

# A note on the Wehrheim-Woodward category

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## Abstract

Wehrheim and Woodward have shown how to embed all the canonical relations between symplectic manifolds into a category in which the composition is the usual one when transversality and embedding assumptions are satisfied. A morphism in their category is an equivalence class of composable sequences of canonical relations, with composition given by concatenation. In this note, we show that every such morphism is represented by a sequence consisting of just two relations, one of them a reduction and the other a coreduction.

## 1 Introduction

The problem of quantization, i.e., the transition from classical to quantum physics, may be formulated mathematically as the search for a functor from a “classical” category whose objects are symplectic manifolds to a “quantum” category whose objects are Hilbert spaces (or more general objects, such as spaces of distributions, Fukaya categories or categories of D-modules).

On the classical side, it is useful to include among the morphisms  $X \leftarrow Y$  not only symplectomorphisms, which should produce unitary operators upon quantization, but more general canonical relations, i.e. lagrangian submanifolds of  $X \times \overline{Y}$  (where  $\overline{Y}$  is  $Y$  with its symplectic structure multiplied by  $-1$ ). An immediate difficulty is that the composition of canonical relations can produce relations which are not even smooth submanifolds.

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A solution to the composition problem on the classical side has been given by Wehrheim and Woodward [4], who introduce a category which is generated by the canonical relations, but where composition is merely “symbolic” unless the pair being composed fits together in the best possible sense. In other words, the morphisms in this category are equivalence classes of sequences of canonical relations which are composable as set-theoretic relations. The equivalence relation allows one to shorten a sequence when two adjacent entries compose nicely.

The purpose of this note is to show that any morphism in the WW category may be expressed as a product of just *two* canonical relations. Furthermore, this factorization  $g \circ h$  is analogous to the factorization of an ordinary map  $X \leftarrow Y$  through  $X \times Y$  as the product of the projection  $X \leftarrow X \times Y$  and the graph map  $X \times Y \leftarrow Y$  in that the canonical relation  $g$  is the symplectic analogue of a submersion (it is a kind of symplectic reduction), and  $h$  is analogous to an embedding. In addition, the natural transpose operation on relations, which exchanges source and target, extends to the WW category, and the subcategories to which  $g$  and  $h$  belong are transposes of one another; in this sense, relations have an even nicer structure than maps.

The decomposition  $g \circ h$  is not unique; in the iterative construction which we will describe, the source of  $g$  and target of  $h$  is a product of spaces whose number grows exponentially with  $n$ . Another, due to Katrin Wehrheim, consists of just  $n$  or  $n + 1$  factors, depending on the parity of  $n$ .

In a longer paper in preparation, we will show that the Wehrheim-Woodward construction leads to a rigid monoidal category, in which each morphism  $X \xleftarrow{f} Y$  may be represented by a “hypergraph” which is a lagrangian submanifold of a symplectic manifold  $Q$  of which  $X \times \overline{Y}$  is a symplectic reduction. We will also place the Wehrheim-Woodward construction in an appropriate general setting which is robust enough to apply to categories of operators in which quantization functors may take their values.

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## 2 Relations and their composition

This section is a review of mostly well-known ideas concerning the category of sets and relations, with some new terminology and notation.

We denote by **REL** the category whose objects are sets and for which the morphism space  $\mathbf{REL}(X, Y)$  is the set of all subsets of  $X \times Y$ . We adopt the convention that  $f \in \mathbf{REL}(X, Y)$  is a morphism *to*  $X$  *from*  $Y$ . Thus,  $X$  is the **target** of  $f$  and  $Y$  the **source**, and we write  $X \xleftarrow{f} Y$  for this morphism. The **composition**  $X \xleftarrow{f \circ g} Z$  of  $f \in \mathbf{REL}(X, Y)$  and  $g \in \mathbf{REL}(Y, Z)$  is

$$\{(x, z) \mid (x, y) \in f \text{ and } (y, z) \in g \text{ for some } y \in Y\}.$$

We denote the identity relation on  $X$  by  $X \xleftarrow{1_X} X$ , but also by  $\Delta_X$  when we want to think of it explicitly as a subset of  $X \times X$ .

The natural exchange maps  $X \times Y \leftarrow Y \times X$  define an involutive contravariant **transposition** functor  $f \mapsto f^t$  from **REL** to itself. It is the identity on objects.

