TURAEV-VIRO INVARIANTS AS AN EXTENDED TQFT III

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Abstract. In the third paper in this series, we examine the Reshetikhin-Turaev and Turaev-Viro TQFTs at the level of surfaces. In particular, we show that for a closed surface Σ, $Z_{TV,\mathcal{C}}(\Sigma) \cong Z_{RT,Z(\mathcal{C})}(\Sigma)$, thus extending the equality of 3-manifold invariants proved in [TV II] to an equivalence of TQFTs.

Introduction

In this paper we continue the work from [BK], [Bal] in which we generalized the Turaev-Viro state-sum invariant to manifolds with corners. This gave an extended Topological Quantum Field Theory (TQFT). Using this extended theory, we showed that for a closed 3-manifold $\mathcal{M}$, $Z_{TV,\mathcal{C}}(\mathcal{M}) = Z_{RT,Z(\mathcal{C})}(\mathcal{M})$, where $\mathcal{C}$ is a spherical fusion category, $Z(\mathcal{C})$ is its Drinfeld Center (which is modular) and $Z_{RT,Z(\mathcal{C})}$ is the Reshetikhin-Turaev invariant based on $Z(\mathcal{C})$. In this paper, we show that the TQFTs are isomorphic at the level of surfaces. Namely, if Σ is a closed surface, we show that there is a nautural isomorphism $Z_{TV,\mathcal{C}}(\Sigma) \cong Z_{RT,Z(\mathcal{C})}(\Sigma)$ of vector spaces. We also note that we actually get an equivalence of extended 3-2-1 theories if we impose mild restrictions on the allowed types of manifolds with corners.

It is easy to compute the dimensions of the above spaces:

$$\mathrm{Dim}\, Z_{TV}(\Sigma_g) = \mathrm{Dim}\, Z_{RT}(\Sigma_g) = \mathcal{D}^{2g-2} \sum_{i \in \mathrm{Irr}(\mathcal{C})} d_i^{2-2g} \tag{0.1}$$

where $\mathcal{D}$ is the dimension of $\mathcal{C}$ and d_i is the dimension of simple object X_i. The vector spaces are therefore isomorphic, but this is not enough. We need to exhibit a natural isomorphism between the spaces.

The same question occurs in general when attempting to define any 2D modular functor. For example, in RT theory, one decomposes the surface Σ into a union of punctured spheres[1], evaluates Z_{RT} for each of them, and uses the gluing axiom to obtain $Z_{RT}(\Sigma)$. A priori, this appears to depend on the choice of decomposition of Σ. Refining earlier work by Moore and Seiberg, Bakalov and Kirillov [BK2000] proposed a set of *moves* (The "Lego-Teichmüller Game") relating any two such decompositions. One can show that each of these moves corresponds to a certain natural isomorphism of vector spaces and any two "paths" between two chosen decompositions yield the same map. The space $Z(\Sigma)$ is therefore well defined.

In this paper we apply the results described above to TV theory. In [BK], we constructed an isomorphism

$$Z_{TV}(\Sigma) \cong \mathrm{Hom}_{Z(\mathcal{C})}(\mathbf{1}, Y_1 \otimes \cdots \otimes Y_n) \tag{0.2}$$

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[1]Following [BK2000], we call this a *cut sytem*.

where Σ is an n-punctured sphere with boundary components labeled by $Y_1, \dots Y_n \in Irr(Z(\mathcal{C}))$. Notice that the space on the right of this equation is by definition $Z_{RT,Z(\mathcal{C})}(\Sigma; Y_1, \dots Y_n)$.

It is important to note that RT is defined using "pairs-of-pants" decompositions of surfaces, while TV is defined via cell decompositions. Since the latter is a local construction and the former is inherently nonlocal, comparing the two requires a natural way of passing between them. The solution is simple and is provided immediately by the surface parametrizations defined in [BK]. Using these, we can compute maps between TV state spaces that correspond to each of the moves between cut systems and check that such maps are compatible with the projector $H_{TV}(\Sigma) \longrightarrow Z_{TV}(\Sigma)$. Thus, we get a natural identification $Z_{RT}(\Sigma) \cong Z_{TV}(\Sigma)$.

In both RT and TV theory, once we know the value of the TQFT on a punctured sphere, we can use the gluing axiom to define $Z(\Sigma)$ for any surface. Thus, we get a well-defined vector space, up to natural isomorphism that depends only on the topology of Σ. We do this in each case by defining "intermediate" vector spaces which do depend on some choices[2] and demonstrating that we can identify all such spaces naturally. The key result in this paper is that we can pass between the theories in a natural way, so that $Z_{TV,\mathcal{C}}(\Sigma) \cong Z_{RT,Z(\mathcal{C})}(\Sigma)$ independent of any choices.

The paper is organized as follows. First, we briefly review the theory of parametrized surfaces from [BK2000]. Next, we examine the effect of passing between parametrizations on the associated TV state spaces. In particular, we show that each of the moves yields a natural map between state spaces, which under projection gives the same identification between vector spaces as that in RT. The appendix contains some of the larger diagrams referenced in the paper.

