L^3 ESTIMATES FOR AN ALGEBRAIC VARIABLE COEFFICIENT WOLFF CIRCULAR MAXIMAL FUNCTION

J. ZAHL

Abstract. In 1997, Thomas Wolff proved sharp L^3 bounds for his circular maximal function, and in 1999, Kolasa and Wolff proved certain non-sharp L^p inequalities for a broader class of maximal functions arising from curves of the form $\{\Phi(x,\cdot)=r\}$, where $\Phi(x,y)$ satisfied Sogge's cinematic curvature condition. Under the additional hypothesis that Φ is algebraic, we obtain a sharp L^3 bound on the corresponding maximal function. Since the function $\Phi(x,y)=|x-y|$ is algebraic and satisfies the cinematic curvature condition, our result generalizes Wolff's L^3 bound. The algebraicity condition allows us to employ the techniques of vertical cell decompositions and random sampling, which have been extensively developed in the computational geometry literature.

1. Introduction

1.1. **Background.** Consider the Wolff circular maximal function

$$M^\delta f(r)=\sup_x \frac{1}{|C^\delta(x,r)|}\int_{C^\delta(x,r)}|f|, \tag{1}$$

where $C^\delta(x,r)$ is the δ–neighborhood of the circle centered at x of radius r. In [12], Wolff proved that for each $\epsilon>0$ there exists a constant C_ϵ such that

$$\left\|M^\delta f\right\|_{L^3([1/2,1])}\leq C_\epsilon \delta^\epsilon \left\|f\right\|_{L^3(\mathbb{R}^2)}, \tag{2}$$

which in particular implies that every BRK set (a planar set containing a circle of each radius $r\in[1/2,1]$) must have Hausdorff dimension 2. It is not possible to omit the $\delta^{-\epsilon}$ factor since if (2) held with this factor omitted, it would imply that every BRK set had strictly positive Lebesgue measure, and this is known to be false. Wolff's result built off of his earlier work[1] (jointly with Kolasa) in [7], where he proved the bound

$$\left\|M^\delta f\right\|_q\leq C_{p,q}\delta^{-\frac{1}{2}(\frac{3}{p}-1)}\left\|f\right\|_p, \quad p<\frac{8}{3},\ q\leq 2p'. \tag{3}$$

Equation (3) can almost be obtained by interpolating (2) with the trivial bound

$$\left\|M^\delta f\right\|_\infty\leq C\delta^{-1}\left\|f\right\|_1, \tag{4}$$

though in doing so we pick up an additional $C_\epsilon\delta^\epsilon$ factor.

However, this earlier Kolasa-Wolff result applied not only to circles but to any family of curves satisfying Sogge's cinematic curvature condition first introduced in

Date: December 6, 2010.
Research supported in part by the Department of Defense through the National Defense Science & Engineering Graduate Fellowship (NDSEG) Program.

[1]While [7] was published after [12], [7] was written first.

[11]; let U be a neighborhood of $(a,b)\in\mathbb{R}^2\times\mathbb{R}^2$ and $\Phi\colon U\to\mathbb{R}$ with Φ smooth. Then the family of curves[2] $\digamma(x,r)=\{y\colon \Phi(x,y)=r\}$ is said to satisfy the *cinematic curvature* condition provided

- $$\nabla_y\Phi(a,b)\neq 0. \tag{5}$$
- $$\det\Big(\nabla_x\begin{bmatrix} e\cdot\nabla_y\Phi(x,y)\\ e\cdot\nabla_y\big(\frac{e\cdot\nabla_y\Phi(x,y)}{|\nabla_y\Phi(x,y)|}\big)\end{bmatrix}\Big|_{(x,y)=(a,b)}\Big)\neq 0, \tag{6}$$

  where e is a unit vector orthogonal to $\nabla_y\Phi(a,b)$. While there are two potential choices of vector e, the two choices only differ by a sign, so the veracity of (6) is independent of the choice made.

Informally, the second condition is a quantitative version of the statement that two distinct curves cannot be tangent to second order—it guarantees that if two curves $\digamma$ and $\tilde{\digamma}$ intersect at a point x, then their normal vectors at x or their curvature at x (or both) must differ by at least the distance between $\digamma$ and $\tilde{\digamma}$ in some suitable metric.

Let $\digamma^\delta(x,r)$ be the δ–neighborhood of $\digamma$. Define

$$M_\Phi^\delta f(r)=\sup_{x\in U_1}\frac{1}{|\digamma^\delta(x,r)|}\int_{\digamma^\delta(x,r)}|f|, \tag{7}$$

where U_1 is a sufficiently small neighborhood of a. Then Kolasa and Wolff proved that for any f supported in a sufficiently small neighborhood of b,

$$\big\|M_\Phi^\delta f\big\|_{L^q([1/2,1])}\leq C_{p,q}\delta^{-\frac{1}{2}(\frac{3}{p}-1)}\,\|f\|_p\,,\quad p<\frac{8}{3},\ q\leq 2p'. \tag{8}$$

1.2. New Results.

Theorem 1. *Let Φ be an algebraic function satisfying the cinematic curvature conditions* (5) *and* (6) *at (a,b) and let U_1 be a sufficiently small neighborhood of a. Then for all f supported in a sufficiently small neighborhood of b and for all $\epsilon>0$, there exist a constant C_ϵ depending only on ϵ and Φ such that for all $\delta>0$,*

$$\big\|M_\Phi^\delta f\big\|_{L^3([1/2,1])}\leq C_\epsilon\delta^\epsilon\,\|f\|_{L^3(\mathbb{R}^2)}\,. \tag{9}$$

Remark 2. See Appendix B for the definition of an algebraic function and related concepts.

Remark 3. Theorem 1 generalizes (2). Indeed, $\Phi(x,y)=|x-y|$ is clearly algebraic, and by the rotational, translational, and scale invariance of Φ, in order to verify the cinematic curvature condition it suffices to verify the condition at the point $a=(0,0),\ b=(1,0)$. Then $e=(0,1)$ and the determinant in (6) is 1. Furthermore, if

$$\Phi(x,y)=|x-y|+P(x,y) \tag{10}$$

for P a smooth algebraic function with $\|P\|_{C^3}$ sufficiently small, then Φ satisfies (6) uniformly in the choice of $a,b\in[0,1]^2$. Thus we obtain (9) for any family of smooth algebraically perturbed circles, provided the perturbation is not too large.

[2]Note that we are reversing the role of x and y from the notation of [7].

We shall prove Theorem 1 by modifying Schlag's arguments in [9]. These arguments rely on a key incidence lemma for circles, which is proved by Wolff in [14]. This incidence lemma employs various bounds on the behavior of circle intersections, which do not obviously hold for the more general class of curves we are considering. Luckily, most of the analogous statements were proved by Kolasa and Wolff in [7], so Theorem 1 can largely be obtained by patching together previously known results.

The constraint that Φ be algebraic is quite restrictive and is likely not optimal (indeed it is reasonable to conjecture that it is completely unnecessary). However, this constraint allows us to use a "semi-cylindrical algebraic decomposition" argument from real algebraic geometry. We shall discuss in Section 6 some conjectures about how the algebraic requirements can be weakened.

1.3. **Proof Sketch.** Through standard reductions, it suffices to prove a discretized version of a bound on the adjoint of the maximal operator M_Φ^δ. Roughly speaking, if we have a collection of "tubes" $\{\Gamma^\delta\}$ corresponding to curves with δ–separated radii (see (11) below for the definition of Γ), we need to control the area of the region where many of these tubes overlap. This is Lemma 4 below.

In [9], Schlag showed that (4) holds for families of curves satisfying two conditions. The first is a bound ((19) below) on $|\Gamma^\delta \cap \tilde{\Gamma}^\delta|$ (where here $|\cdot|$ denotes Lebesgue measure) provided we have control over how close Γ and $\tilde{\Gamma}$ are to each other in a suitable parameter space and how close the two curves are to being tangent.

The second requirement, which is made precise in (20) below, controls the number of almost-tangencies that can occur between the elements of $\mathcal{W}$ and $\mathcal{B}$ if $(\mathcal{W}, \mathcal{B})$ is a t–bipartite pair. Informally, two collections of curves $\mathcal{W}$ and $\mathcal{B}$ are called a t–bipartite pair if every two curves in $\mathcal{W}$ (resp $\mathcal{B}$) are close in an appropriate parameter space while those in $\mathcal{W}$ are far from those in $\mathcal{B}$ (there are some additional technical requirements that we shall gloss over here. The full details can be found in Definition 6). The requirement is a quantitative analog of the incidence geometry result that N circles in $\mathbb{R}^2$ can have at most $C_\epsilon N^{3/2+\epsilon}$ tangencies between pairs of circles. The incidence geometry result was proved in [5], and in [14], Wolff obtained the quantitative analog that was then used in Schlag's argument.

The bulk of this paper will be devoted to showing that families of curves arising from algebraic defining functions Φ satisfy the second requirement, i.e. that (20) is true. Once this has been established, one can run Schlag's arguments virtually verbatim to obtain Theorem 1.

2. Definitions and Initial Reductions

First, let us assume $U = U_1 \times U_2$ with U_1, U_2 sufficiently small disks centered at a and b respectively (the requirement that U_1 and U_2 be disks will be relevant—we need U_2 to be a semi-algebraic set). In particular, by selecting U_1, U_2 sufficiently small we can assume that the cinematic curvature conditions hold for every point $(x, y) \in U_1 \times U_2$ with uniform bounds on $\nabla_y \Phi$ and with the determinant in (6) bounded uniformly away from 0.

Throughout this paper, C, C', etc. will denote constants that are allowed to vary from line to line. We will say $X \lesssim Y$ or X is $O(Y)$ if $X < CY$ and $X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$.

Fix $0 < \alpha < C^{-1}\operatorname{diam}(U_2)$. For $x \in U_1, r \in [1/2, 1]$, we define

$$\Gamma(x_0, r_0) = \{y \in B(b, \alpha) \colon \Phi(x_0, y) = r_0\}. \tag{11}$$

We shall call these sets Φ–*circles*, and if Γ is a Φ–circle then Γ^δ will denote its δ–neighborhood. If $\Gamma, \tilde\Gamma$, etc. are Φ–circles, then unless otherwise noted, x_0, r_0 and $\tilde x_0, \tilde r_0$ will refer to their respective centers and radii. The Φ–circles defined here are strict subsets of the sets Ɣ defined in the introduction. However, if the function f is supported on a sufficiently small neighborhood of b then we can define a maximal function analogous to (7) with Γ in place of Ɣ, and the two maximal functions will agree. Thus we shall henceforth work with curves Γ defined by (11).

We shall restrict our attention to those Φ–circles Γ with $x_0 \in U_1,\ r_0 \in (1-\tau, 1)$ for τ a sufficiently small constant which depends only on Φ. By standard compactness arguments, we can recover $L^p([1/2, 1])$ bounds on M_Φ from those on the "restricted" version of M_Φ by considering the supremum over a finite number of scaled versions of the function.

Using standard reductions (see e.g. [9]), in order to prove Theorem 1 it suffices to prove the following estimate.

Lemma 4. *For $\eta > 0$ and δ sufficiently small depending on η, let $\mathcal{A}$ be a collection of Φ–circles with δ–separated radii, with each radius lying in $(1-\tau, 1)$. Then there exists $\tilde{\mathcal{A}} \subset \mathcal{A}$ with $\#\tilde{\mathcal{A}} \geq \frac{1}{C}\#\mathcal{A}$ such that for all $\Gamma \in \tilde{\mathcal{A}}$ and $\delta < \lambda < 1$,*

$$\Big| B(b, C^{-1}\alpha) \cap \{y \in \Gamma^\delta \colon \sum_{\tilde\Gamma \in \mathcal{A}} \chi_{\tilde\Gamma^\delta}(y) > \delta^{-\eta}\lambda^{-2}\} \Big| \leq \lambda |\Gamma^\delta|. \tag{12}$$

In [9], Schlag took Wolff's combinatorial incidence result from [14] and used it in conjunction with an induction on scales argument to prove the analogue of Lemma 4 (in [9], this is Lemma 8). In order to state Schlag's theorem, we first need some additional definitions.

