 be a point in the domain of this parametrization. Fix $m \in M$. Because G is countable, the orbit $G \cdot m$ is countable. But a countable subspace of a manifold is diffeologically discrete. So the smooth parametrization $\varphi \colon r \mapsto f_r(m)$, which takes its values in the countable subspace $G \cdot m$ of M, is locally constant. But locally constant means constant on the connected components of the domain of the parametrization [PIZ05-10, chap. II]. Let B be an open ball in the domain of φ that is centred at r_0. Then $f_r(m) = f_{r_0}(m)$ for all r

in B. Repeating this argument for all m in M, we deduce that $f_r = f_{r_0}$ for all r in B. Thus, the parametrization $r \mapsto f_r$ is locally constant. In conclusion, the image of G by ρ is discrete. It follows immediately that if the action is effective then ρ is an induction. $\square$

3. The case of a discrete stabilizer. *Let M be a manifold, let G be a Lie group, and let $\rho \colon G \to \mathrm{Diff}(M)$ be an effective smooth action of G on M. Suppose that there exists a point m in M such that the stabilizer G_m is discrete. Then ρ is an induction.*

Proof. Let $r \mapsto f_r$ be a parametrization of $\mathrm{Diff}(M)$ that takes its values in $\rho(G)$. Let $r \mapsto g_r$ be the parametrization of G such that $\rho(g_r) = f_r$ for all r. Let r_0 be a point in the domain of these parametrizations. We would like to show that $r \mapsto g_r$ is smooth near r_0.

Let m be a point of M whose stabilizer G_m is discrete. The parametrization $r \mapsto f_r(m)$ of M takes its values in the orbit $G \cdot m$. Since $\overline{m}_\sharp \colon G/G_m \to M$ is an induction (Paragraph 1), and by the definition of the quotient diffeology on G/G_m, there exists a neighbourhood B of r_0 and a smooth map $r \mapsto g'_r$ from B to G such that $f_r(m) = \rho(g'_r)(m)$ for all $r \in B$. The map $r \mapsto \rho(g'_r)^{-1} \circ f_r$ is a plot of $\mathrm{Diff}(M)$ that takes its value in $\rho(G_m)$. But since G_m is assumed to be discrete, $\rho(G_m)$ is also discrete (Paragraph 2). So, $r \mapsto \rho(g'_r)^{-1} \circ f_r$ is locally constant. Therefore, after possibly shrinking the neighbourhood B of r_0, there exists a constant element $\kappa \in G_m$ such that $\rho(g'_r)^{-1} \circ f_r = \rho(\kappa)$, that is, $f_r = \rho(g'_r \kappa)$, for all $r \in B$. But also $f_r = \rho(g_r)$. Because the action is effective, we deduce that $g_r = g'_r \kappa$ for all $r \in B$. Because $r \mapsto g'_r$ is smooth and κ is constant, this implies that $r \mapsto g_r$ is smooth. $\square$

4. Intersecting vector subspaces. *Let E be a finite dimensional vector space. Let $\{E_i\}_{i \in \mathcal{I}}$ be a family of subspaces of E. If $\cap_{i \in \mathcal{I}} E_i = \{0\}$, then there exists a finite set of indices $(i, j, \ldots, k) \subset \mathcal{I}$ such that $E_i \cap E_j \cap \cdots \cap E_k = \{0\}$.*

Proof. We prove the result by induction on the dimension of E. If $E = \{0\}$, the result is trivial: we can take the finite set of indices to be the empty set. Suppose that the result is true for vector spaces of dimension smaller than that of E. Because $\cap_{i \in \mathcal{I}} E_i = \{0\}$, there exists at least one vector subspace E_i such that $E_i \neq E$, that is $\dim(E_i) < \dim(E)$. Let us define $\mathcal{I}' := \mathcal{I} \smallsetminus \{i\}$, $E' := E_i$, and $E'_j := E' \cap E_j$ for all $j \in \mathcal{I}'$. We have $\cap_{j \in \mathcal{I}} E_j = \cap_{j \in \mathcal{I}'} E_j$. Thus, $\{E_j\}_{j \in \mathcal{I}'}$ is a family of vector subspaces of $E'(= E_i)$ with $\cap_{j \in \mathcal{I}'} E'_j = \{0\}$. By the induction hypothesis, there is a finite set of indices $\{j, \ldots, k\} \subset \mathcal{I}'$ such that $E'_j \cap \ldots \cap E'_k = \{0\}$. The union

of this set with the index $\{i\}$ is a finite set of indices $\{i, j, \ldots, k\} \subset \mathcal{J}$ such that $E_i \cap E_j \cap \ldots \cap E_k = \{0\}$. $\qquad\qquad\square$