This paper completes the program outlined in [TV] and continued in [TVII]. The reader is strongly encouraged to read these papers before this one, as they contain much prerequisite material.

Acknowledgments. The author would like to thank Sasha Kirillov for his help in writing this paper.

1. Surface Decompositions

In this section we briefly review the notion of a parametrized surfaces. For a complete exposition, see [BK2000]. Informally, a parametrization is a way of writing a surface Σ as the union on punctured spheres, together with a fixed identification of each punctured sphere with a *standard* sphere. The *standard sphere* with n punctures is defined formally as

$$S_{0,n} = \mathbb{CP}^1 \backslash \{D_1, \dots, D_n\}; D_j = \{z||z - z_j| < \epsilon\}, z_1 < \dots < z_n \tag{1.1}$$

where ϵ is sufficiently small so the boundary circles do not intersect. We also fix a point $p_i \in \partial D_i$. Note that we have fixed an ordering of the boundary circles, so we can refer to the set of boundary components by $\{\mathbf{1}, \dots, \mathbf{n}\}$.

Definition 1.1. An *extended surface* is a compacted oriented surface Σ , possibly with boundary, together with a fixed point p_α on each boundary component $(\partial\Sigma)_\alpha$.

[2] The parametrization in RT and the cell decomposition in TV.

Note that there are several other equivalent ways of defining an extended surface (See [BK2001]). We now give the main definition of this section. Let Σ be an extended surface.

Definition 1.2. A *parametrization* of Σ consists of

(1) A finite set C of non-intersecting simple closed curves on Σ such that $\Sigma \backslash C$ is of genus zero. We call C the set of *cuts* and fix a point on each cut.
(2) For each component Σ_a of $\Sigma \backslash C$, a homeomorphism $\psi : \Sigma_a \to S_{0,n_a}$

Two parametrizations are considered equivalent if they are isotopic [3]. There is a nice graphical way of describing parametrized surfaces. Namely, take the standard sphere with the graph as shown in Figure 1. This graph connects a single internal

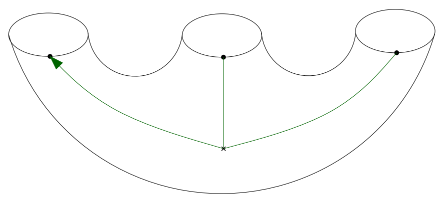


FIGURE 1. The graph on $S_{0,3}$

vertex to each of the points p_α fixed on the boundary and labels the edge connected to circle **1** by an arrow.

To depict a parametrization of any surface Σ, we draw the cuts on Σ. Then for each connected component Σ_α, we pull back the graph on the standard sphere by ψ_α to obtain a graph M_α on Σ_α. Clearly, such data are equivalent (up to isotopy) to specifying a parametrization and henceforth we will refer to a parametrization as a pair (C, M) where C is a set of cuts on Σ and $M = \cup_\alpha M_\alpha$. When possible, we

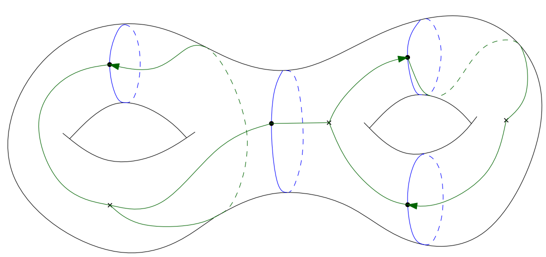


FIGURE 2. A parametrization of a genus two surface $\Sigma = S_{0,3} \sqcup S_{0,3} \sqcup S_{0,2}$. The blue lines are cuts and the green lines are graphs M_α

will often draw the graphs M_α in the plane, ignoring the sufaces into which they are embedded. The reader should have no difficulty passing between such a graph and the surface it represents.

[3] Both the set of cuts and the homemorphisms of boundary components are considered up to isotopy

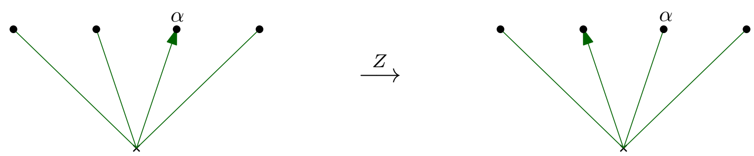


FIGURE 3. Z-move

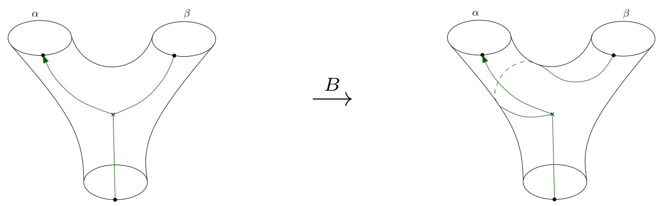


FIGURE 4. B-move

Definition 1.3. Let (Σ, P) be a parametrized surfaces. Then

$$Z_{RT,Z(\mathcal{C})}(\Sigma, P) = \bigoplus_{Y_{1_\alpha},\ldots Y_{n_\alpha}} \bigotimes_{\alpha} Z_{RT,Z(\mathcal{C})}(\Sigma_\alpha; Y_{1_\alpha}, \ldots Y_{n_\alpha}) \tag{1.2}$$

where the product is over all connected components of $\Sigma \backslash C$ and the sum is over all possible labelling boundary disks by irreducible objects of $Z(\mathcal{C})$.