Definition 5. For $X \subset B(b, \alpha)$, we define

$$\Delta_X(\Gamma, \tilde\Gamma) = \inf_{\substack{y \in \overline{X} \colon \Phi(x_0, y) = r_0 \\ \tilde y \in \overline{X} \colon \Phi(\tilde x_0, \tilde y) = \tilde r_0}} |y - \tilde y| + \Big| \frac{\nabla_y \Phi(x_0, y)}{\|\nabla_y \Phi(x_0, y)\|} - \frac{\nabla_y \Phi(\tilde x_0, \tilde y)}{\|\nabla_y \Phi(\tilde x_0, \tilde y)\|} \Big|. \tag{13}$$

Crucially,

$$\Delta_{B(b, C^{-1}\alpha)}(\Gamma, \tilde\Gamma) \geq \Delta_{B(b,\alpha)}(\Gamma, \tilde\Gamma),$$

but there exists a finite family of translates $\{t_i\} \subset \mathbb{R}^2$ (the cardinality of the family depends only on C) so that

$$\inf_i \Delta_{B(b + t_i, C^{-1}\alpha)}(\Gamma, \tilde\Gamma) \leq \Delta_{B(b,\alpha)}(\Gamma, \tilde\Gamma). \tag{14}$$

In the example $\Phi(x, y) = |x - y|$, $\Delta_X(\Gamma, \tilde\Gamma)$ describes how "far" (in (x_0, r_0) parameter space) we would need to move Γ so that $\tilde\Gamma$ and the newly moved curve Γ' are incident at some point in X. Indeed, if $\Phi(x, y) = |x - y|$ and $X = \mathbb{R}^2$ then $\Delta_X(\Gamma, \tilde\Gamma) = \big||x_0 - \tilde x_0| - |r_0 - \tilde r_0|\big|$, provided $x_0, \tilde x_0 \in U_1$ with $\operatorname{diam}(U_1)$ sufficiently small so that in particular, the only way circles can be tangent is if they are internally tangent.

Let

$$d(\Gamma, \tilde\Gamma) = |x_0 - \tilde x_0| + |r_0 - \tilde r_0|. \tag{15}$$

$d(\cdot, \cdot)$ is a metric on the space of curves. Throughout our arguments, the particular choice of metric will not be important since we will not care about multiplicative constants.

Definition 6. Let $\mathcal{W}, \mathcal{B}$ be collections of Φ–circles. We say that $(\mathcal{W}, \mathcal{B})$ is a t–*bipartite pair* if

$$|r_0 - \tilde{r}_0| \geq \delta \text{ for all } \Gamma, \tilde{\Gamma} \in \mathcal{W} \cup \mathcal{B}, \tag{16}$$

$$d(\Gamma, \tilde{\Gamma}) \in (t, 2t) \text{ if } \Gamma \in \mathcal{W},\ \tilde{\Gamma} \in \mathcal{B}, \tag{17}$$

$$d(\Gamma, \tilde{\Gamma}) \in (0, t) \text{ if } \Gamma, \tilde{\Gamma} \in \mathcal{W} \text{ or } \Gamma, \tilde{\Gamma} \in \mathcal{B}. \tag{18}$$

Definition 7. A (δ, t)–*rectangle* R is the δ–neighborhood of an arc of length $\sqrt{\delta/t}$ of a Φ–circle Γ. We say that a Φ–circle Γ is *incident* to R if R is contained in the $C_1\delta$ neighborhood of Γ. We say that R is of type $(\gtrsim \mu, \gtrsim \nu)$ relative to a t–bipartite pair $(\mathcal{W}, \mathcal{B})$ if R is incident to at least $C\mu$ curves in $\mathcal{W}$ and at least $C\nu$ curves in $\mathcal{B}$ for some absolute constant C to be specified later.

We are now able to state Schlag's result.

Proposition 8 (Schlag)**.** *Let $\mathcal{A}$ be a family of Φ–circles with δ–separated radii that satisfy the following requirements:*

(i)

$$|\Gamma^\delta \cap \tilde{\Gamma}^\delta \cap B(b', C^{-1}\alpha)| \lesssim \frac{\delta^2}{(d(\Gamma, \tilde{\Gamma}) + \delta)^{1/2} (\Delta_{B(b,\alpha)}(\Gamma, \tilde{\Gamma}) + \delta)^{1/2}} \tag{19}$$

for any b' in a sufficiently small neighborhood of b.

(ii) For any t–bipartite pair $(\mathcal{W}, \mathcal{B})$, with $t > C\delta$ for an appropriate choice of C; $\mathcal{W}, \mathcal{B} \subset \mathcal{A}$; $\#\mathcal{W} = m$; $\#\mathcal{B} = n$; and for any $\epsilon > 0$, the number of $(\gtrsim \mu, \gtrsim \nu)$ (t, δ)–rectangles is at most

$$C_\epsilon (mn)^\epsilon \Big(\Big(\frac{mn}{\mu\nu} \Big)^{3/4} + \frac{m}{\mu} + \frac{n}{\nu} \Big). \tag{20}$$

Then Lemma 4 holds for the collection $\mathcal{A}$.

Proof. The proof of this theorem can be found in [9], Section 4. However, we need the following minor modifications.

- Schlag actually requires the bound

$$|\Gamma^\delta \cap \tilde{\Gamma}^\delta| \lesssim \frac{\delta^2}{\big(d(\Gamma, \Gamma) + \delta\big)^{1/2} \big(\Delta_{B(b,\alpha)}(\Gamma, \Gamma) + \delta\big)^{1/2}}. \tag{21}$$

  in place of (19). However, (21) can be obtained from (19) by summing over finitely many translates of the ball $B(b, C^{-1}\alpha)$.
- Schlag stipulates that Requirement (ii) in the above theorem hold for all values of t and δ, not merely those for which $t > C\delta$. However, there are at most $\lesssim \delta^{-2}$ (δ, t)–rectangles incident to $(\mathcal{W}, \mathcal{B})$, and if $t < C\delta$ we can use this fact in place the bound from (20). □

The next sections shall be devoted to proving that any δ–separated family of Φ–circles satisfy the two requirements from Proposition 8. Once this has been established we will have proved Theorem 1. The first requirement will not present much difficulty; indeed, it was already proved by Kolasa and Wolff in [7], and it is Property 22 in Section 4 below. Thus the bulk of our efforts will be devoted to proving that the second requirement is satisfied. This will appear as Lemma 44 in Section 5.

3. Algebraic Considerations

Let $\Gamma = \Gamma(x_0, r_0)$ be a Φ–circle and $X \subset B(b, \alpha)$ an open semi-algebraic set of dimension 2 (see Appendix B for the definition of the dimension of a semi-algebraic set); in our discussion below we will only consider closed balls. For $w = (w_1, w_2, w_3) \in \mathbb{R}^3$, let

$$V_{\Gamma,X,w} = \{(x, r, y) \in U_1 \times (1-\tau, 1) \times X \colon \Phi(x_0, y) - r_0 = w_0,\\ \Phi(x, y) - r = w_1, \nabla_y \Phi(x_0, y) \wedge \nabla_y \Phi(x, y) = w_2\}, \qquad (22)$$

where

$$(z^{(1)}, z^{(2)}) \wedge (\tilde{z}^{(1)}, \tilde{z}^{(2)}) = z^{(1)}\tilde{z}^{(2)} - z^{(2)}\tilde{z}^{(1)}.$$

Intuitively, we can think of w_0, w_1, w_2 as being 0. However, setting $w_0, w_1, w_2 = 0$ might cause $V_{\Gamma,X,w}$ to fail to have the correct dimension. Thus we shall choose a very small "generic" choice of w_0, w_1, w_2 which fixes this problem. This will be elaborated upon in Lemma 9.

Let

$$S_{\Gamma,X,w} = \big(\pi_{(x,r)} V_{\Gamma,X,w}\big) \cap \{|x - x_0| > C\delta\} \qquad (23)$$

for an appropriately chosen C, where $\pi_{(x,r)} \colon (x, r, y) \mapsto (x, r)$ is the projection operator. In the example where $\Phi(x, y) = |x - y|$, $S_{\Gamma,X,0}$ is a section of the right-angled "light cone" with vertex $(x_0, r_0) \in \mathbb{R}^3$, i.e. $S_{\Gamma,X,0} \subset \{(x, r) \colon |x - x_0| = |r - r_0|\}$.

Lemma 9. *For an appropriate choice of $0 \leq w_0, w_1, w_2 < C^{-1}\delta$, $S_{\Gamma,X}$ is a semi-algebraic set of bounded complexity. Furthermore, if $X = B(b, \alpha)$ then $S_{\Gamma,X}$ has (semi-algebraic) dimension 2.*

Proof. We shall first show that if w_0, w_1, w_2 are chosen appropriately then $V_{\Gamma,X,w}$ is a semi-algebraic set of codimension 3. It suffices to show that the the defining functions in (22) are algebraic functions whose zero-sets intersect transversely. $\Phi(x_0, y) - r_0$ and $\Phi(x, y) - r$ are immediately seen to be smooth and algebraic since Φ is smooth and algebraic. The components of $\nabla_y \Phi(x_0, y)$ and $\nabla_y \Phi(x, y)$ are smooth and algebraic since the partial derivative of a smooth algebraic function are smooth and algebraic, and thus $\nabla_y \Phi(x_0, y) \wedge \nabla_y \Phi(x, y)$ is smooth and algebraic. The complexity of these functions is clearly independent of the choice of Γ. Finally, by Sard's theorem we can find $w_0, w_1, w_2 < C^{-1}\delta$ such that (w_0, w_1, w_2) is a regular value of the map

$$(x, r, y) \mapsto \big(\Phi(x_0, y) - r_0,\ \Phi(x, y) - r,\ \nabla_y \Phi(x_0, y) \wedge \nabla_y \Phi(x, y)\big).$$

For such a choice of values of w_0, w_1, w_2 we have that $S_{\Gamma,X,w}$ has geometric codimension 3, and thus semi-algebraic codimension 3, as desired (see Appendix B for a review of the relevant real algebraic geometry).

By the Tarski-Seidenberg theorem, $\pi_{(x,r)} V_{\Gamma,X,w}$ is semi-algebraic of bounded complexity, and thus so is $S_{\Gamma,X,w}$. At this point, the dimension of the components of $S_{\Gamma,X,w}$ could be 0,1, or 2. However, we shall show in Corollary 25 below that if $X = B(b, \alpha)$, then $S_{\Gamma,X,w}$ is a smooth manifold of dimension 2 or 3, and thus the components of $S_{\Gamma,X}$ are in fact of (semi-algebraic) dimension 2. □

Remark 10. It is somewhat curious to note that in our proof, we use algebraic considerations to show $\dim(S_{\Gamma,X,w}) \leq 2$ and differential geometric considerations to show $\dim(S_{\Gamma,X,w}) \geq 2$, and thus conclude that $\dim(S_{\Gamma,X,w}) = 2$.

Definition 11. Abusing notation slightly, we shall suppress the dependence of $S_{\Gamma,X,w}$ on w, and we shall define $S_{\Gamma,X}$ to be $S_{\Gamma,X,w}$ for an appropriate choice of w, the existence of which is guaranteed by Lemma 9. None of our arguments below will depend on the specific choice of w, and all of the constants in the estimates below will be independent of the choice of w, provided $|w| < C^{-1}\delta$ for a sufficiently large constant C.

We have defined $S_{\Gamma,X}$ and Δ_X so that

$$S_{\Gamma,X,0} = \{\Gamma' \colon \Delta_X(\Gamma,\Gamma') = 0\}, \tag{24}$$

and thus

$$S_{\Gamma,X} \in \{\Gamma' \colon \Delta_X(\Gamma,\Gamma') = 0\} + B(0, C^{-1}\delta), \tag{25}$$

$$\{\Gamma' \colon \Delta_X(\Gamma,\Gamma') = 0\} \in S_{\Gamma,X} + B(0, C^{-1}\delta), \tag{26}$$

where the + symbol denotes the Minkowski sum. These inclusions are the key facts linking the algebraic and differential geometric properties of Φ. Lemma 9 allows us to use the technique of semi-cylindrical algebraic decompositions (aka vertical algebraic decompositions) to decompose $\mathbb{R}^3$ into a collection of "cells" adapted to a collection of surfaces $\{S_{\Gamma,X}\}$. Informally, a cell is an open subset of $\mathbb{R}^3$ whose boundary consists of pieces of the surfaces from the collection $\{S_{\Gamma,X}\}$ as well as additional surfaces that are added to guarantee that the cells have certain favorable properties. More precisely we have the following result.