For any subset  $T$  of  $Y$ , the **image**  $f(T)$  is the subset  $\{x \in X \mid (x, y) \in f \text{ for some } y \in T\}$  of  $X$ . For a single element  $y \in Y$ , we write  $f(y)$  for the subset  $f(\{y\})$ . The image  $f(Y)$  is called the **range** of  $f$ , and the range  $f^t(X) \subseteq Y$  of the transpose is the **domain**. We denote these by  $\text{Im } f$  and  $\text{Dom } f$  respectively.

$f$  is **surjective** if its range equals its target, and **cosurjective** if its domain equals its source (i.e. if it is “defined everywhere”).  $f$  is **injective** if, for any  $x \in X$ , there is at most one  $(x, y) \in f$ , and **coinjective** if there is at most one  $(x, y) \in f$  for any  $y \in Y$  (i.e. if it is “single valued”). Thus, the cosurjective and coinjective relations in  $X \times Y$  are the graphs of functions to  $X$  from  $Y$ .  $f$  is coinjective [cosurjective] if and only if  $f^t$  is injective [surjective].

Each of the four classes just defined constitutes a subcategory of **REL**, but two intersections of these subcategories will be particularly important. The surjective and coinjective relations (i.e. partially defined but single valued surjections) will be called **reductions**, and those which are injective and cosurjective (i.e. which take all the points of the source to disjoint nonempty subsets of the target) will be called **coreductions**. (In the linear context, these special relations are called reductions and contrareductions by Benenti and Tulczyjew [1], who characterize them by several categorical properties.) We will sometimes indicate that a morphism is of one of these types by decorating the arrow which represents it:  $X \leftarrow Y$  for a reduction and  $X \leftarrow Y$  for a coreduction. The arrows decorated at both ends are the invertible relations, i.e., the bijective functions.

The composition of relations  $X \xleftarrow{f} Y$  and  $Y \xleftarrow{g} Z$  involves two steps. The first is to form the fibre product  $f \times_Y g$ , i.e. the intersection in  $X \times Y \times Y \times Z$  of  $f \times g$  with  $X \times \Delta_Y \times Z$ . The second is to project  $f \times_Y g$  into  $X \times Z$ . We will call the pair  $(f, g)$  **monic** if this projection is injective, i.e. if there is only one  $y \in Y$  which accounts for each  $(x, z)$  belonging to  $f \circ g$ .

A composable pair  $(f, g)$  is automatically monic if  $f$  is injective or  $g$  is coinjective. Thus, all compositions within the categories of injective relations or coinjective relations, and hence of reductions or coreductions, are monic.

### 3 Smooth relations

The objects on which smooth relations operate are not just sets. To get a category, we begin with all relations between manifolds before singling out the smooth ones.

**Definition 3.1** *The objects of the category **MREL** are the smooth differentiable manifolds, and the morphisms are the set-theoretic relations; i.e.,  $\mathbf{MREL}(X, Y)$  consists of all subsets of the product manifold  $X \times Y$ . A relation  $f \in \mathbf{MREL}(X, Y)$  is **smooth** if it is a closed submanifold of  $X \times Y$ . The manifold whose only element is the empty set, carrying its unique smooth structure, will be denoted by  $\mathbf{1}$ .*

The forgetful functor  $\mathbf{REL} \leftarrow \mathbf{MREL}$  which forgets the differentiable structure is full as well as faithful.

**Example 3.2** The graph of any smooth map  $X \xleftarrow{f} Y$  is a smooth relation. The smooth relations in  $\mathbf{MREL}(Y, \mathbf{1})$  correspond to the closed submanifolds  $M \subseteq Y$ . The composition in  $\mathbf{MREL}$  of the relations corresponding to  $f$  and  $M$  corresponds to the subset  $f(M)$ , which is in general neither closed nor a submanifold, hence not a smooth relation.

We see already from the example above that we will need to impose extra conditions on pairs of smooth relations to insure that their composition is smooth. To describe these conditions, it is useful to look at the tangent “operation”, which is not a functor since its domain is not (yet) a category.

Since the tangent bundle of any submanifold is a smooth submanifold of the pulled back tangent bundle of the ambient manifold, the tangent bundle  $Tf$  of any smooth relation  $f \in \mathbf{MREL}(X, Y)$  smooth relation  $\mathbf{SREL}(TX, TY)$ . We call it the **differential** of  $f$ .