We will typically denote a simple object $Y_i \in Z(\mathcal{C})$ by its index i. In all that follows, i^* represents the dual object Y_i^*, which is also simple. To simplify formulas, many authors attempt to pick a function $f : Irr(\mathcal{C}) \to Irr(\mathcal{C})$ so that $Y_i^* = Y_{f(i)}$, but one should avoid doing this at all costs since it is often impossible to do so in a consistent manner (See [BK2001], Remark 2.4.2).

Example 1.4. Let Σ be the torus with one puncture and parametrization as shown on the left hand side of Figure 1 and boundary disk labeled by Y. Then $Z_{RT,Z(\mathcal{C})}(\Sigma) = \bigoplus_{i \in Irr(Z(\mathcal{C}))} \mathrm{Hom}_{Z(\mathcal{C})}(\mathbf{1}, Y \otimes i \otimes i^*)$.

Now we describe a set of *moves* between parametrizations of a surface. As we'll see below, we can relate any two decompositions by a finite composition of these moves:

(1) The Z-move cyclically permutes the boundary components.
(2) The B-move *braids* one boundary component about an adjacent one.
(3) The F-move removes a cut. If a cut separates $S_{0,n}$ and $S_{0,m}$, deleting the cut gives a component homeomorphic to $S_{0,m+n-2}$ together with a graph inherited from the original components. Notice that we connect circle **1** from the one sphere to circle **m** of the other, thus resulting in a graph which inherits a natural ordering of boundary circles.
(4) The S-move interchanges meridians and longitudes of the punctured torus.

Theorem 1.5. *Let $A = (\Sigma, C, M)$ and $A' = (\Sigma, C', M')$ be two parametrizations of a surface. Then A and A' are related by a finite sequence of Z, B, F and S moves described above.*

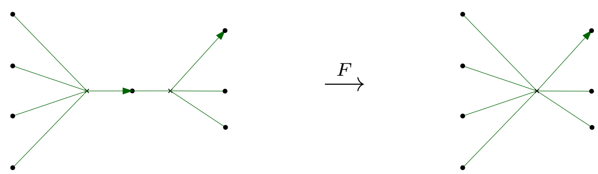


FIGURE 5. F-move

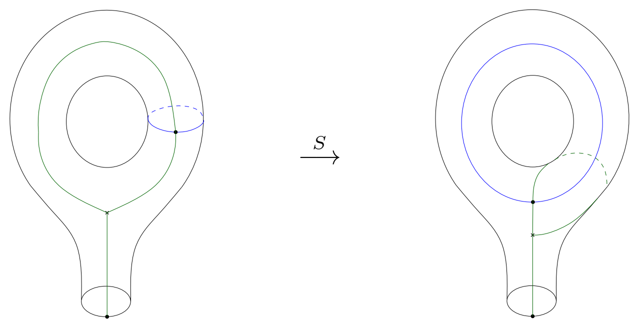


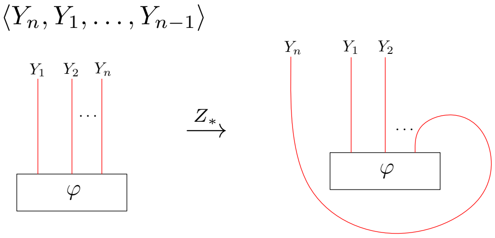
FIGURE 6. S-move

This result has its origins in conformal field theory. It was conjectured by Moore and Seiberg and rigorously proved in [BK2000]. It is a generalization of a result by Hatcher and Thurston ([HT1980]), which describes moves between surfaces decomposed into spheres, cylinders and pairs-of-pants, but doesn't take into account the full data of a parametrization. The theorem in [BK2000] does a lot more in fact: it provides a complete set of relations between the above moves, but we will not need this part explicitely.

