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Conservative interacting particles system with anomalous rate of ergodicity. *

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Abstract

We analyze certain conservative interacting particle system and establish ergodicity of the system for a family of invariant measures. Furthermore, we show that convergence rate to equilibrium is exponential. This result is of interest because it presents counterexample to the standard assumption of physicists that conservative system implies polynomial rate of convergence.
Keywords: Hörmander type generators, conservative interacting particle system, ergodicity.

1 Introduction

In this paper we present an example of the conservative interacting particle system with exponential rate of convergence to equilibrium. This system naturally appears in the dyadic model of turbulence (see [2]). In [2] it has been established that the system has anomalous dissipation. This result seems to be the reason behind exponential rate of convergence to equilibrium. Similar systems naturally appear in the models of heat conduction and quantum spin systems ([3],[4],[9],[10],[11]).

Ergodic properties of systems of interacting particles is one of the central topics of statistical mechanics. They have been studied starting from the works of Spitzer [19] and Dobrushin [7]. The literature of the subject is huge and we will not attempt to list it here, see [17] and references therein.

Interacting particle systems are usually divided into two classes: conservative and nonconservative ones. Conservative ones are presumed to have at most polynomial rate of convergence to equilibrium and dissipative ones exponential one ([16]).

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In the same time rigorous mathematical results about rates of convergence to equilibrium of conservative systems has been established only in the handful of cases such as Kawasaki dynamics ([5],[6]), Ginzburg-Landau type processes ([14],[16]) and Brownian moment processes ([13]). The result of this paper shows that existence of formal conservation law does not necessarily imply polynomial rate of convergence. Consequently, "meta" theorem that conservative interacting particle systems are ergodic with polynomial rate of convergence to equilibrium is not correct.

2 The system

Let (Ω, F_t, P) be a filtered probability space and (W_n) be a sequence of independent Brownian motions. Consider the equation

$$dX_n = k_{n-1}X_{n-1} \circ dW_{n-1} - k_n X_{n+1} \circ dW_n, \qquad X_n(0) = X_n^{(0)} \tag{1}$$

for all $n \geq 0$, with $X_0 = 0$, $k_0 = 0$, and $k_n = \lambda^n, n \in \mathbb{N}$ for some $\lambda > 1$, $X_n^{(0)}$ deterministic or F_0-adapted. The stochastic integral in the system (1) is in Stratonovich sense.

Remark 1 *The assumption $k_n = \lambda^n, n \in \mathbb{N}$ has been imposed for simplicity. It can be relaxed to the assumption that the sequence $\left\{\frac{k_{n+1}}{k_n}\right\}_{n=1}^{\infty}$ is nondecreasing and the first term of the sequence is bigger than* 1.

Consider the space

$$W = \left\{ (x_n)_{n\in\mathbb{N}} : \|x\|_W^2 := \sum k_n^{-2} x_n^2 < \infty \right\}.$$

Definition 2 *We say that a sequence of continuous adapted processes (X_n) is a weak (in the analytical sense) solution in W of equation (1) if (X_n) is $L^\infty\left([0,T]; L^2(\Omega; W)\right)$ and*

$$dX_n = k_{n-1}X_{n-1}dW_{n-1} - k_n X_{n+1} dW_n - \frac{1}{2}\left(k_n^2 + k_{n-1}^2\right) X_n dt$$

for each $n \geq 1$. If we have

$$E\left[\|X(t)\|_W^2\right] \leq E\left[\left\|X^{(0)}\right\|_W^2\right], \qquad a.e.\ t \geq 0$$

we say it is a Leray solution in W.