A smooth relation  $f$  will be called a **surmersion** if it is surjective and coinjective, along with  $Tf$ , and if the projection  $Y \leftarrow f$  is not only injective but proper.  $f$  will be called an **embedding** if  $f^t$  is a surmersion. Note that a surmersion can be partially defined but is single valued, along with its differential; an embedding can be multiply defined but, along with its differential, must be defined everywhere. The collection of smooth relations is clearly invariant under transposition, and so we may define **cosurmersion**s and **coembeddings** as the transposes of surmersions and embeddings. In this

smooth setting, we will reserve the decorated arrows  $X \leftarrow Y$  and  $X \hookrightarrow Y$  and the terms “reduction” and “coreduction” for surmersive coembeddings and cosurmersive embeddings. The surmersions, embeddings, cosurmersions, and coembeddings do each form subcategories of **SREL**, as will follow from the considerations below. We will also reserve the term “monic” for compositions  $X \xleftarrow{f} Y \xleftarrow{g} Z$  where the projection  $X \times Z \leftarrow f \times_Y g$  is not only injective but also proper.

For compositions of smooth relations, we may impose a transversality condition which does not seem to have a useful counterpart in **REL**.

**Definition 3.3** *When  $X \xleftarrow{f} Y$  and  $Y \xleftarrow{g} Z$  are smooth, the pair  $(f, g)$  is **transversal** if  $f \times g$  is transversal to  $X \times \Delta_Y \times Z$ , insuring that  $f \times_Y g$  is again a manifold. A transversal pair is **strongly transversal** if the projection map  $X \times Z \leftarrow f \times_Y g$  is an embedding onto a closed submanifold (which is  $f \circ g$ ). This means that  $(f, g)$  and  $(Tf, Tg)$  are both monic, and a consequence of these conditions is that  $X \xleftarrow{f \circ g} Z$  is again a smooth relation.*

**Remark 3.4** For smooth transversal  $(f, g)$ , the monicity condition at each  $(x, y, y, z) \in f \times_Y g$  applied to  $T_{(x,y)}f$  and  $T_{(y,z)}g$  means that the projection from  $f \times_Y g$  to  $f \circ g$  is an immersion, but we need  $(f, g)$  itself to be monic to insure that this immersion is an embedding, so that  $f$  is strongly transversal to  $g$ .

Note that a composition  $X \leftarrow Y \leftarrow Z$  is strongly transversal if either arrow is decorated at the junction point  $Y$ , i.e. if we have either  $X \hookrightarrow Y$  or  $Y \hookrightarrow Z$  (or both). The smooth embeddings, surmersions, reductions, and coreductions between manifolds each form subcategories of **MREL**. Other subcategories are given by the graphs of smooth maps and by the transposes of such graphs.

**Example 3.5** If  $f$  and  $g$  are smooth, and their composition  $f \circ g$  happens to be smooth, it may still not be the case that  $T(f \circ g) = Tf \circ Tg$ .

For instance, let  $C$  and  $D$  be submanifolds in a manifold  $X$  which intersect in a single point  $x$ , but whose tangent spaces there are equal. Let  $f$  be the reduction  $C \leftarrow X$  whose domain is  $C$  (transpose of the inclusion), and let  $g$  be the inclusion  $X \hookrightarrow D$ . The composition  $f \circ g$  is the single point  $(x, x)$  in  $C \times D$ , but the composition  $Tf \circ Tg$  is the graph in  $TC \times TD$  of the isomorphism  $T_x C \leftarrow T_x D$  given by their inclusion in  $T_x X$ .

## 4 Canonical relations

A **canonical relation** (called in [4] a “lagrangian correspondence”) between symplectic manifolds  $(X, \omega_X)$  and  $(Y, \omega_Y)$  is a smooth relation  $X \xleftarrow{f} Y$  which is lagrangian as a submanifold of  $(X, \omega_X) \times (Y, -\omega_Y)$ . We will often omit the symbol for the symplectic structure and will use the notation  $\overline{X}$  for the **dual**  $(X, -\omega_X)$  when  $X$  is  $(X, \omega_X)$ .

If we drop the condition that  $f$  be canonical, or even that it be smooth, we obtain the category **SREL** of relations between symplectic manifolds. The canonical relations are morphisms in this category, and the starting point for their study is the important fact that if  $X \xleftarrow{f} Y$  and  $Y \xleftarrow{g} Z$  are canonical, and if  $f$  is strongly transversal to  $g$ , so that  $f \circ g$  is smooth, then  $X \xleftarrow{f \circ g} Z$  is canonical as well. The graph  $\gamma_f$  of  $X \xleftarrow{f} Y$  will be considered as an element of **SREL** $(X \times \overline{Y}, \mathbf{1})$ .