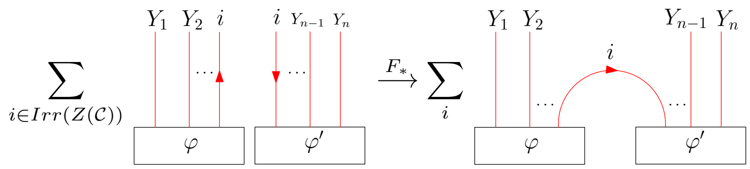
These moves are important for defining a 2-dimensional (extended) modular functor $\mathcal{F}$. Given the vector space associated to the punctured sphere, one should be able to use the gluing axiom to describe $\mathcal{F}(\Sigma)$ for a surface of any genus. Different parametrizations should give naturally isomorphic vector spaces; one can check that this is so by verifying that it is true for each of the *simple* moves between parametrizations. If P, P' are two parametrizations of a surface Σ related by a single Z, B, F or S move, we can explicitely describe the correspondence between associated vector spaces in RT theory:

Lemma 1.6. *Let P, P' be two parametrizations of a surface Σ and let $X : (\Sigma, P) \longrightarrow (\Sigma, P')$ be any composition of Z, B, F and S moves connecting P and P'. Then X induces an isomorphism $X_* : Z_{RT,Z(\mathcal{C})}(\Sigma, P) \xrightarrow{\cong} Z_{RT,Z(\mathcal{C})}(\Sigma, P')$. This isomorphism is independent of the choice of X. In terms of the generators,*

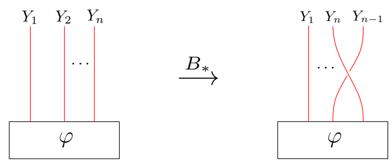
(1) *The Z-move corresponds to the rotation isomorphism:*
$\langle Y_1, \dots, Y_n \rangle \to \langle Y_n, Y_1, \dots, Y_{n-1} \rangle$



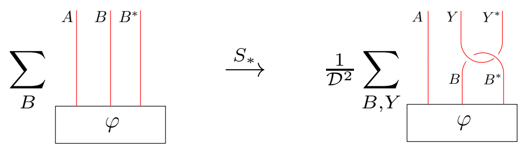
(2) *The F-move gives the composition isomorphism. That this is an isomorphism follows directly from semisimplicity.*



(3) *The B-move gives the braiding isomorphism*



(4) *The S-move gives multiplication by the S-matrix*



2. Parametrized surfaces and cell decompositions

In this section, we state and prove the main result of the paper: TV and RT theories assign the same vector space (up to natural isomorphism) to any closed surface Σ.

Given two cell decompositions Δ, Δ' of a surface Σ, there is a natural map

$$\Psi_{\Delta',\Delta} : H(\Sigma, \Delta) \longrightarrow H(\Sigma, \Delta') \tag{2.1}$$

obtained by computing a state sum on the cylinder $\Sigma \times I$, with a decomposition chosen to agree with Δ on $\Sigma \times 0$ and Δ' on $\Sigma \times 1$. Note, that this map does not depend on the choice of the internal decomposition. We will refer to this map as the *cylinder* map.

The cylinder map is not an isomorphism in general since the dimension of $H_{TV,\mathcal{C}}(\Sigma, \Delta)$ depends on the number of edges of Δ, but it is almost an isomorphism. More precisely, define the space $Z_{TV,\mathcal{C}}(\Sigma, \Delta) = Im(\Psi_{\Delta,\Delta})$. Then

$$\Psi_{\Delta,\Delta'} : Z_{TV,\mathcal{C}}(\Sigma, \Delta) \longrightarrow Z_{TV,\mathcal{C}}(\Sigma, \Delta') \tag{2.2}$$

is a natural isomorphism. We can refer to this space as $Z_{TV,\mathcal{C}}(\Sigma)$, since up to natural isomorphism it doesn't depend on the cell decomposition.

Given a parametrized surface Σ, there is a natural way to obtain a cell decomposition of Σ . We have a fixed collection of closed curves dividing Σ into the union of punctured spheres. These cuts become 1-cells in the cell decomposition. Further, for each punctured sphere thus obtained, we have a graph from our parametrization terminating at fixed points on the boundary circles. Each edge of this graph becomes a 1-cell and the points at which the 1-cells terminate become vertices. It is easy to see that these choices define a cell decomposition in the sense of [BK] [4]. We call the cell decomposition obtained in this way, the *associated* cell decomposition to parametrization P.

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[4] See, in particular, Figure 27

Recall that for a punctured sphere with standard cell decomposition (Figure 1), we have a projection $H_{TV,\mathcal{C}}(S^2) \xrightarrow{\pi} Z_{TV,\mathcal{C}}(S^2) \cong Z_{RT,Z(\mathcal{C})}(S^2)$. The associated inclusion map i can be described graphically. The normalization factors are chosen

$$\varphi \xrightarrow{i} \bigoplus_{x_1,\ldots,x_n} \prod_{j=1}^{n} \sqrt{d_{x_j}}$$

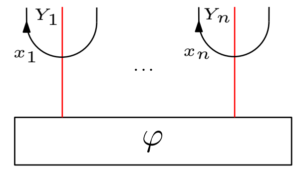


FIGURE 7. $i : Z_{RT,Z(\mathcal{C})}(S^2) \hookrightarrow H_{TV,\mathcal{C}}(S^2)$

to agree with that in [BK], so that $\pi \circ i = Id$.