Theorem 3 *For every $X^{(0)} \in L^2(\Omega; W)$, F_0-measurable, there exists a unique weak Leray solution in W of equation (1).*

Remark 4 *We use Galerkin type finite dimensional approximation to show existence of solution of system (1). Different way would be to apply results of Holevo [12].*

Proof. Step 1 (existence). For each $N \in \mathbb{N}$, consider the finite dimensional system

$$dX_n^N = k_{n-1} X_{n-1}^N dW_{n-1} - k_n X_{n+1}^N dW_n - \frac{1}{2}\left(k_n^2 + k_{n-1}^2\right) X_n^N dt, \qquad n = 1, ..., N$$

with $X_0^N = X_{N+1}^N = 0$ and the initial condition $X_n^N(0)$ equal to $X_n^{(0)}$, $n = 1, ..., N$. This system has a unique strong solution, with all moments finite. Indeed, it immediately follows from the Theorem 3.3, p. 7 of [1]. Set

$$q_n^N = E\left[\left(X_n^N\right)^2\right].$$

By Itô formula (we need finite fourth moments to have that the Itô terms are true martingales, then they disappear taking expected value) we have (we drop N)

$$\begin{aligned} q_n' &= -\left(k_n^2 + k_{n-1}^2\right) q_n + k_{n-1}^2 q_{n-1} + k_n^2 q_{n+1} \\ &= -k_{n-1}^2 \left(q_n - q_{n-1}\right) + k_n^2 \left(q_{n+1} - q_n\right) \end{aligned}$$

for $n = 1, ..., N$, with $q_0 = q_{N+1} = 0$. Denote by $\|\cdot\|_W^2$ the same norm introduced above also in the case of a finite number of components. We have

$$\begin{aligned} \frac{d}{dt} E\left[\left\|X^N\right\|_W^2\right] &= \sum_{n=1}^N k_n^{-2} \frac{d}{dt} q_n^N = \\ &= -\sum_{n=1}^N k_n^{-2} k_{n-1}^2 \left(q_n - q_{n-1}\right) + \sum_{n=1}^N k_n^{-2} k_n^2 \left(q_{n+1} - q_n\right) \\ &= -\lambda^{-2} \sum_{n=1}^N \left(q_n - q_{n-1}\right) + \sum_{n=1}^N \left(q_{n+1} - q_n\right) \le -q_1. \end{aligned}$$

Since $q_1 \ge 0$ by definition, we have

$$E\left[\left\|X^N(t)\right\|_W^2\right] \le E\left[\left\|X^N(0)\right\|_W^2\right], \qquad t \ge 0. \tag{2}$$

Thus the sequence $\left(X^N\right)_{N\ge 0}$ is bounded in $L^\infty\left([0,T]; L^2(\Omega; W)\right)$. Therefore, there exists a subsequence $N_k \to \infty$ such that $\left(X_n^{(N_k)}\right)_{n\ge 1}$ converges weakly to some $(X_n)_{n\ge 1}$ in $L^2(\Omega \times [0,T]; W)$ and also weak star in $L^\infty\left([0,T]; L^2(\Omega; W)\right)$. Now the proof proceeds by standard arguments typical of equations with monotone operators (which thus apply to linear equations), presented in [18], [15]. The subspace of $L^2(\Omega \times [0,T]; W)$ of progressively measurable processes is strongly closed, hence weakly closed, hence $(X_n)_{n\ge 1}$ is progressively measurable. The one-dimensional stochastic integrals which appear in each equation of the system are (strongly) continuous linear operators from the subspace of $L^2(\Omega \times [0,T])$ of progressively measurable processes to $L^2(\Omega)$, hence they are weakly continuous, a fact that allows us to pass to the limit in each one of the linear equations of the system. A posteriori, from these integral equations, it follows that there is a modification such that all components are continuous. The proof of existence is complete.

Step 2 (uniqueness). Assume that $X^{(i)}$, $i = 1, 2$ are two weak solutions in W. Then $Y = X^{(1)} - X^{(2)}$ is a weak solution in W, but with zero initial condition. Set similarly as above

$$q_n = E\left[Y_n^2\right].$$

By Itô formula we have

$$\begin{aligned}\frac{1}{2}dY_n^2 &= k_{n-1}Y_{n-1}Y_n dW_{n-1} - k_n Y_n Y_{n+1} dW_n \\ &\quad - \frac{1}{2}\left(k_n^2 + k_{n-1}^2\right) Y_n^2 dt + k_{n-1}^2 Y_{n-1}^2 dt + k_n^2 Y_{n+1}^2 dt.\end{aligned}$$

Define family of stopping times

$$\tau_m^n = \inf_{t\geq 0}\{\int_0^t Y_n^4(s)ds \geq m\}, m, n \in \mathbb{N}.$$

