

Solution for the problem of the game *heads or tails*

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Abstract

In this paper, we describe the solution for a problem dealing with definite properties of binary sequences. This problem, proposed by Xavier Grandsart in the form of a mathematical contest [1], has been solved also by Maher Younan, Ph.D. student of Theoretical Physics at the University of Geneva, and Pierre Deligne, professor at Princeton and Field Medals, using different approaches with respect to the one presented in this work. All the proofs can be found in [1] (in French).

1 Introduction

Consider a repetition of n throwings in the game heads or tails: 0 if heads, 1 if tail. This can be represented as a binary sequence $D = d_0, d_1, \dots, d_{n-1}$ of length n .

We consider the 4 pairs made by inverse sequences of order 4 with periodical dimensions: 0100 – 0010, 1101 – 1011, 0011 – 1100, 1010 – 0101. We compute the difference between the number of occurrences of the two terms in each pair for our binary sequence D . Only the inverse sequences made of consecutive terms are considered. Furthermore, the binary sequence is circular. Hence we consider only the inverse sequences of order 4 having this form: $d_{i|n}, d_{(i+1)|n}, d_{(i+2)|n}, d_{(i+3)|n}$, $i \in \{0, \dots, n-1\}$ (where $i|n$ means $i \bmod n$).

Prove that, for each random binary sequence D , the difference between the number of occurrences of the two terms in each pair of inverse sequences of order 4 is preserved. For instance, why if the number of occurrences of

the term 0011 is equal to the number of occurrences of the term 1100 minus t , the number of occurrences of the term 1101 is equal to the number of occurrences of the term 1011 minus t ?

2 The solution

We consider a binary random sequence $D = d_0, d_1, \dots, d_{n-1}$ with length n . We have to prove that, for every sequence D , if we consider the four pairs of inverse sequences of order 4 with periodical dimension (0100 – 0010, 1101 – 1011, 0011 – 1100, 1010 – 0101), the difference between the number of occurrences of the two terms in each pair is the same. To do this, we use the mathematical induction on the size of the sequence D .

THEOREM 1. For every $D = d_0, d_1, \dots, d_{n-1}, d_i \in \{0, 1\}, 0 \leq i < n$, the difference between the number of occurrences of the two terms 0100 – 0010, 1101 – 1011, 1010 – 0101, 0011 – 1100 is always the same.

Proof. We consider as basis of induction the sequence of size 4, then we will prove that for every $n > 4$ the hypothesis is fulfilled.

Basis step.

There are 16 possible binary sequences D of size 4. The table 1 shows the number of inverse sequences and the difference between the number of occurrences of the 2 terms. As can be seen, this difference is equal to 0 for each sequence.

Inductive step.

Suppose that the theorem is valid for a sequence $D = d_0, d_1, \dots, d_{n-1}$. We will show that this is also true for the sequence $D' = d_0, d_1, \dots, d_{n-1}, d_n$. The difference in the inverse sequences of order 4 with periodical dimension between D and D' is that, by moving from D to D' , the following sequences are lost:

Sequence	0100	0010	1101	1011	1010	0101	0011	1100	Difference
0000	0	0	0	0	0	0	0	0	0
0001	1	1	0	0	0	0	0	0	0
0010	1	1	0	0	0	0	0	0	0
0011	0	0	0	0	0	0	1	1	0
0100	1	1	0	0	0	0	0	0	0
0101	0	0	0	0	2	2	0	0	0
0110	0	0	0	0	0	0	1	1	0
0111	0	0	1	1	0	0	0	0	0
1000	1	1	0	0	0	0	0	0	0
1001	0	0	0	0	0	0	0	0	0
1010	0	0	0	0	2	2	0	0	0
1011	0	0	1	1	0	0	0	0	0
1100	0	0	0	0	0	0	1	1	0
1101	0	0	1	1	0	0	0	0	0
1110	0	0	1	1	0	0	0	0	0
1111	0	0	0	0	0	0	0	0	0

Table 1: Sequences of size 4

- $d_{n-3}, d_{n-2}, d_{n-1}, d_0$
- $d_{n-2}, d_{n-1}, d_0, d_1$
- d_{n-1}, d_0, d_1, d_2

However, there are 4 new sequences:

- $d_{n-3}, d_{n-2}, d_{n-1}, d_n$
- $d_{n-2}, d_{n-1}, d_n, d_0$
- d_{n-1}, d_n, d_0, d_1
- d_n, d_0, d_1, d_2 .

Hence, the digits involved in that change are $d_{n-3} - d_{n-2} - d_{n-1} - d_0 - d_1 - d_2$ and d_n . Since d_i is 0 or 1, and there are 7 digits involved, there are 2^7 possible cases to be considered: we will show that, for every one, the theorem 1 is valid.