We have two parallel notions in TV and RT theory. On the RT side, we have surface parametrizations and passing between any two parametrizations gives an isomorphism as described earlier in Lemma 1.6. On the TV, side, we have cell decompositions; passing between any two cell decompositions gives a natural isomorphism obtained from a cylinder as described earlier. The following theorem, which implies the main result in this paper, shows that these two notions are the same, up to projection.

Theorem 2.1. *Let P, P' be two parametrizations of a surface Σ with associated cell decompositions Δ, Δ' respectively. Then the diagram in Figure 8 commutes.*

Here, X_* is the map described in Lemma 1.6, j is the map described in Figure 7 followed by projection to $Z_{TV,\mathcal{C}}(\Sigma)$and Ψ is the isomorphism described in (2.2).

Proof. To show the diagram commutes, we will verify that it does for each of the generators Z, B, F and S. The Z and B moves are essentially immediate, while the F and S moves require some work. Throughout the proof, our convention will be that diagrams of surfaces represent the vector spaces associated to them. In particular, a parametrized surface (Σ, P) represents $Z_{RT,Z(\mathcal{C})}(\Sigma, P)$ and a cell-decomposed surface (Σ, Δ) represents $Z_{TV,\mathcal{C}}(\Sigma, \Delta)$. In the diagrams below, we have written Ψ from Figure 8 as the composition of several elementary steps for the reader's edification. We have moved several of the large diagrams to Section 3.

The Z-move. This follows directly from the natural isomorphism from Lemma 1.6(1).

The B-move. A proof of this fact may be found in [BK] (lemma 2.1), where we provide an explicit computation.

$$\begin{tikzcd}
Z_{TV,\mathcal{C}}(\Sigma,\Delta) \arrow[r, "\Psi"] & Z_{TV,\mathcal{C}}(\Sigma,\Delta') \\
Z_{RT,Z(\mathcal{C})}(\Sigma,P) \arrow[u, "j"] \arrow[r, "X_*"] & Z_{RT,Z(\mathcal{C})}(\Sigma,P') \arrow[u, "j"']
\end{tikzcd}$$

FIGURE 8.

The F-move. We will show that the diagram in Figure 11 commutes . The arrow labeled F is the isomorphism described in Lemma 1.6, those labeled i are inclusion maps (Figure 7), and G is the gluing isomorphism at the level of state-spaces (Theorem 7.3, [BK]) . The other maps are all *cylinder* maps 2.1. Notice that this can be done in fewer steps, but the cylinder maps will be more difficult to realize. Figure 11 by contrast contains cylinder maps that are all easy to compute.

To check that this diagram commutes we begin with a vector in $Z_{RT,Z(\mathcal{C})}$ and proceed about the diagram in two ways. In Figure 12 we give the answer. The explicit computation at each stage left to the reader.

The S-move. We will show that the diagram in Figure 13 commutes. We have omitted some intermediate steps on the right side of the diagram as they are much the same as those on the left. Notice that the diagrams connected by the horizontal arrow labeled S are parametrized surface while the others are of cell-decomposed surfaces. We have chosen a convenient cell decomposition as the terminating point of the diagram which is easy to work with since there are simple maps α, β to this space which can be though of as contractions along edges u_1 and u_2 respectively (Figure 9). If we start on the bottom left of figure Figure 13 and proceed around

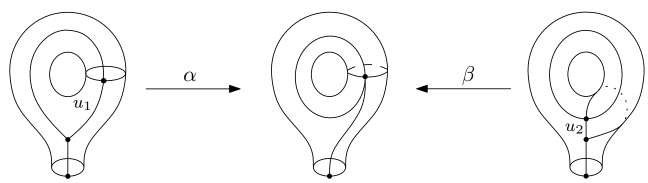


FIGURE 9. To identify the spaces on the left and the right, we use cylinder maps α, β to the space in the center and compare the images of these maps.

in two different ways, we get two vectors, φ_1, φ_2 in the same space as shown in Figure 10

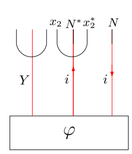
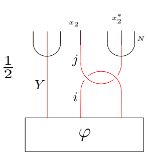
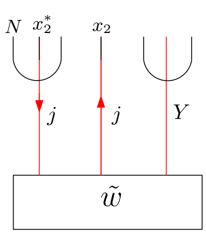
$\varphi_1 = \frac{1}{\mathcal{D}} \sum_{x_1, x_2, N} (d_{x_1} d_{x_2} d_N)^{\frac{1}{2}}$  $; \varphi_2 = \frac{1}{\mathcal{D}^3} \sum_{x_1, x_2, N} (d_{x_1} d_{x_2} d_N)^{\frac{1}{2}}$ 

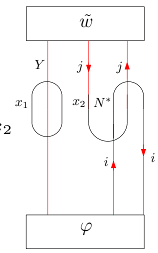
FIGURE 10.