It is a well-known and useful fact that any smooth map  $X \xleftarrow{f} Y$  may be factored as the composition  $g \circ h$  of a surmersion  $g$  and an embedding  $h$ , where  $g$  and  $h$  are also smooth maps. We merely let  $g$  be the natural projection  $X \leftarrow X \times Y$  and  $h$  be the embedding  $X \times Y \leftarrow Y$  of  $Y$  onto the graph of  $f$ . A similar factorization applies in many other categories mapping, but canonical relations require the following slightly more complicated construction, since  $g$  and  $h$  above are not canonical relations, even when  $f$  is a symplectomorphism.

**Proposition 4.1** *Any canonical relation  $X \xleftarrow{f} Y$  may be factored as*

$$X = X \times \mathbf{1} \xleftarrow{1_X \times \epsilon_Y} X \times \overline{Y} \times Y \xleftarrow{\gamma_f \times 1_Y} \mathbf{1} \times Y = Y.$$

*This composition is strongly transversal, the relation  $1_X \times \epsilon_Y$  is a reduction, and  $\gamma_f \times 1_Y$  is a coreduction.*

**Proof.** In fact, the factorization works for any smooth relation (if we replace  $\overline{Y}$  by  $Y$  in the diagram).

We first look set-theoretically. A pair  $(x, y)$  lies in the composition if and only if there is a triple  $(x, y', y) \in X \times \overline{Y} \times Y$  for which  $y' = y$  and  $(x, y') \in f$ , i.e. if and only if  $(x, y) \in f$ . Since this triple is determined by  $x$  and  $y$ , the composition is monic. The composed relation, being just  $f$ , is a closed submanifold, so we have proven monicity.

Next, we show that the composition is monic on the level of tangent spaces. In fact, we may simply apply the tangent functor to the factorization above. Identifying  $T\mathbf{1}$  with  $\mathbf{1}$ , and using the facts that  $T1_X = 1_{TX}$ ,  $T\epsilon_Y = \epsilon_{TY}$ ,  $T\gamma_f = \gamma_{Tf}$ , and  $T1_Y = 1_{TY}$ , we obtain

$$TX = TX \times \mathbf{1} \xleftarrow{1_{TX} \times \epsilon_{TY}} TX \times \overline{TY} \times TY \xleftarrow{\gamma_{Tf} \times 1_{TY}} \mathbf{1} \times TY = TY.$$

But this is the same factorization as above, applied to  $Tf$  rather than  $f$ , so it is monic, too.

For canonical relations, monicity implies transversality by symplectic duality. In the general smooth case, transversality may be verified directly; we omit the details.

Finally,  $1_X \times \epsilon_Y$  is a reduction because  $1_X$  and  $\epsilon_Y$  are, and  $\gamma_f \times 1_Y$  is a coreduction because  $\gamma_f$  and  $1_Y$  are.

□

## 5 The Wehrheim-Woodward construction

Wehrheim and Woodward [4] construct a category containing all the canonical relations between symplectic manifolds, and in which composition coincides with set-theoretic composition in the strongly transversal case.

The construction of their category, which we will denote by  $WW(\mathbf{SREL})$ , begins with a category of “paths”. If we think of a category as a directed graph with the objects as vertices and morphisms as edges, these are paths in the usual sense, where we allow “weakly monotonic” reparametrization.

**Definition 5.1** *The support of an infinite composable sequence*

$$\mathbf{f} = (\dots f_{-1}, f_0, f_1, \dots)$$

*in any category is the set of integers  $j$  for which  $f_j$  is not an identity morphism. A **canonical path in SREL** is an infinite composable sequence of canonical relations with finite support. The target and source of  $f_j$  for all sufficiently large negative  $j$  is thus a fixed object  $X$  and, for all sufficiently large positive  $j$ , a fixed object  $Y$ . We call  $X$  the **target** and  $Y$  the **source** of the path  $\mathbf{f}$ .*