Since $Y \in L^\infty\left([0,T]; L^2(\Omega; W)\right)$ we infer that for all $n \in \mathbb{N}$

$$\int_0^t Y_n^4(s)\, ds < \infty, \mathbb{P} - a.s.$$

Therefore,

$$\lim_{m\to\infty} \tau_m^n = t, \mathbb{P} - a.a$$

Define family of local martingales $\{M_n\}$ by

$$M_n(t) := \int_0^t k_{n-1}Y_{n-1}Y_n dW_{n-1} - k_n Y_n Y_{n+1} dW_n, t \geq 0.$$

Then, $\{M_n^m(\cdot \wedge \tau_m^n)\}$ is a family of martingales. Denote

$$q_{n,m}(\cdot) = E\left[Y_n^2(\cdot \wedge \tau_m^n)\right]$$

Consequently

$$\begin{aligned}q_{n,m}' &= -\left(k_n^2 + k_{n-1}^2\right) q_{n,m} + k_{n-1}^2 q_{n-1,m} + k_n^2 q_{n+1,m} \\ &= -k_{n-1}^2\left(q_{n,m} - q_{n-1,m}\right) + k_n^2\left(q_{n+1,m} - q_{n,m}\right)\end{aligned}$$

and from this (and the positivity of $q_{n,m}$'s) we can deduce $q_{n,m} = 0$. Taking the limit $m \to \infty$ by the Lebesgue Dominated Convergence Theorem we can conclude that $q_n = 0$.

Step 3 (Leray solution). From (2) and the definition of $X^N(0)$ we have

$$E\left[\left\|X^N(t)\right\|_W^2\right] \leq E\left[\left\|X^{(0)}\right\|_W^2\right], \qquad t \geq 0.$$

Hence

$$E\left[\int_a^b \left\|X^N(t)\right\|_W^2 dt\right] \leq (b-a)\, E\left[\left\|X^{(0)}\right\|_W^2\right], \qquad 0 \leq a \leq b.$$

Weak convergence in $L^2(\Omega \times [0,T]; W)$ implies that

$$E\left[\int_a^b \|X(t)\|_W^2 dt\right] \leq (b-a)\, E\left[\left\|X^{(0)}\right\|_W^2\right], \qquad 0 \leq a \leq b.$$

By Lebesgue differentiation theorem, we get $E\left[\|X(t)\|_W^2\right] \leq E\left[\|X^{(0)}\|_W^2\right]$ for a.e. t. The proof is complete. ■

3 Continuous dependence and Markov property

Proposition 5 *The unique Leray solution in W depends continuously on its initial condition in the following sense. If X^η and X^ρ are the solutions corresponding to the initial conditions $\eta, \rho \in L^2(\Omega; W)$, F_0-measurable, then:*

i)

$$E\left[\|X^\eta(t) - X^\rho(t)\|_W^2\right] \leq E\left[\|\eta - \rho\|_W^2\right], \qquad a.e.\ t \geq 0$$

ii) for every $N > 0$

$$E\left[\sum_{n=1}^N k_n^{-2}\left(X_n^\eta(t) - X_n^\rho(t)\right)^2\right] \leq E\left[\|\eta - \rho\|_W^2\right], \qquad \text{for all } t \geq 0.$$

Proof. The difference $X^\eta - X^\rho$ is a weak solution in W with initial condition $\eta - \rho$, hence it is Leray. This implies (i). Then (ii) holds by continuity of single components and Fatou theorem. ■

We have proved that, for every $x \in W$ there is a unique weak solution $(X_n^x(t))$ in W. Let us prove that the family X^x is a Markov process. Define the operator P_t on $B_b(W)$ as

$$(P_t\varphi)(x) := E[\varphi(X^x(t))].$$

By the previous result, P_t is well defined also from $C_b(W)$ to $C_b(W)$.

Proposition 6 *We have*

$$E[\varphi(X^x(t+s))\,|F_t] = (P_s\varphi)(X^x(t)) \tag{3}$$

for all $\varphi \in C_b(W)$, hence the family X^x is a Markov process. The Markov semigroup P_t is Feller.