Let P be the sequence $d_{n-3}, d_{n-2}, d_{n-1}, d_0, d_1, d_2$; The first step to do is to compute the number of sequences lost for each case. Table 2 shows this.

P	0100	0010	1101	1011	1010	0101	0011	1100
000000	0	0	0	0	0	0	0	0
000001	0	0	0	0	0	0	0	0
000010	0	1	0	0	0	0	0	0
000011	0	0	0	0	0	0	1	0
000100	1	1	0	0	0	0	0	0
000101	0	1	0	0	0	1	0	0
000110	0	0	0	0	0	0	1	0
000111	0	0	0	0	0	0	1	0
001000	1	1	0	0	0	0	0	0
001001	1	1	0	0	0	0	0	0
001010	0	1	0	0	1	1	0	0
001011	0	1	0	1	0	1	0	0
001100	0	0	0	0	0	0	1	1
001101	0	0	1	0	0	0	1	0
001110	0	0	0	0	0	0	1	0
001111	0	0	0	0	0	0	1	0
010000	1	0	0	0	0	0	0	0
010001	1	0	0	0	0	0	0	0
010010	1	1	0	0	0	0	0	0
010011	1	0	0	0	0	0	1	0
010100	1	0	0	0	1	1	0	0
010101	0	0	0	0	1	2	0	0
010110	0	0	0	1	0	1	0	0

010111	0	0	0	1	0	1	0	0
011000	0	0	0	0	0	0	0	1
011001	0	0	0	0	0	0	0	1
011010	0	0	1	0	1	0	0	0
011011	0	0	1	1	0	0	0	0
011100	0	0	0	0	0	0	0	1
011101	0	0	1	0	0	0	0	0
011110	0	0	0	0	0	0	0	0
011111	0	0	0	0	0	0	0	0
100000	0	0	0	0	0	0	0	0
100001	0	0	0	0	0	0	0	0
100010	0	1	0	0	0	0	0	0
100011	0	0	0	0	0	0	1	0
100100	1	1	0	0	0	0	0	0
100101	0	1	0	0	0	1	0	0
100110	0	0	0	0	0	0	1	0
100111	0	0	0	0	0	0	1	0
101000	1	0	0	0	1	0	0	0
101001	1	0	0	0	1	0	0	0
101010	0	0	0	0	2	1	0	0
101011	0	0	0	1	1	1	0	0
101100	0	0	0	1	0	0	0	1
101101	0	0	1	1	0	0	0	0
101110	0	0	0	1	0	0	0	0
101111	0	0	0	1	0	0	0	0
110000	0	0	0	0	0	0	0	1
110001	0	0	0	0	0	0	0	1
110010	0	1	0	0	0	0	0	1
110011	0	0	0	0	0	0	1	1
110100	1	0	1	0	1	0	0	0

110101	0	0	1	0	1	1	0	0
110110	0	0	1	1	0	0	0	0
110111	0	0	1	1	0	0	0	0
111000	0	0	0	0	0	0	0	1
111001	0	0	0	0	0	0	0	1
111010	0	0	1	0	1	0	0	0
111011	0	0	1	1	0	0	0	0
111100	0	0	0	0	0	0	0	1
111101	0	0	1	0	0	0	0	0
111110	0	0	0	0	0	0	0	0
111111	0	0	0	0	0	0	0	0

Table 2: Sequences lost

After adjoining d_n , there are four new sequences, as previously described. Let $d_n = 0$ and $P = d_{n-3}, d_{n-2}, d_{n-1}, 0, d_0, d_1, d_2$; table 3 shows the number of sequences adjoined for each case.

P'	0100	0010	1101	1011	1010	0101	0011	1100
0000000	0	0	0	0	0	0	0	0
0000001	0	0	0	0	0	0	0	0
0000010	0	1	0	0	0	0	0	0
0000011	0	0	0	0	0	0	1	0
0000100	1	1	0	0	0	0	0	0
0000101	0	1	0	0	0	1	0	0
0000110	0	0	0	0	0	0	1	0
0000111	0	0	0	0	0	0	1	0
0010000	1	1	0	0	0	0	0	0
0010001	1	1	0	0	0	0	0	0
0010010	1	2	0	0	0	0	0	0
0010011	1	1	0	0	0	0	1	0