*Two canonical paths will be considered as equivalent if one may be obtained from the other by inserting and removing finitely many identity morphisms. This does not change the target or source. The set of equivalence classes is the **path category**  $P(\mathbf{SREL})$ . We will denote the equivalence class of  $(\dots f_{-1}, f_0, f_1, \dots)$  by  $\langle \dots f_{-1}, f_0, f_1, \dots \rangle$  and will also use the notation  $\langle f_r, \dots, f_s \rangle$  when the support of the sequence  $\mathbf{f}$  is contained in the interval  $[r, s]$ .*

*To compose  $\langle \mathbf{f} \rangle \in P(\mathbf{SREL})(X, Y)$  and  $\langle \mathbf{g} \rangle \in P(\mathbf{SREL})(Y, Z)$ , choose representative sequences, remove all but finitely many copies of  $1_Y$  from the positive end of the first sequence and the negative end of the second, and then concatenate the truncated sequences.*

*The identity morphism in  $P(\mathbf{SREL})$  of any symplectic manifold  $X$  is (represented by) the constant sequence with all entries equal to  $1_X$ .*

**Remark 5.2** One could work as well with finite sequences, but the infinite version is more convenient when it comes to defining rigid monoidal structures.

**Remark 5.3** Every morphism in  $P(\mathbf{SREL})$  has a unique “minimal” representative for which  $f_i$  is an identity morphism for all  $i \leq 0$  and for which there are no identity morphisms in between nonidentity morphisms.

**Remark 5.4** A useful way to carry out the composition of two sequences is to shift the first one (which does not change its equivalence class) so that its support (the set of  $j$  such that  $f_j$  is not an identity morphism) is contained in the negative integers, and to shift the second so that its support is contained in the positive integers. The composition is then represented by the sequence whose value at  $j$  is  $f_j$  for  $j \leq 0$  and  $g_j$  for  $j \geq 0$ .

One may use a similar idea to verify associativity of composition; given three sequences, shift them so that their supports are contained in disjoint, successive intervals of integers.

We leave to the reader the proof of the following result.

**Proposition 5.5** *There is a unique functor  $\mathbf{SREL} \xrightarrow{\mathcal{L}'} P(\mathbf{SREL})$  which is the identity on objects and which takes each morphism  $\langle \dots, f_{-1}, f_0, f_1, \dots \rangle$  to the composition  $\dots \circ f_{-1} \circ f_0 \circ f_1 \circ \dots$  in  $\mathbf{SREL}$ . (The “infinite tails” of identity morphisms may be ignored here.)*

We now define the Wehrheim-Woodward category  $WW(\mathbf{SREL})$  by permitting the actual composition of strongly transversal pairs.

**Definition 5.6** *The Wehrheim-Woodward category  $WW(\mathbf{SREL})$  is the quotient category obtained from the category  $P(\mathbf{SREL})$  of canonical paths by the smallest equivalence relation for which two paths are equivalent if a sequence representing one is obtained from a sequence representing the other by replacing successive entries forming a strongly transversal pair  $(p, q)$  by the single entry  $pq$ . The equivalence class in  $WW(\mathbf{SREL})$  of  $\langle \mathbf{f} \rangle \in P(\mathbf{SREL})$  will be denoted by  $[\mathbf{f}]$ .*

The composition functor  $c'$  above descends to a functor  $\mathbf{SREL} \xrightarrow{\mathcal{L}'} WW(\mathbf{SREL})$ , i.e.  $c([\dots, f_{-1}, f_0, f_1, \dots]) = \dots \circ f_{-1} \circ f_0 \circ f_1 \circ \dots$ .

The canonical relations themselves embed naturally in  $WW(\mathbf{SREL})$  by the map  $s$  (a cross section to the composition functor  $c$ ) which maps each smooth morphism  $f$  to the equivalence class of sequences containing one entry equal to  $f$  and all the others equal to identity morphisms.  $WW(\mathbf{SREL})$  is then characterized by the universal property that any map which takes canonical relations to morphisms in some category  $\mathbf{C}$ , which takes units to units, and which takes strongly transversal compositions to compositions, extends uniquely to a functor from  $WW(\mathbf{SREL})$  to  $\mathbf{C}$ .



## 6 Simplifying WW morphisms

We prove here the main result of this note, that any morphism in  $WW(\mathbf{SREL})$  may be represented by a sequence of just two nontrivial canonical relations.