Proof. We have just to prove the identity (3), the other claims being obvious or classical. Indeed, Feller property follows from part i) of Proposition 5.

Consider the equation on a generic interval $[s,t]$ with initial condition $\eta \in L^2(\Omega; W)$, F_s-measurable, at time s and call $X^{s,\eta}(t)$ the solution. Consider the function

$$Y(t) := \begin{cases} X^x(t) & \text{for} \quad t \in [0,s] \\ X^{s,X^x(s)}(t) & \text{for} \quad t \geq s. \end{cases}$$

Direct substitution into the equations prove that Y is a solution with initial condition x, hence equal to $X^x(t)$ also for $t \geq s$. This proves the evolution property

$$X^{s,X^x(s)}(t) = X^x(t), \qquad t \geq s.$$

Thus

$$E\left[\varphi\left(X^x(t+s)\right)|F_t\right] = E\left[\varphi\left(X^{t,X^x(t)}(t+s)\right)|F_t\right].$$

If we prove that

$$E\left[\varphi\left(X^{t,\eta}(t+s)\right)|F_t\right] = (P_s\varphi)(\eta)$$

for all $\eta \in L^2(\Omega; W)$, F_t-measurable, we are done. If $\eta = x$, a.s. constant, it is true, by exploiting the fact that the increments of the Brownian motions W_n from t to $t+s$ are independent of F_t; and because the dynamics is autonomous. From constant values one generalizes to $\eta = \sum_{i=1}^n x_i 1_{A_i}$, $A_i \in F_t$; indeed, for such η, we have

$$X^{t,\eta}(t+s) = \sum_{i=1}^n X^{t,x_i}(t+s)\, 1_{A_i}.$$

Finally we have the identity for all η by the continuity result above. ■

4 Invariant measures

Consider the measures μ_r, parametrized by $r \geq 0$, formally defined as

$$\mu_r(dx) = \frac{1}{Z}\exp\left(-\frac{\sum_{n=1}^\infty x_n^2}{2r}\right)dx.$$

The rigorous definition is: μ_r is the Gauss measure on l^2, namely the Gaussian measure on W having covariance equal to identity. For every function f of the first n coordinates only of l^2, the measure μ_r is given by

$$\int_Y f(x_1,...,x_n)\,\mu_r(dx) = \frac{1}{Z_n}\int_{\mathbb{R}^n} f(x_1,...,x_n)\exp\left(-\frac{\sum_{k=1}^n x_k^2}{2r}\right)dx_1...dx_n$$

where $Z_n = (2\pi r)^{n/2}$. This formula identifies μ_r.

Moreover, for technical reasons, we need

$$\widetilde{W} = \left\{(x_n)_{n\in\mathbb{N}} : \|x\|_W^2 := \sum k_n^{-4}x_n^2 < \infty\right\}.$$

Notice that

$$\mu_r\left(\widetilde{W}\right) = 1$$

and the embedding of W in $\widetilde{W}$ is compact.

Proposition 7 *For every $r > 0$, μ_r is invariant for the Markov semigroup P_t defined above on W.*

Proof. It is sufficient to prove

$$\int_Y (P_t\varphi)(x)\,\mu_r(dx) = \int_Y \varphi(x)\,\mu_r(dx)$$

for all φ of the form $\varphi(x) = f(x_1, ..., x_n)$, with bounded continuous f. We have

$$\int_Y (P_t\varphi)(x)\,\mu_r(dx) = E\left[\int_Y f\left(X_1^x(t), ..., X_n^x(t)\right)\mu_r(dx)\right].$$

The strategy now is the following one. On an enlarged probability space, if necessary, we can define an F_0-measurable r.v. $X^{(0)} \in L^2(\Omega; W)$ with law μ_r. For every $N > 0$ denote by X^N the Galerkin approximations used to prove existence above, with initial condition $X_n^N(0) = X_n^{(0)}$. Since such sequence has only one weak limit point in $L^2([0,T] \times \Omega; W)$, the full sequence X^N weakly converges to the Leray solution X having initial condition $X^{(0)}$. Denote by ρ_N the law of X^N and by ρ the law of X, on $L^2([0,T]; W)$. We shall prove that the sequence ρ_N is tight in $L^2\left([0,T]; \widetilde{W}\right)$. Then there exists a subsequence ρ_{n_k} weakly convergent to some probability measure on $L^2\left([0,T]; \widetilde{W}\right)$. Such measure must be ρ.