Lemma 12. *Let $\mathcal{D}$ be a collection of Φ–circles, $\#\mathcal{D} = N$. Then there exists an algorithm for creating a vertical decomposition of $U_1 \times (1-\tau, 1)$ (recall that U_1 and τ were specified in Section 2 and depend only on Φ) into $\lesssim N^3 \log N$ open (in $\mathbb{R}^3$) cells $\{\Omega_i\}$ such that $U_1 \times (1-\tau, 1)$ is the union of sets of the following types:*

- *cells,*
- *the dividing surfaces $\{S_{\Gamma,B(b,\alpha)} \colon \Gamma \in \mathcal{D}\}$,*
- *vertical walls: 2–dimensional semi-algebraic sets whose projections under the map $\pi_x \colon (x,r) \mapsto x$ are 1–dimensional semi-algebraic sets.*

The cells in this decomposition have the property that

$$\Omega \cap S_{\Gamma,B(b,\alpha)} = \emptyset \text{ for all cells } \Omega \text{ and all } \Gamma \in \mathcal{D}. \tag{27}$$

Furthermore, for each cell Ω in the decomposition, there is a bounded number (6 will suffice) of dividing surfaces such that Ω is one of the cells arising from the decomposition algorithm applied to this subcollection of surfaces (i.e. the existence of the other $N-6$ surfaces is irrelevant if all we care about is the cell Ω).

Proof. This statement follows from the techniques developed by Chazelle, Edelsbrunner, Guibas, and Sharir in [4]. Unfortunately, while Theorem 12 is claimed in [4] and follows (with some effort) from the methods described in [1, chapter 8], we are unaware of a complete and detailed proof of Theorem 12 in the literature. The author intends to present such a proof in his forthcoming PhD thesis. In the interests of keeping this paper self contained, we will give a brief expository sketch of the vertical algebraic decomposition in Appendix A. □

Lemma 13. *Let $\mathcal{B}$ be a collection of Φ–circles, $\#\mathcal{B} = n$. Randomly select (see Remark 14) a subset $\mathcal{D} \subset \mathcal{B}$ with $\#\mathcal{D} = N < C^{-1}n$, and let $\{\Omega_i\}_1^M$, $M \leq N^3 \log N$*

be the cells from Lemma 12. Then with high probability (see Remark 15) we have that for each i,

$$\#\{\Gamma \in \mathcal{B} \colon S_{\Gamma, B(b,\alpha)} \cap \Omega_i \neq \emptyset\} \lesssim \frac{N \log n}{n}. \tag{28}$$

Remark 14. To obtain our random selection we shall take a uniformly distributed random sample with replacement from $\mathcal{B}$. However, our algorithm will only work if the elements of the sample are all distinct. By requiring that $N \leq \frac{1}{C} n$ for C sufficiently large, we can ensure that this will occur with high probability, so this assumption will not cause difficulty.

Remark 15. By "high probability" we mean that for any probability $P < 1$ we can select a choice of constant C in the quasi-inequality in (28) so that the decomposition satisfies (28) with probability at least P. Later in the proof of Theorem 1 we shall need the above decomposition to satisfy additional properties which also occur with high probability (relative to another set of constants that we can weaken at will). We can ensure that all of these properties are simultaneously satisfied by requiring that each of the properties are separately satisfied with sufficiently high probability and using the trivial union bound.

Proof. Lemma 13 follows from Lemma 12 by the technique of random sampling (see e.g. [5]). Again, we shall briefly review this technique in Appendix A. □

Lemma 12 (which is only used to prove Lemma 13) is the only place where Lemma 9 is used, and it is thus the only place where we use the requirement that Φ be algebraic. We shall discuss in Section 6 some conjectures about how to obtain Lemma 12 through other (less algebraic) means, though our best attempts in this direction have thus far yielded only provisional results.

4. Cinematic Curvature and its Implications

Many of Wolff's arguments from [12] rely on the local differential properties of families of circles. The relevant properties are captured by the notion of cinematic curvature defined in the introduction. In [7], Kolasa and Wolff establish several key properties of families of curves with cinematic curvature which we shall recall below.

Property 16 (Straightening out). Let $x_0 \in U_1$. Then we can find a diffeomorphism $\psi_{x_0} \colon U_2' \to U_2$ and a choice of $r_0 = r_0(x_0)$ such that

$$\Phi(x_0, \psi_{x_0}(y)) - r_0 = y^{(2)}$$

where U_2' is an appropriately chosen domain (which may no longer be a disk). Furthermore for fixed y_0,

$$\psi_{x_0}(y_0) \text{ and } r_0(x_0) \text{ are continuous in } x. \tag{29}$$

This is discussed in [7, p 126]. To simplify notation, we shall say that Φ has been *straightened out* around x_0 if we (temporarily) replace the function $\Phi(x_0, \cdot)$ with $\Phi(x_0, \phi_{x_0}(\cdot)) - r_0(x_0)$, i.e. in "straightened out" coordinates, $\Phi(x_0, y) = y^{(2)}$. Note that if we straighten out around x_0 then in this new coordinate system Φ might no longer be algebraic. This will not pose any problems to our analysis below; we shall only be straightening out to simplify the proofs of certain diffeomorphism-invariant statements, and the statement can then be "pulled back" to the original (semi-algebraic) Φ. This process may change some of the constants involved in the relevant statements. However (29) will guarantee that the constants are worsened by at most a bounded amount so we can safely ignore this problem.

Property 17 (Derivative bounds). If we straighten out Φ at x_0 then for $y \in B(0,\alpha)$,

$$|\partial_{y^{(1)}}\Phi(x,y)| + |\partial^2_{y^{(1)}}\Phi(x,y)| \sim |x - x_0|, \tag{30}$$

$$|\partial_{y^{(2)}}\Phi(x,\psi_{x_0,r_0}(y))| \sim 1, \tag{31}$$

where $\partial_{y^{(1)}}$ denotes the partial derivative in the $y^{(1)}$–direction, etc. The constants in the quasi-equalities above are uniform in all variables. Indeed, since the cinematic curvature condition is diffeomorphism invariant, (30) and (31) are equivalent to the cinematic curvature condition. This is addressed in Equation (21) and the surrounding discussion of [7].

Property 18 (Unique point of parallel normals). Let $\Gamma, \tilde{\Gamma}$ be Φ–circles with

$$\Delta_{B(b,C^{-1}\alpha)}(\Gamma,\tilde{\Gamma}) \le {C'}^{-1}|x_0 - \tilde{x}_0|$$

for a sufficiently large constant C'. Then there is a unique point $\xi = \xi(x_0, r_0, \tilde{x}_0) \in \Gamma \cap B(0,\alpha)$ such that

$$\nabla_y\Phi(x_0,\xi) \wedge \nabla_y\Phi(\tilde{x}_0,\xi) = 0. \tag{32}$$

Furthermore,

$$|\Phi(\tilde{x}_0,\xi) - \tilde{r}_0| \lesssim \Delta_{B(b,C^{-1}\alpha)}(\Gamma,\tilde{\Gamma}), \tag{33}$$

and

$$\Gamma \cap \tilde{\Gamma} \cap B(b, C^{-2}\alpha) \subset B\left(\xi, C\Big(\frac{\Delta_{B(b,C^{-1}\alpha)}(\Gamma,\tilde{\Gamma})}{|x_0 - \tilde{x}_0|}\Big)^{1/2}\right). \tag{34}$$

Equations (33) and (34) are Equations (26) and (27) in [7].

Property 19 (Appolonius-type bounds). Let $t > C\delta$. Fix three Φ–circles $\Gamma_1, \Gamma_2, \Gamma_3$, let $B_0 = B(b, C^{-2}\alpha)$, and let

$$\begin{aligned} Y = \Big\{\Gamma\colon\, & \Delta_{B(b,C^{-1}\alpha)}(\Gamma,\Gamma_i) < C_1\delta,\ i = 1,2,3; \\ & d(\Gamma \cap B_0, \Gamma_i \cap B_0) > t,\ i = 1,2,3; \\ & \Gamma^\delta \cap \Gamma_i^\delta \cap B_0 \ne \emptyset,\ i = 1,2,3; \\ & \mathrm{dist}(\Gamma^{C_1\delta} \cap \Gamma_i^{C_1\delta} B_0, \Gamma^\delta \cap \Gamma_j^\delta \cap B_0) > C_3\sqrt{\delta/t},\ i \ne j\Big\}. \end{aligned} \tag{35}$$

Informally, Y is the collection of curves that are almost tangent to each of the curves $\Gamma_1, \Gamma_2, \Gamma_3$, with the additional requirement that the three regions of almost-tangency not be too close to each other.

If we identify Φ–circles Γ with points $(x_0, r_0) \in \mathbb{R}^3$ then

$$Y \text{ is the union of two sets, each of diameter } \lesssim t. \tag{36}$$

This is is Lemma 3.1(ii) in [7].

Property 20. For three fixed curves $\Gamma_1, \Gamma_2, \Gamma_3$, and a given curve $\Gamma = \Gamma(x_0, r_0)$, we say that Φ is Γ–*adapted* if there exists points a_1, a_2, a_3, with $a_j \in \Gamma_j$ such that

$$|a_j - \xi_j(x_0)| \le C^{-1}\sqrt{\delta/t},$$

and

$$\Phi(x, a_1) = 0,$$

$$\nabla_x\Phi(x, a_2) = (e \cdot (a_2 - a_1))\beta$$

for all x, where e is the tangent vector to Γ_1 at a_1, β is a vector independent of y with $|\beta| \sim 1$, and

$$\xi_i(y) = \xi(x_i, r_i, x_0).$$

Remark 21. Informally, the notion of a Γ–adapted defining function is a way of getting around the problem that we are forced to work with a defining function Φ, but we are actually interested in its level sets $\{\Phi(x, \cdot) = r\}$. Thus we are free (within certain constraints to be dealt with below) to modify Φ provided that our new defining function has the same level sets as the old one. Choosing a Γ–adapted defining function (provided a suitable one exists) simplifies many of the technicalities in our estimates.

Lemma 3.6 in [7] tells us that if $\Gamma \in Y$ then by composing Φ with suitable diffeomorphisms, a Γ–adapted defining function Φ exists which satisfies uniform derivative bounds, and this function Φ has the same level sets as our original Φ (i.e. it gives rise to the same Φ–circles), so the corresponding maximal functions are identical (the adapted defining function may not be algebraic, but this will not affect our analysis).

Now, if Φ is Γ–adapted, define

$$T(x) = \begin{pmatrix} \nabla_x \Phi(x, \xi_1(x)) & -1 \\ \nabla_x \Phi(x, \xi_2(x)) & -1 \\ \nabla_x \Phi(x, \xi_2(x)) & -1 \end{pmatrix}. \tag{37}$$

Informally, if we fix a choice of Γ and select a defining function adapted to Γ, then for x in a neighborhood of x_0, $T(x)$ describes how changing x affects how close $\Gamma(x, r_0)$ is to being tangent with each of $\Gamma_1, \Gamma_2, \Gamma_3$.

Lemma 3.8 in [7] tells us that when restricted to each connected component of Y (individually), T is boundedly conjugate to its linear part, i.e. if Γ, and $\tilde{\Gamma}$ lie in the same connected component of Y, then

$$T(x_0) T(\tilde{x}_0)^{-1} = I + E(\tilde{x}_0), \tag{38}$$

where (say) $\|E(\tilde{x}_0)\| < 1/100$. Furthermore, for the same choice of $\Gamma, \tilde{\Gamma}$,

$$|\xi_1(\tilde{x}_0) - \xi_1(x_0)| \lesssim \sqrt{\delta/t}. \tag{39}$$

Equation (39) is a consequence of Equation 45 in [7] once we note that $\Gamma \in Y$ implies that $|T(x_0)(x - x_0, r - r_0)| < C\delta$.