5. OBTAINING A DISCRETE INTERSECTION OF STABILIZERS. *Let* M *be a manifold,* G *a Lie group, and* $\rho \colon G \to \mathrm{Diff}(M)$ *an effective smooth action. Then there exists a finite set of points* $\{m_1, \ldots, m_N\} \subset M$ *such that the intersection of their stabilizers,* $G_{m_1} \cap \ldots \cap G_{m_N}$, *is discrete.*

Proof. Denote by $\mathcal{G}$ the Lie algebra of G, and, for every point m of M, denote by $\mathcal{G}_m$ the Lie algebra of the stabilizer G_m of m. For any smooth parametrization $P \colon U \to G$, if P takes values in G_m, then, for every $r \in U$, the composition of the differential $dP_r \colon T_r U \to T_{P(r)} \mathcal{G}$ with the left translation by $P(r)^{-1}$ takes values in the subspace $\mathcal{G}_m$ of $\mathcal{G}$.

Because the action is effective, $\cap_{m \in M} G_m = \{1\}$. Recall that $\mathcal{G}_m = \{\xi \in \mathcal{G} \mid \exp(t\xi) \in G_m \text{ for all } t \in \mathbf{R}\}$. It follow that $\cap_{m \in M} \mathcal{G}_m = \{0\}$. According to Paragraph 4, there exists a finite set of points, say, $\{m_1, \ldots, m_N\} \subset M$, such that $\mathcal{G}_{m_1} \cap \cdots \cap \mathcal{G}_{m_N} = \{0\}$.

Let $P \colon U \to G$ be a smooth parametrization that takes values in $G_{m_1} \cap \cdots \cap G_{m_N}$. For every $r \in U$, the composition of the differential $dP_r \colon T_r U \to T_{P(r)} G$ with the left translation by $P(r)^{-1}$ takes values in $\mathcal{G}_{m_1} \cap \ldots \cap \mathcal{G}_{m_N}$. So the differential dP_r is zero for every $r \in U$, and so the map P is locally constant. Therefore, $G_{m_1} \cap \cdots \cap G_{m_N}$ is discrete. $\qquad\qquad\square$

6. SMOOTH LIE GROUP ACTIONS. *Let* M *be a manifold,* G *a Lie group, and* $\rho \colon G \to \mathrm{Diff}(M)$ *an effective smooth action. Then* ρ *is an induction. That is,* ρ *is a diffeomorphism onto its image, when the image is equipped with the subset diffeology.*

NOTE. Let M be a manifold, G a Lie group, and $\rho \colon G \to \mathrm{Diff}(M)$ a smooth action that is not necessarily effective. Let $K = \ker \rho$. Applying Claim 6 to the quotient G/K, we deduce that ρ induces a diffeomorphism from G/K to its image in $\mathrm{Diff}(M)$.

Proof. By Paragraph 5, there exists a natural number N and a point $\mu = (m_1, \ldots, m_N)$ in M^N whose stabilizer G_μ under the diagonal action of G on M^N is discrete. Write this diagonal action as $\rho_N \colon G \to \mathrm{Diff}(M^N)$, that is, $\rho_N(g)(m_1, \ldots, m_N) = (g \cdot m_1, \ldots, g \cdot m_N)$. By Paragraph 3, the map $\rho_N \colon G \to \mathrm{Diff}(M^N)$ is an induction. Now, let $r \mapsto f_r$ be a plot of $\mathrm{Diff}(M)$ that takes its values in $\rho(G)$, and, for every r, let g_r be the element of G such that $f_r = \rho(g_r)$. Let $f_{N,r}(m_1, \ldots, m_N) := (f_r(m_1), \ldots, f_r(m_N))$. Then $r \mapsto f_{N,r}$ is a plot of $\mathrm{Diff}(M^N)$ that takes its values in $\rho_N(G)$, and $f_{N,r} = \rho_N(g_r)$ for all r.

Because ρ_N is an induction, it follows that $r \mapsto g_r$ is smooth. Thus, ρ is an induction. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Box$

3. A simple case of embedding

We have proved the theorem that we stated in the introduction: an effective smooth Lie group action $\rho\colon G \to \mathrm{Diff}(M)$ is an induction.