It is enough to show that the sequence X^N of Galerkin approximations is bounded in $L^2(\Omega, L^2([0,T], W)) \cap L^2(\Omega, W^{\alpha,2}([0,T], \widetilde{W})), \alpha \in (0,1)$. That implies that laws $\{\rho_N\}_{N=1}^{\infty}$ are bounded in probability on

$$L^2([0,T], W) \cap W^{\alpha,2}([0,T], \widetilde{W}), \alpha \in (0,1).$$

Since embedding

$$L^2([0,T], W) \cap W^{\alpha,2}([0,T], \widetilde{W}) \subset L^2([0,T], \widetilde{W}), \alpha \in (0,1).$$

is compact by Theorem 2.1, p. 370 of [8](applied with $B_0 = W$, $B = B_1 = \widetilde{W}$, $p = 2$) we shall conclude that the sequence ρ_N is tight in $L^2\left([0,T]; \widetilde{W}\right)$.

Since the sequence (X^N) is bounded in $L^\infty([0,T], L^2(\Omega, W))$ it remains to show that the sequence (X^N) is bounded in $L^2(\Omega, W^{\alpha,2}([0,T], \widetilde{W}))$ for some $\alpha \in (0,1)$.

Decompose X^N as

$$X^N(t) = X^N(0) - \int_0^t A^N X^N(s)ds + \int_0^t B^N(X^N)dW(s) = J_1^N(t) + J_2^N(t) + J_3^N(t)$$

where

$$(A^N x)_{n,m} = -\frac{\delta_{n,m}}{2}(k_{n-1}^2 + k_n^2)x_n,$$
$$(B^N x)_{n,m} = k_{n-1}x_{n-1}\delta_{n,m+1} - k_n x_{n+1}\delta_{n,m}, x \in P_N(W),\ m, n = 1, \dots, N.$$

We have

$$\mathbb{E}|J_N^1|^2_{W^{1,2}(0,T;W)} \le T\mathbb{E}|X^{(0)}|^2_W. \tag{4}$$

Since $|A^N|_{\mathcal{L}(W,\widetilde{W})} \le K = 1 + \sup_n \frac{k_{n-1}^2}{k_n^2}$ we infer that

$$\mathbb{E}|J_2^N|^2_{W^{1,2}(0,T;\widetilde{W})} \le C(T,K)\mathbb{E}|X^N|^2_{L^2([0,T],W)} \le C(T,K)\mathbb{E}|X^{(0)}|^2_W. \tag{5}$$

Fix $\alpha \in (0, \frac{1}{2})$. By Lemma 2.1, p. 369 of [8] we have that

$$\mathbb{E}|J_2^N|^2_{W^{\alpha,2}(0,T;\widetilde{W})} \le \mathbb{E}\int_0^T |B^N(X^N)|^2_{L_{HS}(l^2,\widetilde{W})}\, ds \tag{6}$$

Notice that

$$\begin{aligned}|B(x)|^2_{L_{HS}(l^2,\widetilde{W})} &= \sum_{n=1}^{\infty} |B(x)(e_n)|^2_{\widetilde{W}} \\ &\le \sum_{n=1}^{\infty} k_n^{-4}k_n^2x_{n+1}^2 + k_{n+1}^{-4}k_n^2x_n^2 \le (K+K^2)|x|^2_W, x \in W.\end{aligned} \tag{7}$$

where $(e_n)_{n=1}^{\infty}$ is ONB in l^2.