Property 22 (Bounds on intersection area). Let $\Gamma, \tilde{\Gamma}$ be Φ circles. Then

$$|\Gamma^\delta \cap \tilde{\Gamma}^\delta \cap B(b, C^{-2}\alpha)| \lesssim \frac{\delta^2}{\big(d(\Gamma, \tilde{\Gamma}) + \delta\big)^{1/2} \big(\Delta_{B(b, C^{-1}\alpha)}(\Gamma, \tilde{\Gamma}) + \delta\big)^{1/2}}, \tag{40}$$

$$\operatorname{diam}(\Gamma^\delta \cap \tilde{\Gamma}^\delta \cap B(b, C^{-2}\alpha)) \lesssim \Big(\frac{\Delta_{B(b, C^{-1}\alpha)}(\Gamma, \tilde{\Gamma}) + \delta}{d(\Gamma, \tilde{\Gamma}) + \delta}\Big)^{1/2}. \tag{41}$$

This is Lemma 3.1(i) in [7].

As noted above, when $\Phi(x, y) = |x - y|$, then $S_{\Gamma, B(b,\alpha)}$ is a section of the right-angled light-cone with focus at (x_0, r_0). We shall establish several lemmas that show that certain key properties of light cones are preserved when we consider the set $S_{\Gamma, B(b,\alpha)}$ for Φ a general defining function satisfying the requirements from Theorem 1.

Lemma 23. *Let $\Gamma, \tilde{\Gamma}$ be Φ–circles with*

$$\Delta_{B(b,C^{-1}\alpha)}(\Gamma,\tilde{\Gamma}) < {C'}^{-1}|x_0 - \tilde{x}_0|. \tag{42}$$

Then there exists Γ' with $x_0' = \tilde{x}_0,\ |r_0' - \tilde{r}_0| \lesssim \Delta_{B(b,C^{-1}\alpha)}(\Gamma,\tilde{\Gamma})$ such that

$$\Gamma' \in S_{\Gamma,B(b,\alpha)}. \tag{43}$$

Furthermore,

$$\Delta_{B(b,\alpha)}(\Gamma,\tilde{\Gamma}) \lesssim \operatorname{dist}(S_{\Gamma,B(b,\alpha)},\tilde{\Gamma}) \lesssim \Delta_{B(b,C^{-1}\alpha)}(\Gamma,\tilde{\Gamma}). \tag{44}$$

Remark 24. Note that we have to use different sets X in the subscript of Δ on the right and left sides of (44). In the case where $\Phi(x,y) = |x-y|$ (and thus we can define Φ over (say) a large dilate of the unit circle),

$$\Delta_{(B(0,100))}(\Gamma,\tilde{\Gamma}) = \big||x_0 - \tilde{x}_0| - |r_0 - \tilde{r}_0|\big|,$$

provided $\Gamma, \tilde{\Gamma}$ lie in suitably restricted sets, and if two circles are nearly incident, we can always change one of them slightly so that they are exactly incident. In the more general case we are considering, however, it may not always be possible to make two almost-incident curves exactly incident by changing one of them slightly; it is possible that when we try to move one of the curves to make the two curves incident, the "point of incidence" occurs outside the domain of definition of Φ (and thus there is no point of incidence). Thus, we need to be more careful about how we define incidence and almost-incidence. This consideration will occur frequently in the lemmas below, and it will significantly lengthen our analysis.

Proof. By (25) and (26), in order to obtain (44), it suffices to establish the esimate

$$\Delta_{B(b,\alpha)}(\Gamma,\tilde{\Gamma}) \lesssim \operatorname{dist}(S_{\Gamma,B(b,\alpha),0},\tilde{\Gamma}) \lesssim \Delta_{B(b,C^{-1}\alpha)}(\Gamma,\tilde{\Gamma}). \tag{45}$$

First, note that $\Delta_{B(b,\alpha)}(\cdot,\cdot)$ is jointly smooth in both variables with uniformly bounded derivatives. Since $\Delta_{B(b,\alpha)}(\Gamma,\tilde{\Gamma}) = 0$ for $\tilde{\Gamma} \in S_{\Gamma,B(b,\alpha)}$, we immediately obtain the first inequality in (45). The second inequality in (45) follows from (43), which we shall now prove.

Straighten out Φ around $\tilde{x}_0$. From Property 18 of Φ, there exists $\xi \in B(b,\alpha)\cap\Gamma$ such that

$$\nabla_y\Phi(x_0,\xi) \wedge \nabla_y\Phi(\tilde{x}_0,\xi) = 0, \tag{46}$$

i.e. (in straightened out coordinates)

$$\frac{\nabla_y\Phi(x_0,\xi)}{|\nabla_y\Phi(x_0,\xi)|} = (\pm 1, 0),$$

and

$$|\Phi(\tilde{x},\xi) - \tilde{r}_0| \lesssim \Delta_{B(b,C^{-1}\alpha)}(\Gamma,\tilde{\Gamma}),$$

where here and below the implicit constants are uniform in the choice of $\Gamma, \tilde{\Gamma}$ provided (42) is satisfied uniformly. Thus if we select $x_0' = \tilde{x}_0,\ r_0' = \tilde{r}_0 + \Phi(\tilde{x}_0,\xi)$ then ξ lies on Γ', which establishes (43). □

Corollary 25. *$S_{\Gamma,B(b,\alpha)}$ is a smooth manifold and* $\dim(S_{\Gamma,B(b,\alpha)}) \geq 2$.

Proof. For each $(x_0,r_0) \in S_{\Gamma,B(b,\alpha)}$, for $|x - x_0|$ sufficiently small we have that the map $(x,r_0) \mapsto (x,r')$ described in Lemma 23 gives us a smooth embedding of $(x_0,r_0) \ni B(b,\alpha) \subset \mathbb{R}^2$ into $S_{\Gamma,B(b,\alpha)}$. Since $B(b,\alpha)$ is a 2–dimensional smooth manifold, the result follows. □

Corollary 26. *There exists* C_0 *such that for all* Φ*–circles* Γ*, all* $(x,r)\in S_{\Gamma,B(b,\alpha)}$*, and all* $t<C^{-1}|x-x_0|$*,*

$$\pi_x\big(S_{\Gamma,B(b,\alpha)}\cap\{(x',r')\colon |x-x'|<t,|r-r'|<C_0t\}\big)=\{x'\colon |x-x'|<t\},\quad (47)$$

i.e. the cylindrical section centered at $(x,r)\in S_{\Gamma,B(b,\alpha)}$ *of radius* t *and height* Ct *contains all of (or possibly all of one of the sheets of)* $S_{\Gamma,B(b,\alpha)}$ *confined to the corresponding truncated cylinder.*

5. Counting incidences between bipartite pairs of curve families

Recall the definition of a t–bipartite pair $(\mathcal{W},\mathcal{B})$, a (δ,t)–rectangle, and a rectangle of type $(\gtrsim\mu,\gtrsim\nu)$ relative to $(\mathcal{W},\mathcal{B})$ (Definition 7).

Definition 27. We shall say that a (δ,t) rectangle R is of type $(\sim\mu,\sim\nu)$ if it is of type $(\gtrsim\mu,\gtrsim\nu)$, but is not of type $(\gtrsim C\mu,\gtrsim\nu)$ or $(\gtrsim\mu,\gtrsim C\nu)$ for some absolute constant C which shall be determined later.

Definition 28. We say that two (δ,t)–rectangles are *close* if there is a $(2\delta,t)$ rectangle containing both of them and are *comparable* if there is a $(C_0\delta,t)$–rectangle containing both of them.

For $(\mathcal{W},\mathcal{B})$ a t–bipartite pair with $t>C\delta$ and X a set, define

$$\mathcal{I}_X=\{(\Gamma,\tilde\Gamma)\in(\mathcal{W},\mathcal{B})\colon \Delta_X(\Gamma,\tilde\Gamma)<\delta\},$$
$$\tilde{\mathcal{I}}_X=\{(\Gamma,\tilde\Gamma)\in(\mathcal{W},\mathcal{B})\colon \Delta_X(\Gamma,\tilde\Gamma)<C\delta\},$$

for some constant C to be determined later, where we recall that Δ_X is defined in (13).

We shall state and prove a series of lemmas that are analogous to Lemmas 1.5–1.16 in [14]. If the proof of a lemma is the same as that of the corresponding lemma in [14] we shall omit it. Throughout the discussion below, $(\mathcal{W},\mathcal{B})$ is a t–bipartite pair with $\#\mathcal{W}=m,\ \#\mathcal{B}=n$.

Lemma 29.

(i) *If* $\Delta_{B(b,C^{-1}\alpha)}(\Gamma,\tilde\Gamma)<\delta$*, then there exists a* (δ,t)*–rectangle* $R\subset B(b,\alpha)$ *such that* Γ *and* $\tilde\Gamma$ *are tangent to any* (t,δ)*–rectangle close to* R*.*

(ii) *Conversely, if* $\Gamma,\tilde\Gamma$ *are tangent to a common* (δ,t)*–rectangle* $R\in B(b,\alpha)$ *then* $\Delta_{B(b,\alpha)}(\Gamma,\tilde\Gamma)\leq C\delta$*, and if* $\Gamma,\tilde\Gamma$ *are tangent to comparable* (δ,t)*–rectangles* $R,R'\in B(b,\alpha)$ *then* $\Delta_{B(b,\alpha)}(\Gamma,\tilde\Gamma)\lesssim\delta$*.*

Lemma 30. *Let* $\Gamma\in\mathcal{W},\tilde\Gamma\in\mathcal{B}$*. Then there are at most* $O(1)$ *incomparable* (δ,t)*–rectangles* $R\subset B(b,\alpha)$ *tangent to both* Γ *and* $\tilde\Gamma$*.*

Proof. Since $d(\Gamma,\tilde\Gamma)\sim t$, (40) gives us the bound

$$|B(b',C^{-1}\alpha)\cap\Gamma\cap\tilde\Gamma|\lesssim\delta^{3/2}t^{-1/2}\qquad(48)$$

for all b' in a sufficiently small neighborhood of b. Each (δ,t)–rectangle has area $\sim\delta^{3/2}t^{-1/2}$ and incomparable (δ,t)–rectangles are pairwise disjoint. The lemma follows by applying (48) to $O(1)$ choices of $b'=b+t_i$. □

Lemma 31.

(i) *Let* $\mathcal{R}\subset B(b,\alpha)$ *be a collection of pairwise nonclose rectangles. Then*

$$\#\tilde{\mathcal{I}}_{B(b,\alpha)}\gtrsim\#\{(R,\Gamma,\tilde\Gamma)\in\mathcal{R}\times\mathcal{B}\times\mathcal{W}\colon \Gamma \text{ and } \tilde\Gamma \text{ are tangent to } R\}.$$

(ii) There exists a a collection $\mathcal{R}$ of pairwise incomparable (δ, t)–rectangles $R \in B(b,\alpha)$ such that

$$\#\tilde{\mathcal{I}}_{B(b,C^{-1}\alpha)} \lesssim \#\{(R,\Gamma,\tilde{\Gamma}) \in \mathcal{R}\times\mathcal{B}\times\mathcal{W} \colon \Gamma \text{ and } \tilde{\Gamma} \text{ are tangent to } R\}.$$

Proof. The first statement is immediate. The second statement follows from (33). □

Lemma 32. *Let $\Gamma_1, \Gamma_2, \Gamma_3$ be three Φ–circles. Let $\mathcal{R}$ be a collection of pairwise incomparable rectangles $R \in B(b,\alpha)$ with the property that for each $R \in \mathcal{R}$ there is a Φ–circle Γ such that:*

- $d(\Gamma, \Gamma_i) \geq t,\ \ i = 1, 2, 3.$
- *Γ, Γ_1 are tangent to R.*
- *There exist two (δ, t)–rectangles $R_2, R_3 \in B(b,\alpha)$ such that Γ and Γ_i are tangent to R_i, $i = 2, 3$ and such that R_1, R_2, R_3 are pairwise incomparable.*

Then $\#\mathcal{R} \lesssim 1$.