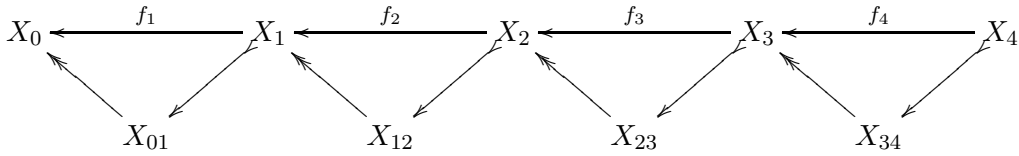
**Theorem 6.1** *Let  $(f_1, \dots, f_r)$  be a composable sequence in  $\mathbf{SREL}$ , with  $f_{i-1,i} \in \text{Hom}(X_{i-1}, X_i)$  for  $i = 1, \dots, n$ . Then there is a symplectic manifold  $Q$  and canonical relations  $A \in \text{Hom}(X_0, Q)$  and  $B \in \text{Hom}(Q, X_n)$  such that  $A$  is a reduction,  $B$  is a coreduction, and  $[f_1, \dots, f_r] = [A, B]$  in  $WW(\mathbf{SREL})$ .*

**Proof.** We illustrate the proof with diagrams for the case  $n = 4$ , which is completely representative of the general case.

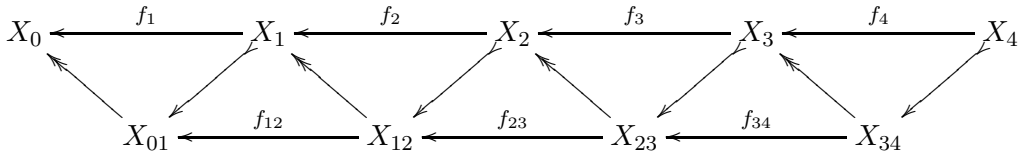
First, we write  $[f_1, f_2, f_3, f_4]$  as a composition:

$$X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \xleftarrow{f_3} X_3 \xleftarrow{f_4} X_4$$

By Proposition 4.1, we may factor each arrow as the strongly transversal composition of a reduction and a coreduction. The top row of the next diagram is then equivalent to the zigzag line below it.

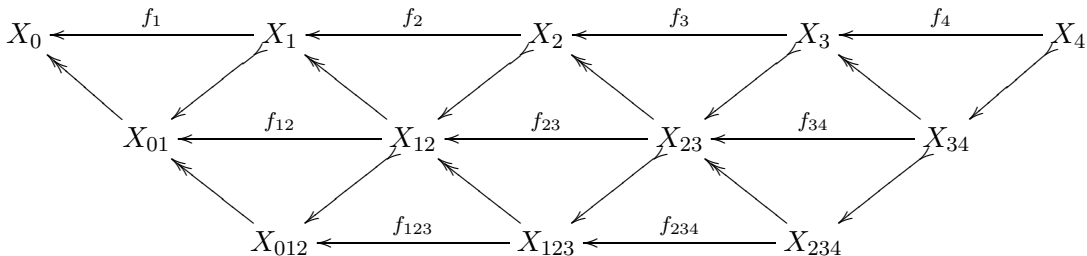


Next, we compose pairs of diagonal arrows to produce the bottom row below.

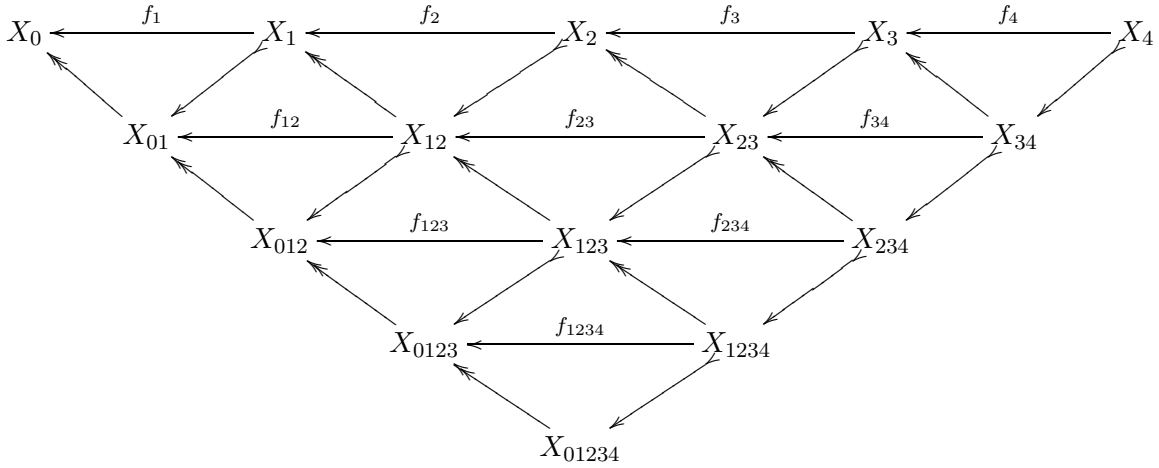


Each of these compositions is strongly transversal, even “doubly so”, thanks to the decorations on the arrows identifying them as reductions and coreductions. It follows that the original composition on the top row is equivalent to the composition of the bottom row with the outer diagonal edges.