Combining inequalities (6) and (7) we infer that

$$\mathbb{E}|J_2^N|^2_{W^{\alpha,2}(0,T;\widetilde{W})} \le C\mathbb{E}\int_0^T |X^N(s)|^2_W\, ds \le C(T,K,\alpha)\mathbb{E}|X^{(0)}|^2_W. \tag{8}$$

Hence, inequalities (4), (5) and (8) imply that for some $\alpha \in (0, \frac{1}{2})$

$$\mathbb{E}|X^N|^2_{W^{\alpha,2}([0,T],\widetilde{W})} \le C(T,K,\alpha)\mathbb{E}|X^{(0)}|^2_W, \tag{9}$$

and the result follows. ■

Corollary 8 *The semigroup $(P_t)_{t\ge 0}$ acting on $C_b(W)$ can be extended to $L^p(W,\mu_r)$ for any $p \ge 1$. Generator of the semigroup $(P_t)_{t\ge 0}$ is given by the formula*

$$\mathcal{L} = \frac{1}{2}\sum_{j=1}^{\infty} k_j^2 D_{j,j+1}^2$$

with $D_{j,j+1} = x_j\partial_{x_{j+1}} - x_{j+1}\partial_{x_j}, j \in \mathbb{N}$.

Proof. It follows from Itô formula. ■

5 Symmetry of the generator in the Sobolev spaces

Let

$$L = \sum_{i=1}^{\infty} (\frac{\partial^2}{\partial x_i^2} - x_i \frac{\partial}{\partial x_i})$$

be Ornstein-Uhlenbeck operator and

$$\mathcal{H}^n = \left\{ f \in L^2(W, \mu_r) : |f|^2_{\mathcal{H}^n} = |f|^2_{L^2(W,\mu_r)} + ((-L)^n f, f)_{L^2(W,\mu_r)} < \infty \right\}, n \in \mathbb{Z},$$

$$\mathcal{C} = \left\{ \phi : W \to \mathbb{R}, \phi(x) = f(x_1, \ldots, x_n), f \in C^4(\mathbb{R}^n, \mathbb{R}), n \in \mathbb{N} \right\}.$$

We have

$$[D_{i,i+1}, L]\phi = 0, \phi \in \mathcal{C}.$$

Consequently,

$$[\mathcal{L}, L]\phi = 0, \phi \in \mathcal{C},$$

and

Proposition 9 *For all $f, g \in \mathcal{C}$ we have*

$$(f, \mathcal{L}g)_{\mathcal{H}^n} = (g, \mathcal{L}f)_{\mathcal{H}^n} = -\sum_{l=1}^{\infty} k_l^2 (D_{l,l+1} f, D_{l,l+1} g)_{\mathcal{H}^n}, n \in \mathbb{Z}.$$

Fix $n \in \mathbb{N} \cup 0$.

Corollary 10 *The operator $\mathcal{L}$ is closable in $\mathcal{H}^n$ and its closure has bounded from above self-adjoint extension, which we continue to denote by the same symbol $\mathcal{L}$. Moreover, the self-adjoint extension $\mathcal{L}$ generates a strongly continuous contraction semigroup $T_t = e^{t\mathcal{L}} : \mathcal{H}^n \to \mathcal{H}^n$ such that $T_t = P_t|_{\mathcal{H}^n}$.*

6 Ergodicity

Define

$$\mathcal{A}_r(f) = \sum_{n=1}^{\infty} |\partial_n f|^2_{L^2(W,\mu_r)} = (-Lf, f)_{L^2(W,\mu_r)},$$

$$\nu = \sum_{n=1}^{\infty} \frac{n}{k_n^2}.$$

Theorem 11 *There exist $C = C(\{k_n\}_{n=1}^{\infty}) > 0$ such that for any $f \in \mathcal{H}^1$ and $t \geq 0$*

$$\mathcal{A}_r(P_t f) \leq C \mathcal{A}_r(f) e^{-\frac{t}{\nu}}, f \in \mathcal{H}^1. \tag{10}$$

Proof.

It is enough to show (10) for $f \in C_b^4(W)$. Indeed, $C_b^4(W)$ is dense in $\mathcal{H}^1$ and $(P_t)_{t\geq 0}$ is a contraction on $\mathcal{H}^1$ by 10.