Proof. We shall establish the proof with the additional restriction that R must lie in $B(b', C^{-2}\alpha)$ for b' in a sufficiently small neighborhood of b. Once this has been established, we can recover the full result by selecting $O(1)$ choices of b' such that $B(b,\alpha) \subset \bigcup_{b'} B(b', C^{-2}\alpha)$.

Let $R \in \mathcal{R}$ and let Γ be a Φ–circle satisfying the above conditions. Then we must have $\Gamma \in Y$, where Y is as defined in (35); indeed the above requirements on Γ are precisely those needed to ensure that $\Gamma \in Y$. By (40),

$$\Gamma \cap \Gamma_1 \cap B(b', C^{-2}\alpha) \subset B(\xi(x_0, r_0, x_1), C\delta^{1/2}t^{-1/2}). \tag{49}$$

Now, let $\Gamma_0 \in Y$ and let $\tilde{\Phi}$ be a Γ_0–adapted defining function with the same level sets as Φ. Since $\tilde{\Phi}$ has the same level sets as Φ and the gradient of $\tilde{\Phi}$ is comparable to that of Φ, it suffices to prove the lemma for $\tilde{\Phi}$. However, by (39) we have that if Γ is in the same connected component of Y as Γ_0 then

$$|\xi(x_1, r_1, x_0) - \xi(x_1, r_1, x)| \lesssim \sqrt{\delta/t}. \tag{50}$$

Since Y contains only two connected components, (49) and (50) imply that

$$\bigcup_{(x_0,r_0)\in Y} \Gamma(x_0, r_0) \cap \Gamma_1 \cap B(b', C^{-2}\alpha) \subset \Big(B(z_0, C\delta^{1/2}t^{-1/2}) \cap \Gamma_1\Big) \cup \Big(B(z_1, C\delta^{1/2}t^{-1/2}) \cap \Gamma_1\Big), \tag{51}$$

where z_0, z_1 are points in the two connected components of Y respectively. In particular, the set on the right hand side of (51) has measure $\lesssim \delta^{3/2}t^{-1/2}$. Since every $R \in \mathcal{R}$ must lie in this set, and pairwise incomparable rectangles must be disjoint, we obtain $\#\mathcal{R} \lesssim 1$. □

Lemma 33. *Let $\Gamma, \tilde{\Gamma}$ be Φ–circles with $d(\Gamma, \tilde{\Gamma}) = t > C\delta$ and $r_0 \geq \tilde{r}_0$. Let $R, \tilde{R} \in B(b, C^{-1}\alpha)$ be comparable (δ, t)–rectangles with $\Gamma, \tilde{\Gamma}$ tangent to $R, \tilde{R}$ respectively. Then*

(i) $\tilde{\Gamma} \cap B(b, C^{-1}\alpha)$ is contained in the $C\delta$–neighborhood of

$$\{y \in B(b,\alpha) \colon \Phi(x_0, y) \leq r_0\}.$$

(ii) For any constant A there is a constant $C(A)$ such that the cardinality of any set of pairwise incomparable (δ, t)–rectangles $R \in B(b, C^{-1}\alpha)$ each of which is tangent to Γ and intersects the $A\delta$–neighborhood of

$$\{y \in B(b, \alpha)\colon \Phi(\tilde{x}_0, y) \leq r_0\}$$

does not exceed $C(A)$.

Proof. Straighten Φ around x_0. By Lemma 29.(ii), with α replaced by $C^{-1}\alpha$, we have $\Delta_{B(b,C^{-1}\alpha)}(\Gamma, \tilde{\Gamma}) \leq C'\delta$. Thus if we choose the value of C in the statement of the lemma to be sufficiently large (depending on C'), then $|x_0 - \tilde{x}_0| > C''\Delta_{B(b,C^{-1}\alpha)}(\Gamma, \tilde{\Gamma})$, so by Property 18 of cinematic curvature, there exists a unique point $\xi(\tilde{x}_0, \tilde{r}_0, x_0) \in \tilde{\Gamma}$ satisfying (32), i.e.

$$\nabla_y \Phi(\tilde{x}_0, \xi) = (0, \pm 1),$$

so ξ is a global maximum of the function $y^{(1)} \mapsto \Phi(\tilde{x}_0, (y^{(1)}, y^{(2)}))$ in the domain $(y^{(1)}, y^{(2)}) \in B(b, \alpha)$, where $y^{(2)} = y^{(2)}(y^{(1)})$ is implicitly defined by $(y^{(1)}, y^{(2)}(y^{(1)})) \in \tilde{\Gamma}$ (we can verify without difficulty that this is well-defined). By (33) (noting that in the straightened out coordinate system, $\Gamma = \{y^{(2)} = 0\} \cap U_2'$),

$$\begin{aligned} \Phi(\tilde{x}_0, (\xi, y^{(2)}(\xi)) &\lesssim \Delta_{B(b,C^{-1}\alpha)}(\Gamma, \tilde{\Gamma}) \\ &\lesssim \delta, \end{aligned}$$

and thus for an appropriate choice of C,

$$\tilde{\Gamma} \cap U_2' \subset \{y^{(2)} < C\delta\}.$$

Returning to our original coordinate system, this is Statement (i) of the lemma.

To obtain the second statement, note that by the same reasoning as above,

$$\Gamma^\delta \cap \{y \in B(b, \alpha)\colon \Phi(x_2, y) \leq r_1\} + B(b, A\delta) \subset \tilde{\Gamma}^{C(A)\delta}$$

for a suitable constant $C(A)$, where the $+$ in the above equation denotes the Minkowski sum. The result then follows from (40) and the fact that incomparable rectangles are disjoint. □

Lemma 34.

(i) The cardinality of any set of $(\sim \mu, \sim \nu)$ rectangles is $\lesssim \frac{mn^{2/3}}{\mu\nu^{2/3}}$.

(ii) The cardinality of any set of $(\gtrsim \mu, \gtrsim \nu)$ rectangles is $\lesssim \frac{mn^{2/3}}{\mu\nu^{2/3}} + \frac{n}{\nu}\log\frac{m}{\mu}$.

Remark 35. Recall that a rectangle of type $(\gtrsim \mu, \gtrsim \nu)$ is a rectangle that is incident to at most $C\mu$ curves in $\mathcal{W}$ and at most $C\mu$ curves in $\mathcal{B}$ for some absolute constant C (a rectangle of type $(\sim \mu, \sim \nu)$ is defined similarly), so the statement of the lemma is well defined.

Proof. Combined with the previous lemmas, Statement (i) is simply the graph theory theorem that a $m \times n$ matrix with entries 0 and 1 which has a forbidden 2×3 submatrix of 1s has $\lesssim mn^{2/3}$ 1s in total (see e.g. [5] for a proof of this theorem). Statement (ii) is obtained from Statement (i) by dyadic summation. □

Lemma 36. *Let $(\mathcal{W}, \mathcal{B})$ be a t–bipartite pair that has no $(\gtrsim 1, \gtrsim \nu_0)$ or $(\gtrsim \mu_0, \gtrsim 1)$ rectangles $R \in B(b, \alpha)$. Then*

$$\#\mathcal{I}_{B(b,C^{-1}\alpha)}(\mathcal{W}, \mathcal{B}) \lesssim \mu_0^{1/3} nm^{2/3} \log \nu_0 + \nu_0 m \log \mu_0. \tag{52}$$

Lemma 37. *Let $\mathcal{W}, \mathcal{B}$ be a t–bipartite pair with $\#\mathcal{B} = n$. Randomly select a subset $\mathcal{D} \subset \mathcal{B}$ with $\#\mathcal{D} = N < \frac{1}{C}n$. (we shall call the elements of $\mathcal{D}$* dividing circles*), and let $\mathcal{S} = \{S_{\Gamma, B(b,\alpha)} : \Gamma \in \mathcal{D}\}$. Then with high probability (relative to our random selection of $\mathcal{D} \subset \mathcal{B}$), we can partition*

$$\mathcal{W} = \mathcal{W}^* \sqcup \bigsqcup_1^M \mathcal{W}_i \tag{53}$$

so that the decomposition has the following properties.

(i) $M \lesssim N^3 \log N$.

(ii) For each i,

$$\#\{\Gamma \in \mathcal{B} : \Delta_{B(b,C^{-1}\alpha)}(\Gamma, \tilde{\Gamma}) \lesssim \delta \text{ for some } \tilde{\Gamma} \in \mathcal{W}_i\} \lesssim \frac{n \log n}{N}.$$

(iii) For each $\Gamma \in \mathcal{W}^*$ there exists a dividing Φ–circle $\tilde{\Gamma}$ such that $\Delta_{B(b,\alpha)}(\Gamma, \tilde{\Gamma}) \lesssim \delta$.

Remark 38. The implicit constants appearing above depend only on Φ and the probability that a randomly selected $\mathcal{D} \subset \mathcal{B}$ has the desired properties. In particular, by worsening the implicit constants we can make the probability arbitrarily close to 1.

Proof. Perform the cell decomposition of the arrangement $\mathcal{D}$, as described in Lemma 12. Let

$$\mathcal{W}^* = \{\Gamma \in \mathcal{W} : \operatorname{dist}(\Gamma, S_{\tilde{\Gamma}, B(b,\alpha)}) \leq C\delta \text{ for some } \tilde{\Gamma} \in \mathcal{D}\}, \tag{54}$$

and for each $i = 1, \dots, M$, let

$$\mathcal{W}_i = \{\Gamma \in \mathcal{W} \backslash \mathcal{W}^* : \Gamma \in \overline{\Omega}_i\}. \tag{55}$$

If some Γ is present in more than one $\mathcal{W}_i$, remove it from all but one of the $\mathcal{W}_i$ (the choice is irrelevant). We shall now verify that this decomposition satisfies the properties claimed in the lemma. Claim (i) is immediate from Lemma 12, and Claim (iii) follows from (44).

Now, suppose $\Gamma \in \mathcal{W}_i, \tilde{\Gamma} \in \mathcal{B}$ with $\Delta_{B(b,C^{-1}\alpha)}(\Gamma, \tilde{\Gamma}) \leq \delta$. Then by (44), $\operatorname{dist}(\Gamma, S_{\tilde{\Gamma}, B(b,\alpha)}) \leq C\delta$ and so we can select $(x, r) \in S_{\tilde{\Gamma}, B(b,\alpha)}$ with $d((x, r), \Gamma) \leq C\delta$ (for possibly a larger constant C). Furthermore, since $(\mathcal{W}, \mathcal{B})$ is a t–bipartite pair, we have that $|x - \tilde{x}_0| \gtrsim t > C\delta$, and thus by Corollary 26, there exists

$$(x_0', r_0') \in S_{\tilde{\Gamma}, B(b,\alpha)} \cap \{(x', r') : |x' - \tilde{x}_0| < C_1\delta, |r' - \tilde{r}_0| < C_2\delta\} \tag{56}$$

such that $x_0' = x_0$. However, (56) implies that $|r_0' - r_0| < C_2\delta$, and selecting constants appropriately in the definition of $\mathcal{W}^*$, this is less than $\operatorname{dist}(\Gamma, S_{\Gamma', B(b,\alpha)})$ for any $\Gamma' \in \mathcal{D}$. Since the boundary of each cell Ω consists only of dividing surfaces $S_{\Gamma', B(b,\alpha)}$ and vertical manifolds, we conclude that $(x_0', r_0') \in \Omega_i$, and thus $S_{\tilde{\Gamma}, B(b,\alpha)} \cap \Omega_i \neq \emptyset$. Equation (28) bounds the number of dividing surfaces that can intersect each cell Ω_i, and this in turn gives us property (ii). □

Lemma 39. *With high probability,*

$$\#\mathcal{W}^* \lesssim \frac{n \# \tilde{\mathcal{I}}_{B(b,\alpha)}(\mathcal{W}, \mathcal{B})}{N}. \tag{57}$$

Proof. This follows from property (iii) of Lemma 37. Indeed, the probability of a given $w \in \mathcal{W}$ being in $\mathcal{W}^*$ is bounded by

$$\frac{n}{N}\{b \in \mathcal{B} \colon \Delta_{B(b,\alpha)} < C\delta\},$$

so the expected size of $\mathcal{W}^*$ is $\frac{n\#\tilde{\mathcal{I}}_{B(b,\alpha)}(\mathcal{W},\mathcal{B})}{N}$, from which the result follows. □

Definition 40. We define a *cluster* of Φ–circles analogously to Wolff's definition in [14]: A cluster is a subset $\mathcal{C} \subset \mathcal{W}$ (or $\mathcal{B}$) with the property that there exists a (δ, t)–rectangle R such that every $\Gamma \in \mathcal{C}$ is tangent to a (δ, t)–rectangle comparable to R.