We can easily verify that these vectors are the same by picking some vector w in the dual space and comparing the pairings $\langle \varphi_1, w \rangle$ and $\langle \varphi_1, w \rangle$. Let w be given by

$\frac{1}{\mathcal{D}} \sum_{i, N, x_1, x_2} (d_N d_{x_1} d_{x_2})^{\frac{1}{2}}$ 

where $\tilde{w}$ is some vector in $\bigoplus_j \mathrm{Hom}_{Z(\mathcal{C})}(\mathbf{1}, j^* \otimes j \otimes Y)$. Then

$$\langle w, \varphi_1 \rangle = \frac{1}{\mathcal{D}^2} \sum d_N d_{x_1} d_{x_2}$$



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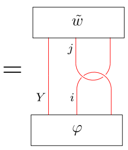
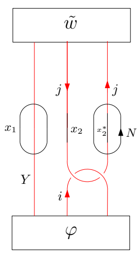


$$= \langle \tilde{w}, S_* \varphi \rangle$$

$$= \frac{1}{\mathcal{D}^4} \sum d_N d_{x_1} d_{x_2}$$



$$= \langle w, \varphi_2 \rangle \qquad \square$$

As an immediate corollary, we get

Theorem 2.2. *For any closed surface, Σ, we have a natural isomorphism* $Z_{RT,Z(\mathcal{C})}(\Sigma) \cong Z_{TV,\mathcal{C}}(\Sigma)$.

The results in [BK], [Bal] and in this paper actually show that RT and TV theories are equivalent as extended (3-2-1) theories in a suitably restricted sense. In particular, we allow 3-manifolds with embedded tubes inside, but not with arbitrary embedded graphs. Keeping this in mind, the two theories are equivalent; they associate the same category to a closed 1-manifold ($Z(\mathcal{C})^{\boxtimes n}$ to a union of n circles), and all the above arguments, particularly Theorem 2.2 work in the same exact way for surfaces with boundary.

3. Diagrams

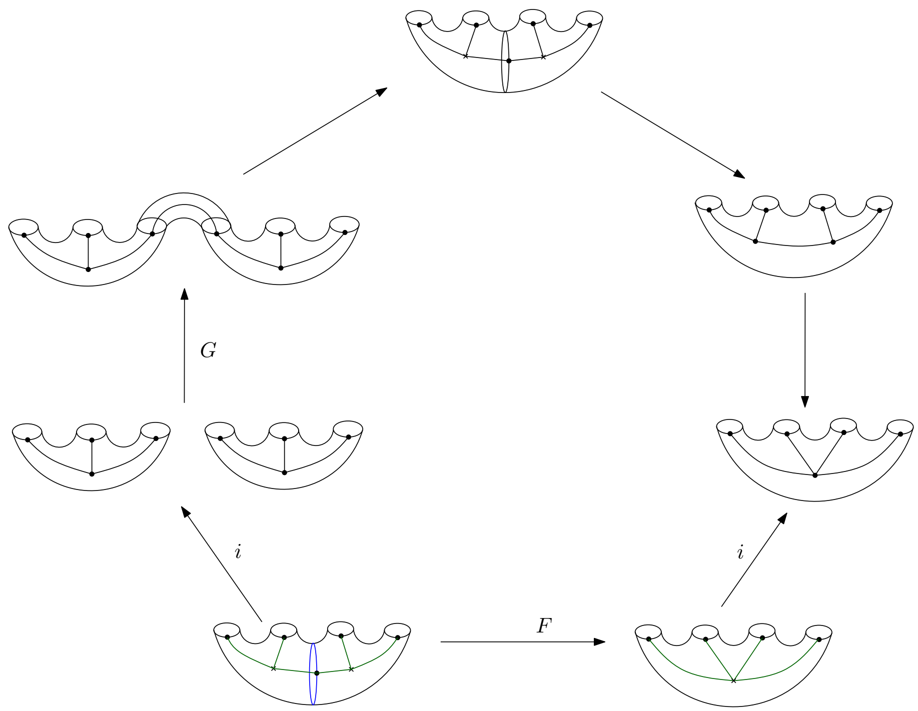


FIGURE 11. The F-move fuses together two spheres along a boundary component. In terms of parametrized surfaces this is realized by simply removing a cut separating the spheres. At the level of cell decompositions we want to identify the result with the standard sphere decomposition (Figure 1). We include several intermediate steps to make the computation more transparent.