Denote $f_t = P_t f$ for $t \geq 0$. For $i \in \mathbb{N}$, we can calculate that

$$
\begin{aligned}
|\partial_i f_t|^2 - P_t|\partial_i f|^2 &= \int_0^t \frac{d}{ds} P_{t-s}|\partial_i f_s|^2 ds \\
&= \int_0^t P_{t-s}(-\mathcal{L}(|\partial_i f_s|^2) + 2\partial_i f_s \mathcal{L}\partial_i f_s + 2\partial_i f_s[\partial_i, \mathcal{L}]f_s) ds \\
&= \int_0^t P_{t-s}\Big(- \sum_{m\in\mathbb{N}} k_m^2 |D_{m,m+1}(\partial_i f_s)|^2 \\
&\quad + \partial_i f_s(-(k_i^2 + k_{i-1}^2)\partial_i f_s + 2k_{i-1}^2 D_{i,i-1}\partial_{i-1} f_s + 2k_i^2 D_{i,i+1}\partial_{i+1} f_s)\Big) ds.
\end{aligned}
\tag{11}
$$

Integrating (11) with respect to the invariant measure μ_r yields

$$
\begin{aligned}
\mu_r|\partial_i f_t|^2 - \mu_r|\partial_i f|^2 = \int_0^t \Big(&- \sum_{m\in\mathbb{N}} k_m^2 |D_{m,m+1}(\partial_i f_s)|^2 \\
&- (k_i^2 + k_{i-1}^2)\mu_r|\partial_i f_s|^2 + 2k_{i-1}^2 \mu_r(\partial_i f_s D_{i,i-1}\partial_{i-1} f_s) \\
&+ 2k_i^2 \mu_r(\partial_i f_s D_{i,i+1}\partial_{i+1} f_s)\Big) ds.
\end{aligned}
\tag{12}
$$

Notice that the operators $D_{i,j}, i, j \in \mathbb{N}$, are antisymmetric in $L^2(\mu_r)$. Therefore

$$
\begin{aligned}
\mu_r|\partial_i f_t|^2 - \mu_r|\partial_i f|^2 = \int_0^t \Big(&- \sum_{m\in\mathbb{N}} k_m^2 \mu_r|D_{m,m+1}(\partial_i f_s)|^2 \\
&- (k_i^2 + k_{i-1}^2)\mu_r|\partial_i f_s|^2 - 2k_{i-1}^2 \mu_r(D_{i,i-1}(\partial_i f_s)\partial_{i-1} f_s) \\
&- 2k_i^2 \mu_r(D_{i,i+1}(\partial_i f_s)\partial_{i+1} f_s)\Big) ds.
\end{aligned}
\tag{13}
$$

Hence, by Young's inequality we deduce that

$$
\begin{aligned}
\mu_r|\partial_i f_t|^2 - \mu_r|\partial_i f|^2 \leq \int_0^t \Big(&- \sum_{m\in\mathbb{N}} k_m^2 \mu_r|D_{m,m+1}(\partial_i f_s)|^2 \\
&- (k_i^2 + k_{i-1}^2)\mu_r|\partial_i f_s|^2 + k_{i-1}^2 \mu_r|D_{i,i-1}\partial_i f_s|^2 + k_{i-1}^2\mu_r|\partial_{i-1} f_s|^2 \\
&+ k_i^2 \mu_r|D_{i,i+1}\partial_i f_s|^2 + k_i^2\mu_r|\partial_{i+1} f_s)|^2\Big) ds \leq \\
\leq \int_0^t \Big(&- \sum_{m\neq i,i-1} k_m^2 \mu_r|D_{m,m+1}(\partial_i f_s)|^2 \\
&- (k_i^2 + k_{i-1}^2)\mu_r|\partial_i f_s|^2 + k_{i-1}^2\mu_r|\partial_{i-1} f_s|^2 + k_i^2\mu_r|\partial_{i+1} f_s)|^2\Big) ds.
\end{aligned}
\tag{14}
$$

Let $\triangle^k$ denote the operator on $\mathbb{R}^{\mathbb{N}}$ given by

$$
\triangle^k f(i) = k_i^2(f(i+1) - f(i)) + k_{i-1}^2(f(i-1) - f(i)), i \in \mathbb{N}, f : \mathbb{N} \to \mathbb{R}, k_0 = 0,
\tag{15}
$$

and set $F(i,t) = \mu_r|\partial_i(P_t f)|^2$ for $t \geq 0, i \in \mathbb{N}$. Then we can rewrite (14) as

$$F(t) \leq F(0) + \int_0^t \triangle^k F(s)\, ds, \qquad t \in [0,\infty). \tag{16}$$