Lemma 41. *Let $\mathcal{C} \subset \mathcal{W}$ be a cluster and let $\Gamma \in \mathcal{B}$. Then then any set of pairwise incomparable (δ, t)–rectangles each of which is tangent to some circle in $\mathcal{C}$ and to Γ has cardinality $O(1)$.*

Remark 42. Lemma 33 is used to prove this lemma.

Lemma 43. *Given a value of μ_0, we can write*

$$\mathcal{W} = \mathcal{W}_g \sqcup \mathcal{W}_b \tag{58}$$

where

(i) $\mathcal{W}_g$ and $\mathcal{B}$ have no (δ, t)–rectangles of type $(\gtrsim \mu_0, \gtrsim 1)$.
(ii) $\mathcal{W}_b$ is the union of $\lesssim \frac{\#\mathcal{W}}{\mu_0}(\log m)(\log n)$ clusters.

Lemma 44. *Let $(\mathcal{W}, \mathcal{B})$ be a t–bipartite pair with $m = |\mathcal{W}|,\ n = |\mathcal{B}|$. Let $\mathcal{R}$ be a set of pairwise incomparable $(\geq \mu, \geq \nu)$ (δ, t)–rectangles $R \subset B(b, \alpha)$.*

For any $\epsilon > 0$,

$$\#\mathcal{R} \lesssim_\epsilon (mn)^\epsilon\Big(\Big(\frac{mn}{\mu\nu}\Big)^{3/4} + \frac{m}{\mu} + \frac{n}{\nu}\Big). \tag{59}$$

In order to prove Lemma 44, it suffices to consider the case where $\mu = \nu = 1$ and establish the bound

$$\#\mathcal{R} \lesssim_\epsilon (mn)^\epsilon\Big((mn)^{3/4} + m\log n + n\log m\Big). \tag{60}$$

To obtain (59) from (60) we apply a random sampling argument. The details of this random sampling argument are in [14, p1253], so we shall not reproduce them here. We shall call the Φ–circles $\Gamma \in \mathcal{W}$ "white" Φ–circles and those in $\mathcal{B}$ "black" Φ–circles. By Lemma 30, each pair $(\Gamma, \tilde{\Gamma}) \in (\mathcal{W}, \mathcal{B})$ of white and black Φ–circles are jointly incident to at most $O(1)$ incomparable (δ, t)–rectangles, so $\#\mathcal{R} \lesssim mn$. Thus if $(mn)^{1/C} < \log(mn)$ then (60) holds immediately (with an implicit constant depending on C). Thus we may assume

$$(mn)^{1/C} > \log(mn) \tag{61}$$

for some fixed choice of C which will be determined below.

We shall closely follow [14] and substitute our lemmas above for Wolff's analogous ones. Wolff's induction argument allows him to control the number of incomparable (δ, t)–rectangles of type $(\gtrsim 1, \gtrsim 1)$ relative to a collection $(\mathcal{W}, \mathcal{B})$ over the region $B(b, \alpha)$ if he has similar control over smaller collections $(\mathcal{W}', \mathcal{B}')$. Our argument will allow us to control the number of incomparable $(\gtrsim 1, \gtrsim 1)$ rectangles in a small region $B(b, C^{-1}\alpha)$ if we have control over the number of incomparable rectangles in a much larger region $B(b, \alpha)$, but luckily we only require this control for smaller collections of circles. Since the control is uniform in b, we can apply this result to finitely many translates $\{b+t_i\}$ of b to recover the result over the larger region $B(b,\alpha)$, which allows us to iterate the induction step. We shall focus on the key steps where our arguments differ from Wolff's, and refer readers to [14] for the details of those arguments which are identical.

Assume (60) holds for all pairs $(\mathcal{W}',\mathcal{B}')$ with $(\#\mathcal{W}')(\#\mathcal{B}') < mn/2$. The base case of the induction is taken care of by (61).

If $m \leq n^{\frac{1}{3}+\epsilon}$ or vice versa, then Lemma 44 follows from Lemma 34. Thus we may assume

$$m^{1/3+\epsilon} < n < m. \tag{62}$$

Let $\mathcal{W} = \mathcal{W}_g \cup \mathcal{W}_b,\ \mathcal{B} = \mathcal{B}_g \cup \mathcal{B}_b$ be the decomposition from Lemma 43 with $\mu_0 = \nu_0 = (mn)^{1/4}$. From property (ii) of the decomposition we have

$$\#\tilde{\mathcal{I}}_{B(b,\alpha)}(\mathcal{W}_b,\mathcal{B}) < \log m \log n (mn)^{3/4}, \tag{63}$$

$$\#\tilde{\mathcal{I}}_{B(b,\alpha)}(\mathcal{W},\mathcal{B}_b) < \log m \log n (mn)^{3/4}. \tag{64}$$

These quantities are $< \frac{1}{1000}(mn)^{\epsilon}(mn)^{3/4}$ provided that we choose the appropriate constant C in (61).

We shall now obtain the bound

$$\#\tilde{\mathcal{I}}_{B(b',C^{-1}\alpha)}(\mathcal{W}_g,\mathcal{B}_g) \leq C_\epsilon (mn)^{\epsilon}(mn)^{3/4} C_0^{-1}, \tag{65}$$

where we can make C_0 arbitrarily large at the cost of increasing C_ϵ. Furthermore, this bound will be independent of the choice of $b' \in B(b,\alpha)$. Thus we shall apply (65) with $b' = b + t_i$ for $\{t_i\}$ a finite family of translates such that $B(b,\alpha) \subset \bigcup B(b+t_i, C^{-1}\alpha)$. Then

$$\tilde{\mathcal{I}}_{B(b,\alpha)}(\mathcal{W}_g,\mathcal{B}_g) \subset \bigcup \tilde{\mathcal{I}}_{B(b+t_i,C^{-1}\alpha)}(\mathcal{W}_g,\mathcal{B}_g), \tag{66}$$

since if $\Delta_{B(b,\alpha)}(\Gamma,\tilde{\Gamma}) < C\delta$ then we must have $\Delta_{B(b+t_i,C^{-1}\alpha)}(\Gamma,\tilde{\Gamma}) < C\delta$ for some i. Thus if we apply (65) for each t_i and select C_0 sufficiently small we obtain

$$\#\tilde{\mathcal{I}}_{B(b,\alpha)}(\mathcal{W}_g,\mathcal{B}_g) \leq C_\epsilon (mn)^{\epsilon}(mn)^{3/4} C_0^{-1}. \tag{67}$$

Combining (67), (63), and(64) and using Lemma 31 we obtain (60). It thus suffices to prove (65).

Let us perform the decomposition $\mathcal{W}_g = \mathcal{W}_g^* \sqcup \bigsqcup_1^M \mathcal{W}_g^i$ as given by Lemma 37, with α replaced by $C^{-1}\alpha$ and selecting a value of N satisfying

$$C\log(mn)^{1/\epsilon} < N < C^{-1}\min\Big(n^{3/4}m^{-1/4}\log(mn), m^{1/4}n^{-1/12}\log(mn)\Big). \tag{68}$$

Such a value of N exists by assumption (62) and by selecting a sufficiently large constant in (61).

We claim:

$$\#\mathcal{W}_g^* \leq \frac{1}{1000C_0}\#\mathcal{W}_b. \tag{69}$$

Indeed, $(\mathcal{W}_g,\mathcal{B}_g)$ contain no (δ,t)–rectangles of type $(\gtrsim \mu_0 \gtrsim 1)$ or $(\gtrsim 1, \gtrsim \nu_0)$ so by Lemma 36 (with δ replaced by $C\delta$ for a suitable constant C),

$$\#\tilde{\mathcal{I}}_{B(b,C^{-1}\alpha)}(\mathcal{W}_g,\mathcal{B}_g) \lesssim m^{5/4}n^{1/4}\log m + m^{3/4}n^{13/12}\log n,$$

and thus by Lemma 39 (recall that now α is replaced by $C^{-1}\alpha$ and $C^{-1}\alpha$ is replaced by $C^{-2}\alpha$) we can select our decomposition of $\mathcal{W}_g$ so that

$$\#\mathcal{W}_g^* \lesssim \frac{n}{N}\Big(m^{5/4}n^{1/4}\log m + m^{3/4}n^{13/12}\log n\Big). \tag{70}$$

Using (68) and selecting a sufficiently large constant in (61) (of course the choice of constant in (61) will depend on the desired constant C_0 in (69)) we obtain (69). Since $(\#\mathcal{W}_g^*)(\#\mathcal{B}) < mn/2$ we can apply the induction hypothesis to obtain

$$\#\tilde{\mathcal{I}}_{B(b,\alpha)}(\mathcal{W}_g^*,\mathcal{B}_g) \leq \frac{1}{1000C_0}C_\epsilon(mn)^\epsilon\Big((mn)^{3/4}+m\log n+n\log m\Big). \tag{71}$$

Now, for each i let

$$\mathcal{B}_g^i = \{\Gamma\in\mathcal{B}_g\colon \Delta_{B(b,C^{-2}\alpha)}(\Gamma,\tilde\Gamma) < C\delta \text{ for some } \tilde\Gamma\in\mathcal{W}_g^i\}. \tag{72}$$

Item (ii) in Lemma 37 implies

$$\#\mathcal{B}_g^i \lesssim \frac{n\log n}{N}. \tag{73}$$

Now, we can apply the induction hypothesis to the pair $(\mathcal{W}_g^i,\mathcal{B}_g^i)$ to conclude

$$\#\tilde{\mathcal{I}}_{B(b,\alpha)}(\mathcal{W}_b^i,\mathcal{B}_b^i) \leq C_\epsilon[(\#\mathcal{W}_b^i)(\#\mathcal{B}_b^i)]^\epsilon[(\#\mathcal{W}_b^i)(\#\mathcal{B}_b^i)]^{3/4}C_0^{-1}. \tag{74}$$

However, $\mathcal{B}_g^i$ was selected so that

$$\#\tilde{\mathcal{I}}_{B(b,C^{-2}\alpha)}(\mathcal{W}_g^i,\mathcal{B}_g) \leq \#\tilde{\mathcal{I}}_{B(b,\alpha)}(\mathcal{W}_g^i,\mathcal{B}_g^i),$$

and thus (74) implies

$$\#\tilde{\mathcal{I}}_{B(b,C^{-2}\alpha)}(\mathcal{W}_g^i,\mathcal{B}_g) \leq C_\epsilon[(\#\mathcal{W}_g^i)(\#\mathcal{B}_g^i)]^\epsilon[(\#\mathcal{W}_g^i)(\#\mathcal{B}_g^i)]^{3/4}C_0^{-1}. \tag{75}$$

Summing (75) over the $M \lesssim N^3\log N$ choices of i and applying Hölder's inequality (see [14, p1252-3], for the details), we obtain

$$\#\bigcup_i\tilde{\mathcal{I}}_{B(b,C^{-2}\alpha)}(\mathcal{W}_g^i,\mathcal{B}_g^i) \leq \frac{1}{1000C_0}C_\epsilon(mn)^\epsilon\Big((mn)^{3/4}+m\log n+n\log m\Big). \tag{76}$$

Combining (76), (68), and (71) we obtain (65).

6. Riemannian metric circles and other generalizations

It is reasonable to ask whether (9) holds for functions Φ which satisfy the cinematic curvature conditions but are not algebraic. An examination of the arguments above reveals that the only place where the algebraic properties of Φ are used is in Lemma 12, where we make use of the fact that the level sets of $\Phi(x,\cdot)$ (and of various functions obtained from Φ) are algebraic curves, and in particular, any two such curves intersect $O(1)$ times.