It is now natural to ask when is such ρ an *embedding*, that is, a homeomorphism of G with its image, when the image is equipped with the subset topology of the D-topology of $\mathrm{Diff}(M)$. In other words, how do these two topologies compare? Paragraph 7 gives a sufficient condition. Applied to $\mathrm{GL}(n, \mathbf{R})$ acting naturally on $\mathbf{R}^n$, this condition shows again that $\mathrm{GL}(n, \mathbf{R})$ is embedded in $\mathrm{Diff}(\mathbf{R}^n)$ [Igl85].

7. An orbit map embedding condition. *Let M be a manifold, G a Lie group, and $\rho\colon G \to \mathrm{Diff}(M)$ an effective smooth action. Suppose that there exists a natural number k and a point $\mu = (m_1, \ldots, m_k)$ in M^k such that the orbit map $\mu_\sharp\colon G \to M^k$ for the diagonal action is an embedding. Then $\rho\colon G \to \mathrm{Diff}(M)$ is an embedding.*

Note 1. Considering the n standard basis vectors of $\mathbf{R}^n$, we conclude that $\mathrm{GL}(n, \mathbf{R})$ is embedded in $\mathrm{Diff}(\mathbf{R}^n)$. The same applies to classical Lie groups such as the orthogonal groups, the unitary groups, the symplectic groups, etc.

Note 2. Every diffeological group is embedded (through left translation) in its group of diffeomorphisms [PIZ05-10, chap. VII]. In particular, T^2 is embedded in $\mathrm{Diff}(T^2)$. The irrational flow $R \to \mathrm{Diff}(T^2)$, given by $\rho(t)([x, y]) = [x + t, y + \alpha t]$ with α irrational, splits as a composition $R \to T^2 \to \mathrm{Diff}(T^2)$. Because the first arrow is not an embedding (cf. Note 2 of Paragraph 1) and the second arrow is an embedding, *a fortiori*, this flow is not embedded in $\mathrm{Diff}(T^2)$. Nevertheless, by our main theorem, this flow *is* a diffeomorphism with its image in $\mathrm{Diff}(T^2)$.

Proof. Let $B \subset G$ be an open set, and let
$$U = \mu_\sharp(B) = \{(g \cdot m_1, \ldots, g \cdot m_k) \in M^k \mid g \in B\}.$$
Since, by assumption, $\mu_\sharp$ is an embedding, U is open in $G\cdot\mu$ with respect to the subset topology. So, there exists an open subset $V \subset M^k$ such that $U = V \cap G \cdot \mu$. Let us introduce the set
$$\Omega = \{f \in \mathrm{Diff}(M) \mid (f(m_1), \ldots, f(m_k)) \in V\}.$$
Let us check now that Ω is open for the D-topology of the functional diffeology and that $\rho(B) = \rho(G) \cap \Omega$. This will prove the claim.

Let $P\colon \mathcal{O} \to \mathrm{Diff}(M)$ be a plot. The preimage $P^{-1}(\Omega)$ consists of those $r \in \mathcal{O}$ such that

$$(P(r)(m_1), \ldots, P(r)(m_k)) \in V.$$

That is, $P^{-1}(\Omega) = (\mu_\natural \circ P)^{-1}(V)$. But $\mu_\natural \circ P$ is smooth, hence D-continuous, and V is open. Thus, $P^{-1}(\Omega)$ is open. Because this is true for every plot $P\colon \mathcal{O} \to \mathrm{Diff}(M)$, we deduce that Ω is D-open.

Let us check now that $\rho(B) = \rho(G) \cap \Omega$. Let $g' \in G$. By the definition of Ω, the diffeomorphism $\rho(g')$ belongs to Ω if and only if the k-tuple $(g' \cdot m_1, \ldots, g' \cdot m_k)$ is in V. But this k-tuple is automatically in $G \cdot \mu$. So $\rho(g')$ belongs to Ω if and only if $(g' \cdot m_1, \ldots, g' \cdot m_k)$ belongs to the intersection $V \cap G \cdot \mu$, which, in turn, is equal to U, which is $\mu_\natural(B)$. Thus, $\rho(g')$ belongs to Ω if and only if there exists $g \in B$ such that $(g' \cdot m_1, \ldots, g' \cdot m_k) = (g \cdot m_1, \ldots, g \cdot m_k)$. This last condition holds exactly if $g^{-1}g' \in G_\mu$. Because G_μ is trivial, $\rho(g')$ belongs to Ω if and only if $g' = g \in B$.

Therefore, Ω is D-open in $\mathrm{Diff}(M)$, and $\rho(B) = \rho(G) \cap \Omega$. So $\rho(B)$ is open with respect to the subset topology on $\rho(G)$ that is induced from the D-topology in $\mathrm{Diff}(M)$. The proof is complete. $\qquad\square$

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