Hence, by the positivity of the semigroup $(e^{t\triangle^k})_{t\geq 0}$, and Duhamel's principle, we can conclude that

$$F(t) \leq G(t) := e^{t\triangle^k} F(0) \tag{17}$$

for $t \in [0,\infty)$. It has been shown in [2] that there exist $C = C(\{k_n\}_{n=1}^{\infty}) > 0$ such that

$$\sum_i G(i,t) \leq Ce^{-\frac{t}{\nu}} \sum_i G(i,0), t \geq 0, \nu = \sum_{n=1}^{\infty} \frac{n}{k_n^2}. \tag{18}$$

Now the result follows from inequalities (17) and (18). ■

Corollary 12

$$\mu_r(P_t f - \mu_r f)^2 \leq C\mathcal{A}_r(f)e^{-\frac{t}{\nu}}, f \in \mathcal{H}^1.$$

Proof. Proof immediately follows from Poincare inequality for Gaussian measure μ_r. ■

Define

$$\overline{\mathcal{H}}^1 = \{f \in L^2(\mu_r) | \int f d\mu_r = 0, ||f||^2_{\overline{\mathcal{H}}^1} = \mathcal{A}_r(f) < \infty\}.$$

Let $D_{\overline{\mathcal{H}}^1}(\mathcal{L})$ domain of operator $\mathcal{L}$ in $\overline{\mathcal{H}}^1$.

Corollary 13 (Poincare inequality in $\overline{\mathcal{H}}^1$) *There exists $C > 0$ such that*

$$||f||^2_{\overline{\mathcal{H}}^1} \leq C < -\mathcal{L}f, f >_{\overline{\mathcal{H}}^1}, f \in D_{\overline{\mathcal{H}}^1}(\mathcal{L}).$$

Corollary 14 *There exists $C > 0$ such that*

$$\mu_r(f - \mu_r f)^2 \leq 2\nu(-\mathcal{L}f, f)_{L^2(\mu_r)}(1 + max(0, \log \frac{C||f||^2_{\overline{\mathcal{H}}^1}}{2\nu(-\mathcal{L}f, f)_{L^2(\mu_r)}})), f \in D(\mathcal{L}) \cap \mathcal{H}^1.$$

Corollaries 13 and 14 can be deduced from the theorem 11 in the same way as Nash-Liggett inequalities has been proven in [13], see proof of Theorem 8.1.

Remark 15 *The convergence in Theorem 11 cannot be improved. Indeed, let $S(l,t) = P_t(x_l^2)$ for $t \geq 0$ and $l \in \mathbb{N}$. Then $\mathcal{L}x_l^2 = k_l^2(x_{l+1}^2 - x_l^2) + k_{l-1}^2(x_{l-1}^2 - x_l^2), l \in \mathbb{N}$, so that,*

$$\frac{\partial S}{\partial t} = \triangle^k S,$$

where $\triangle^k$ is defined by formula (15). Thus

$$S(t) = e^{t\triangle^k} S(0), t \geq 0, \tag{19}$$

so that convergence rate in the Theorem 11 is achieved.

Remark 16 *If* $\sum_{n=1}^{\infty} \frac{n}{k_n^2} = \infty$ *then it is possible to show polynomial rate of convergence to equilibrium in the same way as in the paper [13].*

Remark 17 *We have shown that the exponential rate of convergence for the semigroup* $(P_t)_{t\geq 0}$ *holds if* $k_n = \lambda^n$, $\lambda > 1$. *In the same time, there is no spectral gap if* $\lambda = 1$. *Indeed, it is enough to notice that if* $f_N = \sum_{k=1}^{N}(x_k^2 - 1)$ *then*

$$||f_N||^2_{\overline{\mathcal{H}}^1} \sim N, < -\mathcal{L}f_N, f_N >_{\overline{\mathcal{H}}^1} \text{ is independent upon } N,$$

and corollary 13 *does not hold. Thus, the asymptotic behaviour of our conservative system depends upon the value of the parameter* λ.

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