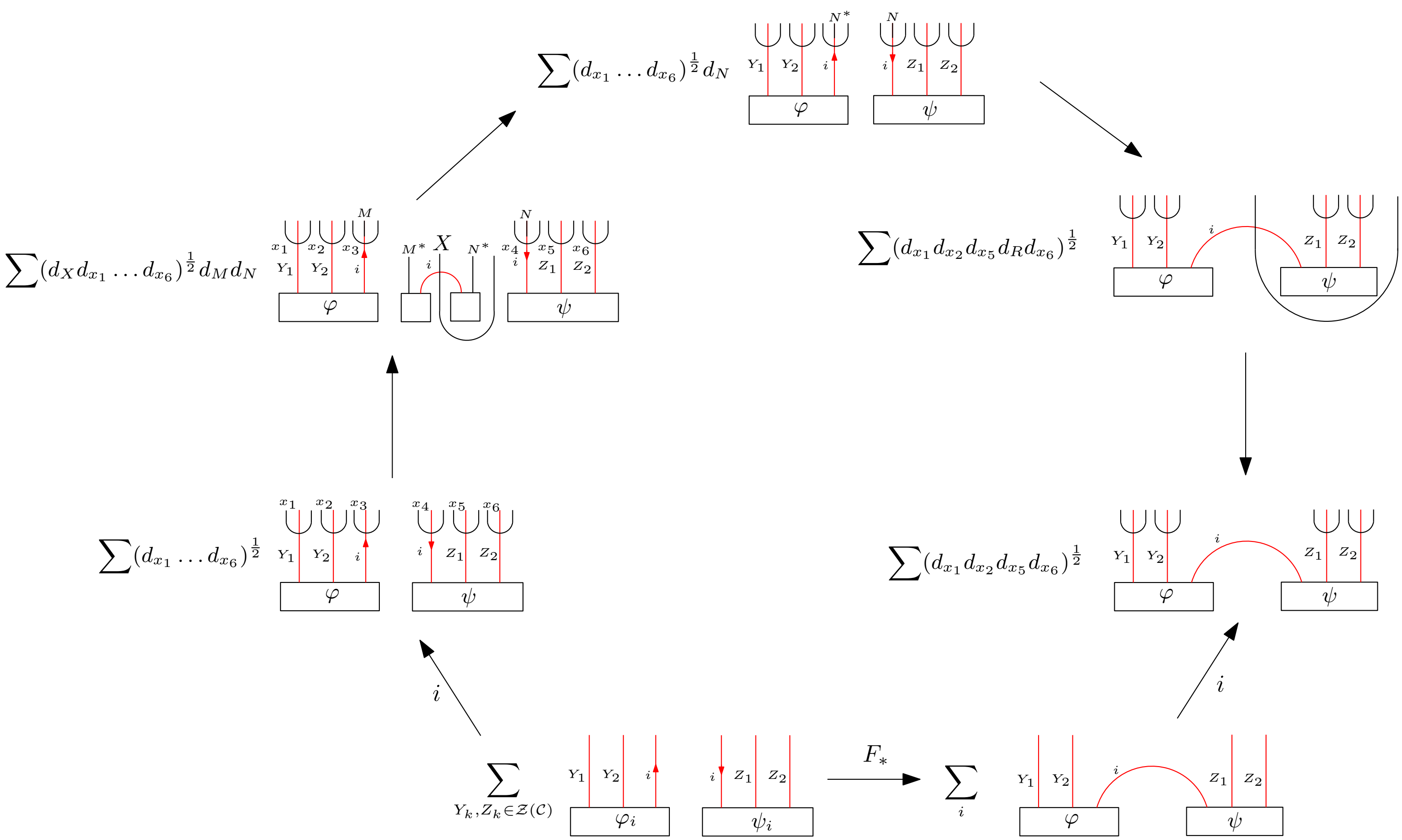


FIGURE 12. A verification that Figure 11 commutes. All unlabeled arrow represent cylinder maps. We start on the lower left and proceed in two ways around the diagram.