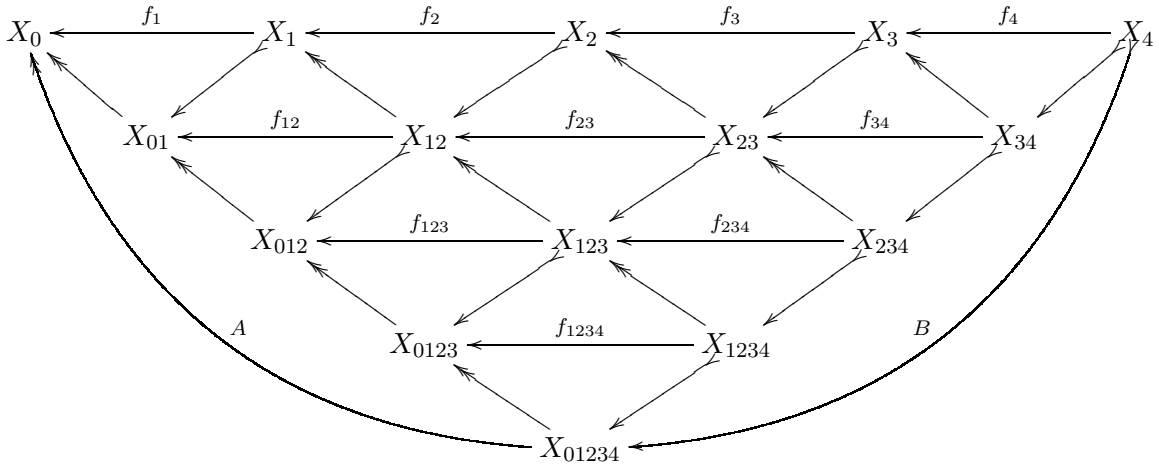
We repeat the process to obtain another row.



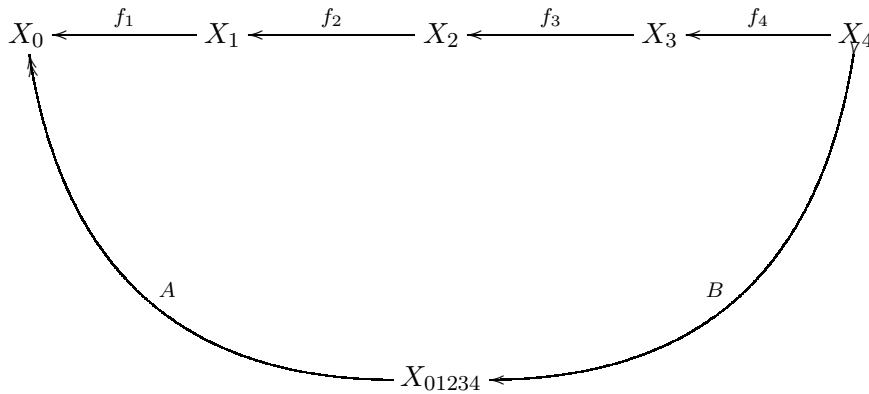
Repeating two more times, we arrive at a triangle, in which the top row is equivalent to the composition of the arrows on the other two sides.



Finally, we observe that all the arrows going up the left-hand side are reductions, so we may compose them all to produce a single reduction  $A$ . Similarly, the coreductions going down on the right yield a coreduction  $B$ .



We may now erase everything in the middle of the diagram to obtain the desired factorization.



□

**Remark 6.2** Sylvain Cappell has pointed out the similarity of this result to ideas of J.H.C. Whitehead on simple homotopy theory, where maps are factored in to collapses and expansions [2]. And Thomas Kragh has noted a resemblance to the theory of Waldhausen categories; the diagram on page 207 of [3] looks very much like the ones in the proof above.

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