One might hope that we could extend (9) to analytic Φ by approximating Φ by the first $\sim|\log\delta|$ terms of its Taylor expansion. Unfortunately, the bounds obtained above are more than superexponential in the degree of Φ, so if we approximate Φ by a polynomial of degree $\sim|\log\delta|$ then the above proof yields maximal function bounds that are worse than the Kolasa-Wolff result (8).

Working through the proof of Lemma 12, we see that the proof requires us to control the number of times certain pairs of curves can intersect. For $x,\tilde x\in U_1,\omega\in\{\pm1\}$, let

$$\gamma_{x,\tilde x,\omega,r} = \{y\colon \Phi(x,y)+\omega\Phi(\tilde x,y) = r\}. \tag{77}$$

We shall call such curves Φ–conics.

Definition 45. We say that Φ has the *bounded conic intersection property* if it satisfies

(i) If $\{x,\tilde{x}\}\neq\{x',\tilde{x}'\}$, then

$$\#(\gamma_{x,\tilde{x},\omega,r}\cap\gamma_{x',\tilde{x}',\omega',r'})\lesssim 1. \tag{78}$$

(ii) All Φ–circles Γ and Φ–conics γ have $O(1)$ $y^{(1)}$–extremal points (defined below).

Definition 46. A $y^{(1)}$–extremal point of a curve ζ is a point $y_0\in\zeta$ such that $\zeta\cap V$ is contained in one of the closed half-spaces $\{y^{(1)}\geq y_0^{(1)}\}$ or $\{y^{(1)}\leq y_0^{(1)}\}$ for V a sufficiently small open neighborhood of y_0.

Requirement (78) is the most difficult to satisfy, and it is the analogue of the Euclidean statement that distinct irreducible conic sections intersect in at most $O(1)$ places (actually 4).

If Φ satisfies the cinematic curvature hypotheses, it need not have the bounded conic intersection property. Indeed, consider the example

$$\Phi(x,y)=y^{(2)}+x^{(1)}y^{(1)}+x^{(2)}(y^{(1)})^2+p(x,y). \tag{79}$$

If $p(x,y)=0$, the Φ–conics

$$\gamma=\{y\colon\Phi((1,0),y)+\Phi((-1,0),y)=r\},$$
$$\tilde{\gamma}=\{y\colon\Phi((0,1),y)+\Phi((0,-1),y)=r\}$$

are identical (both are simply the line $y^{(2)}=r$. Thus we can select p to be a highly oscillatory C^∞ perturbation which causes $\#(\gamma\cap\tilde{\gamma})$ to be arbitrarily large, independent of (say) the C^3–norm of Φ (we could choose some other reasonable norm on Φ and construct similar counter-examples). For example, we could choose

$$\rho(x,y)=C^{-1}\phi(x)\Big(y^{(2)}-\exp\Big[-1/|(y^{(1)})^6|\Big]\sin\Big(\exp\Big[1/|(y^{(1)})^2|\Big]\Big)\Big)$$

for $\phi(x)$ a C^∞ function supported in a small neighborhood of $(1,0)$. This choice of Φ satisfies the cinematic curvature hypothesis, since it satisfies (30) and (31) (provided we choose C sufficiently large so the contributions from ρ do not affect the calculations), but it does not satisfy (78). Of course, the Φ given in (79) may still satisfy (9), but a different proof would be needed. While general Φ need not satisfy (78), we conjecture

Conjecture 47. *Let $\Phi(x,y)=\rho(x,y)$ for ρ a Riemannian metric sufficiently close to Euclidean. Then Φ satisfies the bounded conic intersection property.*

This would imply

Corollary 48 (conditional on conjecture 47)**.** *Let $\Phi(x,y)$ be as in Conjecture* (47)*. Then* (9) *holds for M_Φ.*

Remark 49. Actually, we can still obtain Corollary 48 if we weaken Conjecture 47 to the following statement: If $\Phi(x,y)=\rho(x,y)$ for ρ a Riemannian metric, define a δ–generic Φ–conic to be a curve $\gamma_{x,\tilde{x},\omega,r}$ which is not contained in the δ–neighborhood of any geodesic (this is a quantitative analogue of an (algebraic) conic section being irreducible). Then $\gamma_{x,\tilde{x},\omega,r}$ admits a decomposition $\gamma_{x,\tilde{x},\omega,r}=\cup_i\gamma^i_{x,\tilde{x},\omega,r}$ into $\lesssim|\log\delta|^C$ connected components such that (78) is satisfied for any two components of any two Φ–conics.

Appendix A. The Cell Decomposition

We shall give a brief sketch of the techniques developed by Chazelle et al. in [4] (see also [5] and [1] for a rigorous exposition closer to the one sketched here) on the method of vertical cell decompositions and random sampling.

Let $\mathcal{S} = \{S_1, \ldots, S_N\}$ be a collection of 2–dimensional semi-algebraic sets in $\mathbb{R}^3$ (for which we shall use the coordinates (x, r)).

By subdividing each S_i into a bounded number of pieces if necessary, we may assume that each set S_i may be written in one of the following three forms:

- $S = \mathrm{graph}(f)$, for $f\colon V \to \mathbb{R}$ a smooth algebraic function and $V \subset \mathbb{R}^2$ an open semi-algebraic set. We shall call these sets "surface patches".
- S_i a semi-algebraic set with $\dim(S_i) = 2$ but $\dim(\pi_x(S_i)) = 1$. We shall call these sets "vertical manifolds."
- S_i a semi-algebraic set with $\dim(S_i) < 2$.

To keep our exposition brief, we shall ignore the latter two types of sets, since their presence is merely a technical annoyance that does not contribute significantly to the analysis of the decomposition. Thus we shall assume that the sets in $\mathcal{S}$ consist entirely of surface patches.

Definition 50. For S a surface patch, we shall define $\mathrm{bdry}(S) = \overline{S}\backslash S$, where $\overline{S}$ denotes the closure of S in the Euclidean (rather than Zariski) topology. Note that $\dim(\mathrm{bdry}(S)) = 1$.

Definition 51. A *vertical line segment* $L \subset \mathbb{R}^3$ is a connected 1–dimensional semi-algebraic set with the property that $\pi_x(L)$ is a point. If $(x_0, r_0) \in \mathbb{R}^3$, we say that the (connected) vertical line segment L containing (x_0, r_0) is maximal with respect to $\mathcal{S}$ if L meets no point of any surface in $\mathcal{S}$ except possibly at (x_0, r_0), but any strictly larger line segment does.

If $\gamma \subset \mathbb{R}^3$ is a 1–dimensional semi-algebraic set (i.e. a union of segments of algebraic curves) which is not a union of vertical lines, then if we erect a maximal line segment from every point of γ we obtain a 2–dimensional semi-algebraic set V_γ with $\pi_x(V_\gamma) = \pi_x(\gamma)$. We shall call this set the "maximal vertical wall above γ" (relative to $\mathcal{S}$).

To construct the cell decomposition, erect a maximal vertical wall above $S \cap \tilde{S}$ for every pair of distinct $S, \tilde{S} \in \mathcal{S}$, and a maximal vertical wall above $\mathrm{bdry}(S)$ for each $S \in \mathcal{S}$. If we consider $\mathbb{R}^3$ with the surfaces $S \in \mathcal{S}$ and the above maximal vertical walls removed, then the remaining connected sets (which we shall call pre-cells) each have a unique "top" and "bottom" bounding surface, i.e. for each pre-cell Ω there are unique $S, \tilde{S} \in \mathcal{S}$ such that any maximal line containing $(x, r) \in \Omega$ terminates at points in S and $\tilde{S}$. Thus at this point, each pre-cell is a "cylindrical algebraic set," i.e. it is of the form

$$\Omega = \{(x, r)\colon x \in V_\Omega, f_{1,\Omega}(x) < r < f_{2,\Omega}\}$$

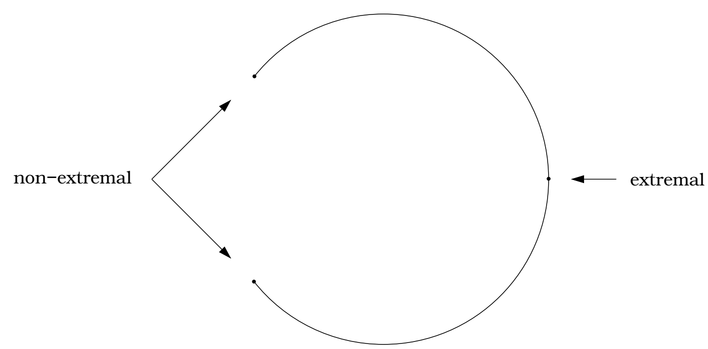
for $V_\Omega \subset \mathbb{R}^2$ an open, semi-algebraic set and $f_{1,\Omega}$, $f_{2,\Omega}$ algebraic functions.

Now, $\mathrm{bdry}(V_\Omega)$ is a 1–dimensional semi-algebraic set, and thus can be written uniquely as an almost disjoint finite union of segments of irreducible algebraic curves such that if any two segments share a boundary point then their defining polynomials are distinct (and thus neither defining polynomial divides the other). We will call the boundaries of these segments the *vertices* of V_Ω. Now, for each vertex $x_0 \in V_\Omega$, erect the wall

$$W_{x_0,\Omega} = \{(x,r) \in \Omega \colon x^{(1)} = x_0^{(1)}\}.$$

Finally, if γ is a 1–dimensional semi-algebraic set, then we say that $x_0 \in \Gamma$ is a $x^{(1)}$–extremal point if there exists an open neighborhood U of x_0 and an irreducible algebraic curve γ' containing $\gamma \cap U$ such that $\gamma' \cap U$ is contained in one of the closed half planes $\{x \colon x^{(1)} \geq x_0^{(1)}\}$ or $\{x \colon x^{(1)} \leq x_0^{(1)}\}$ (see Figure 1).

FIGURE 1. Examples of extremal and non-extremal points of a semi-algebraic curve.



Remark 52. This definition of a $x^{(1)}$–extremal point is consistent with the definition given in Section 6 (Definition 46) for Φ–conics when $\Phi(x,y)$ is a smooth algebraic function. Definition 46 needs to be worded slightly differently since when Φ is not algebraic, there is no analogous notion of an irreducible algebraic component of a semi-algebraic set, so we need to proceed more carefully.

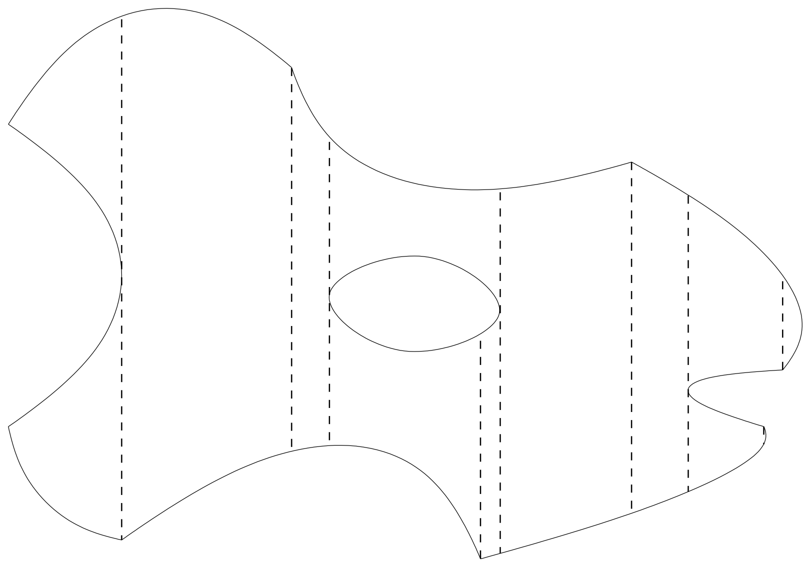
For each extremal point $x_0 \in V_\Omega$, erect the vertical wall $W_{x_0,\Omega}$. Once this has been done, a vertical wall will have been erected in Ω above each of the dashed lines in V_Ω in Figure 2. We also need to add some additional vertical walls $W_{x_0,\Omega}$ with x_0 the endpoint of certain line segments (since the irreducible algebraic curve that contains a line segment is of course a line, which (provided it is not parallel to the $x^{(2)}$–axis) does not have any $x^{(1)}$–extreme points), but in the interest of brevity we shall gloss over this point (we can also ensure that line segments never occur by applying a slight perturbation at an earlier stage of the decomposition).