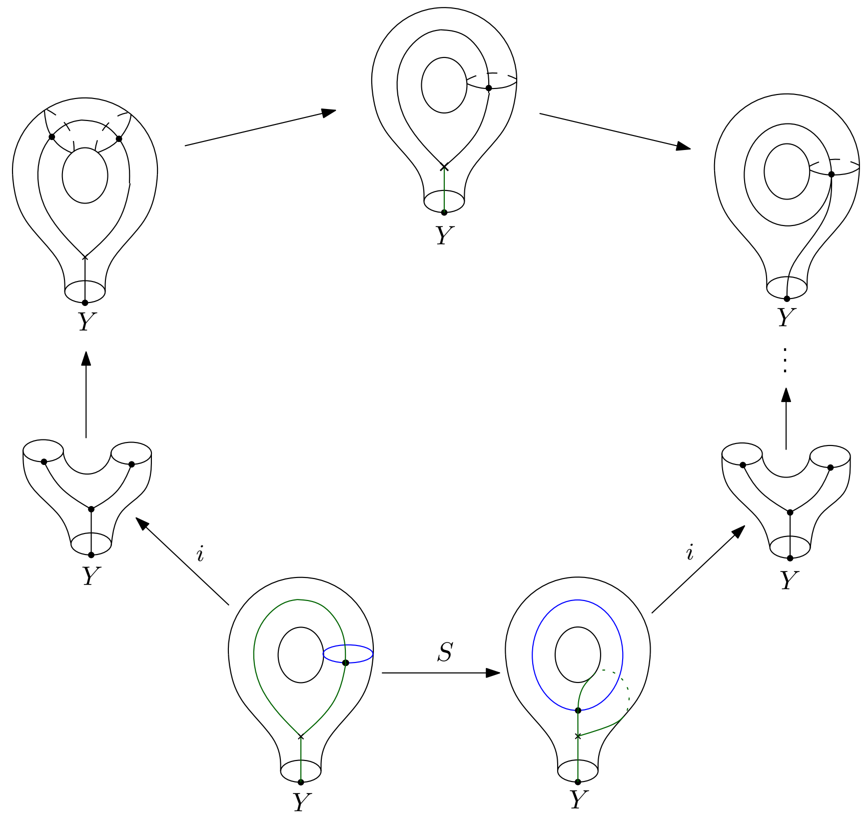


FIGURE 13. The S-move interchanges meridians and longitudes of the 1-punctured torus. Thus, the cut (blue) and the parametrizing graph (green) exchange places under the application of S. On both sides of the diagram, the map i cuts the surface into a pair of pants along the blue edge. Some intermediate steps on the right side of the diagram are omitted since they are identical to those on the left.

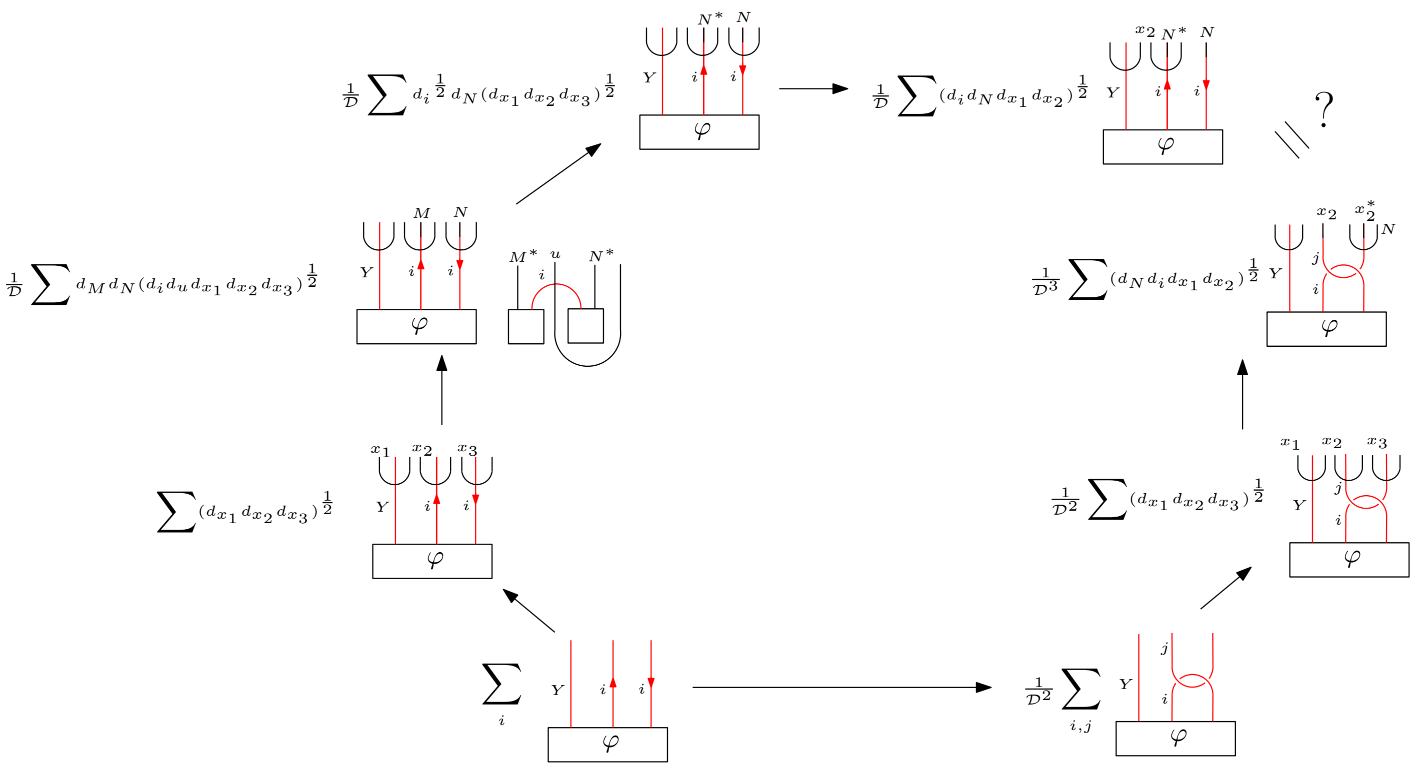


FIGURE 14. A demonstration that Figure 13 commutes. Notice that when we proceed around the diagram we get two different pictures (separated by " $\stackrel{?}{=}$ "), but can verify that they represent the same vector.

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