Once these vertical walls have been erected for each cell Ω, the resulting arrangement of surfaces partitions $\mathbb{R}^3$ into topologically trivial open sets (cells). This partition has the following properties:

(i) Each cell is a semi-algebraic set defined by at most 6 algebraic surfaces.
(ii) For each cell Ω, there is a collection of at most 6 surfaces $S_1, \ldots, S_6 \in \mathcal{S}$ such that if the above cell decomposition algorithm were applied to $\mathcal{S}' = \{S_1, \ldots, S_6\}$, then Ω would be one of the resulting cells in the decomposition.
(iii) There are $\lesssim N^3 \log N$ cells.

Properties (i) and (ii) are immediate from the above cell decomposition algorithm: each cell Ω is contained in a unique pre-cell Ω'. The top and bottom of Ω are the

FIGURE 2. A schematic view of $\pi_x(\Omega)$ after vertical walls have been erected. The dashed lines correspond to vertical walls.



same algebraic surfaces $S, \tilde{S}$ as the top and bottom of Ω'. The "front" and "back" walls of Ω (if the exist) are segments of the vertical wall raised above curves $\gamma, \tilde{\gamma}$ which were obtained by intersecting respectively S and $\tilde{S}$ with two other surfaces $S', \tilde{S}' \in \mathcal{S}$, and the "right" and "left" walls of Ω (if they exist) are walls of the form $W_{x_0,\Omega'}$ where x_0 is a point of intersection of γ_1 and γ_2, where γ_1 is a section of $S \cap S'$ or $\tilde{S} \cap \tilde{S}'$, and γ_2 is a section of $S \cap S_1$ or $\tilde{S} \cap \tilde{S}_1$ for some S_1 or $\tilde{S}_1 \in \mathcal{S}$.

The analysis required to obtain (iii) is somewhat lengthy, but the key idea is as follows. The main step in obtaining property (iii) is to bound the number of vertices in the sets V_Ω, since a bound on the number of vertices leads to a bound on the number of vertical walls $W_{x_0,\Omega}$ added to the arrangement (the contribution from the vertical walls from $x^{(1)}$–extremal points is negligible). These vertices arise when the algebraic curves defining ∂V_Ω intersect. By Bézout's theorem, any two algebraic curves intersect in at most $O(1)$ places (since Φ is of bounded degree, all of the algebraic curves appearing in the cell decomposition are also of bounded degree). This fact allows us to use the theory of Davenport-Schintzel sequences to control the *total* number of intersections between the algebraic curve segments that define the boundaries of the cells (and thus the total number of vertices occurring in the sets V_Ω as Ω ranges over the cells in the decomposition).

Property (ii) of the cell decomposition allows us to use a random sampling argument of the type discussed in [5] to obtain Lemma 12. We shall give a brief sketch of this lemma here. Let $\mathcal{S}$ be a collection of 2–dimensional semi-algebraic surfaces with $\#\mathcal{S} = n$. Randomly select a subset $\mathcal{D} \subset S$ with $\#\mathcal{D} = N < C^{-1}n$ (the requirement $N < C^{-1}n$ allows us to gloss over the distinction between selecting curves from $\mathcal{S}$ with and without replacement, since the probability of the same curve being selected twice is low) Apply the above cell decomposition algorithm to the collection $\mathcal{D}$. For each resulting cell Ω in the decomposition, let

$$Z(\Omega) = \#\{S \in \mathcal{S} \colon S \cap \Omega \neq \emptyset\}.$$

Then,

$$\mathbb{P}\big(Z(\Omega) \geq \lambda \mid \Omega \cap S = \emptyset \text{ for all } S \in \mathcal{D}\big) \leq \Big(1 - \frac{\lambda}{n}\Big)^N. \tag{80}$$

If we set $\lambda = C\frac{n \log n}{N}$, then the right hand side of (80) is $\lesssim n^{-C}$. Thus since our vertical algebraic decomposition gives us an injection from $\mathcal{D}^6$ into the collection of all cells arising from the decomposition of the collection of surfaces $\mathcal{D}$, and since each cell in the resulting decomposition does not intersect any of the surfaces in $\mathcal{D}$ (since the cells are subsets of $\mathbb{R}^3 \backslash \bigcup_{S \in \mathcal{D}} S$), the probability that even a single cell meets more than $\lambda = C\frac{n \log n}{N}$ surfaces is at most $C' n^{6-C}$, which we can make arbitrarily small by choosing C sufficiently large.

Appendix B. Real Algebraic Geometry

In this appendix we shall briefly review a few definitions and theorems from real algebraic geometry. Throughout our discussion, the base field shall be $\mathbb{R}$ and all polynomials shall be assumed to have real coefficients. Unless otherwise noted, all open sets shall be assumed to be open in the Euclidean topology. Many of the results discussed below are applicable to any real fields but we shall not pursue this here. Further details on the material reviewed below can be found in [3], [2], and [8] (see [10] for an English summary of the key results we need from [8]).

Definition 53. A set $S \subset \mathbb{R}^n$ is *semi-algebraic* if

$$S = \bigcup_{i=1}^{n} \{x \colon f_{i,1}(x) = 0, \dots f_{i,\ell_i}(x) = 0, g_{i,1}(x) > 0, \dots, g_{i,m_i}(x) > 0\} \tag{81}$$

for $\{f_{i,j}\}$ and $\{g_{i,j}\}$ polynomials.

Definition 54. The *complexity* of a semi-algebraic set is defined as

$$\min\Big(\sum_{i,j} \deg f_{i,j} + \sum_{i,j} \deg g_{i,j}\Big), \tag{82}$$

where the minimum is taken over all representations of S of the form (81).

Remark 55. This definition of complexity is not standard. In the body of the paper we refer to sets of "bounded complexity." This means that the complexity of the semi-algebraic set is bounded by a number that depends only on the defining function Φ from (9).

Definition 56. A function $f \colon \mathbb{R}^n \to \mathbb{R}^m$ is semi-algebraic if its graph is a semi-algebraic set. The complexity of a semi-algebraic function is the complexity of its graph.

Theorem 57 (Tarski-Seidenberg)**.** *Let $S \subset \mathbb{R}^n$ be semi-algebraic. Then*

$$\pi_{(x_1,\dots,x_{n-1})}(S) \subset \mathbb{R}^{n-1}$$

is semi-algebraic, and the complexity of $\pi_{(x_1,\dots,x_{n-1})}(S)$ is controlled by the complexity of S.

Definition 58. Let $S \subset \mathbb{R}^n$ be a semi-algebraic set. We define

$$\mathcal{I}(S) = \{f \in \mathbb{R}[X_1, \ldots, X_n] \colon f|_S = 0\}. \tag{83}$$

$\mathcal{I}(S)$ is an ideal in $\mathbb{R}[X_1, \ldots, X_n]$.

Definition 59. For an ideal $\mathcal{I}$ in $\mathbb{R}[X_1, \ldots, X_n]$, we define

$$\mathcal{Z}(I) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon f(x_1, \ldots, x_n) = 0 \text{ for all } f \in \mathcal{I}\}, \tag{84}$$

so in particular, $S \subset \mathcal{Z}(\mathcal{I}(S))$.

Definition 60. let S be a semi-algebraic set. We define $\mathcal{P}(S) = \mathbb{R}[X_1, \ldots, X_n]/\mathcal{I}(S)$. Then the dimension of S is given by

$$\dim(S) = \dim(\mathcal{P}(S)),$$

the maximal length of a chain of prime ideals in the ring $\mathcal{P}(A)$ (see e.g. [6] for a discussion of these ideas).

Proposition 61. *Let S be a semi-algebraic set. Then S has the same dimension as its closure in the real Zariski topology, i.e.*

$$\dim(S) = \dim(\mathcal{Z}(\mathcal{I}(S))),$$

and the latter set is algebraic.

Proposition 62. *Let $f(\underline{x}, x_{n+1})$ be a polynomial in $n+1$ variables. Then there exists a partition of $\mathbb{R}^n$ into semi-algebraic sets $A_1, \ldots, A_m$ and for each i a finite number of semi-algebraic functions $\xi_{i,1}, \ldots, \xi_{i,\ell_i} \colon A_i \to \mathbb{R}$ such that*

(i) For each $\underline{x} \in A_i$ such that $f(\underline{x}, \cdot)$ is not identically 0,

$$\{\xi_{i,1}(\underline{x}), \ldots, \xi_{i,\ell_i}(\underline{x})\} = \{x_{n+1} \colon f(\underline{x}, x_{n+1}) = 0\}. \tag{85}$$

(ii)

$$\operatorname{graph}(\xi_{i,j}) \subset \{f = 0\}. \tag{86}$$

The complexity of the A_i and $\xi_{i,j}$ depend only on the complexity of f.

Corollary 63. *Let $S \subset \mathbb{R}^{n+1}$ be an algebraic set. Then we can write*

$$S = \bigcup_1^n S_i \cup \bigcup_1^m T_i, \tag{87}$$

with $S_i = \operatorname{graph}(f_i|_{A_i})$ for f_i a smooth algebraic function and $A_i \subset \mathbb{R}^n$ an open semi-algebraic set, and $\dim \pi_{(x_1,\ldots,x_n)}(T_i) < \dim S$. The complexity of the f_i, A_i, and T_i depend only on the complexity of S.

Remark 64. In addition to Proposition 62, Corollary 63 relies on the the fact that the set of singular points of a semi-algebraic set is itself a semi-algebraic set of strictly lower dimension (see [3, Chapter 2] for a complete discussion of these ideas).

Proposition 65. *Let $S = \bigcup_1^n S_i$ with S_i a semi-algebraic set homeomorphic to $[0,1]^{d_i}$. Then $\dim(S) = \max\{d_1, \ldots, d_n\}$.*

Proposition 66. *Let S be a semi-algebraic set that is also a smooth manifold. Then $\dim(S)$ equals the dimension of S as a smooth manifold.*

References

1. P. Agarwal, M. Sharir. *Davenport–Schinzel Sequences and Their Geometric Applications*, Cambridge University Press, NY. 1995
2. S. Basu, R. Pollack, M Roy. *Algorithms in Real Algebraic Geometry*, Springer, Berlin. 2006
3. J. Bochnak, M. Coste, M. Roy. *Real Algebraic Geometry*, Springer-Verlag, Berlin. 1998
4. B. Chazelle, H. Edelsbrunner, L. Guibas, M. Sharir. A singly-exponential stratification scheme for real semi-algebraic varieties and its applications. Automata, Languages and Programming 372:179–193. 1989
5. K. Clarkson, H. Edelsbrunner, L. Guibas, M. Sharir, E. Welzl. Combinatorial Complexity Bounds for Arrangements of Curves and Surfaces. Discrete and Computational Geometry, 5(1):99–160. 1990.
6. D. Eisenbud. *Commutative Algebra: with a View Toward Algebraic Geometry*, Springer-Verlag, New York. 1995
7. L. Kolasa, T. Wolff. On some variants of the Kakeya problem. Pacific Journal of Mathematics, 190(1):111–154. 1999
8. S. Łojasiewicz. *Ensembles semi-analytiques*. IHES, 1965.
9. W. Schlag. On continuum incidence problems related to harmonic analysis. J. Func. Analysis 201(2):480–521. 2003
10. M. Shiota. *Geometry of subanalytic and semialgebraic sets*. Birkhäuser, Boston. 1997
11. C. Sogge. Propagation of singularities and maximal functions in the plane. Invent. Math. 104:349-376. 1991
12. T. Wolff. A Kakeya-Type Problem for Circles. American Journal of Mathematics, 119(5):985–1026. 1997
13. T. Wolff. Recent work connected with the Kakeya problem. *Prospects In Mathematics*, H. Rossi, ed., AMS 1999
14. T. Wolff. Local smoothing type estimates on L^p for large p. Geometric And Functional Analysis, 10(5):1237–1288. 2000

Department of Mathematics, UCLA, Los Angeles CA 90095-1555, USA
E-mail address: jzahl@math.ucla.edu