0010100	1	1	0	0	1	1	0	0
0010101	0	1	0	0	1	2	0	0
0010110	0	1	0	1	0	1	0	0
0010111	0	1	0	1	0	1	0	0
0100000	1	0	0	0	0	0	0	0
0100001	1	0	0	0	0	0	0	0
0100010	1	1	0	0	0	0	0	0
0100011	1	0	0	0	0	0	1	0
0100100	2	1	0	0	0	0	0	0
0100101	1	1	0	0	0	1	0	0
0100110	1	0	0	0	0	0	1	0
0100111	1	0	0	0	0	0	1	0
0110000	0	0	0	0	0	0	0	1
0110001	0	0	0	0	0	0	0	1
0110010	0	1	0	0	0	0	0	1
0110011	0	0	0	0	0	0	1	1
0110100	1	0	1	0	1	0	0	0
0110101	0	0	1	0	1	1	0	0
0110110	0	0	1	1	0	0	0	0
0110111	0	0	1	1	0	0	0	0
1000000	0	0	0	0	0	0	0	0
1000001	0	0	0	0	0	0	0	0
1000010	0	1	0	0	0	0	0	0
1000011	0	0	0	0	0	0	1	0
1000100	1	1	0	0	0	0	0	0
1000101	0	1	0	0	0	1	0	0
1000110	0	0	0	0	0	0	1	0
1000111	0	0	0	0	0	0	1	0
1010000	1	0	0	0	1	0	0	0
1010001	1	0	0	0	1	0	0	0

1010010	1	1	0	0	1	0	0	0
1010011	1	0	0	0	1	0	1	0
1010100	1	0	0	0	2	1	0	0
1010101	0	0	0	0	2	2	0	0
1010110	0	0	0	1	1	1	0	0
1010111	0	0	0	1	1	1	0	0
1100000	0	0	0	0	0	0	0	1
1100001	0	0	0	0	0	0	0	1
1100010	0	1	0	0	0	0	0	1
1100011	0	0	0	0	0	0	1	1
1100100	1	1	0	0	0	0	0	1
1100101	0	1	0	0	0	1	0	1
1100110	0	0	0	0	0	0	1	1
1100111	0	0	0	0	0	0	0	0
1110000	0	0	0	0	0	0	0	1
1110001	0	0	0	0	0	0	0	1
1110010	0	1	0	0	0	0	0	1
1110011	0	0	0	0	0	0	1	1
1110100	1	0	1	0	1	0	0	0
1110101	0	0	1	0	1	1	0	0
1110110	0	0	1	1	0	0	0	0
1110111	0	0	1	1	0	0	0	0

Table 3: Sequences adjoined, $d_n = 0$

The table 4 shows the number of sequences adjoined for each case, when $d_n = 1$ ($P'' = d_{n-3}, d_{n-2}, d_{n-1}, 1, d_0, d_1, d_2$).

P''	0100	0010	1101	1011	1010	0101	0011	1100
0001000	1	1	0	0	0	0	0	0
0001001	1	1	0	0	0	0	0	0

0001010	0	1	0	0	1	1	0	0
0001011	0	1	0	1	0	1	0	0
0001100	0	0	0	0	0	0	1	1
0001101	0	0	1	0	0	0	1	0
0001110	0	0	0	0	0	0	1	0
0001111	0	0	0	0	0	0	1	0
0011000	0	0	0	0	0	0	1	1
0011001	0	0	0	0	0	0	1	1
0011010	0	0	1	0	1	0	1	0
0011011	0	0	1	1	0	0	1	0
0011100	0	0	0	0	0	0	1	1
0011101	0	0	1	0	0	0	1	0
0011110	0	0	0	0	0	0	1	0
0011111	0	0	0	0	0	0	1	0
0101000	1	0	0	0	1	1	0	0
0101001	1	0	0	0	1	1	0	0
0101010	0	0	0	0	2	2	0	0
0101011	0	0	0	1	1	2	0	0
0101100	0	0	0	1	0	1	0	1
0101101	0	0	1	1	0	1	0	0
0101110	0	0	0	1	0	1	0	0
0101111	0	0	0	1	0	1	0	0
0111000	0	0	0	0	0	0	0	1
0111001	0	0	0	0	0	0	0	1
0111010	0	0	1	0	1	0	0	0
0111011	0	0	1	1	0	0	0	0
0111100	0	0	0	0	0	0	0	1
0111101	0	0	1	0	0	0	0	0
0111110	0	0	0	0	0	0	0	0
0111111	0	0	0	0	0	0	0	0

1001000	1	1	0	0	0	0	0	0
1001001	1	1	0	0	0	0	0	0
1001010	0	1	0	0	1	1	0	0
1001011	0	1	0	1	0	1	0	0
1001100	1	1	0	0	0	0	1	1
1001101	0	0	1	0	0	0	1	0
1001110	0	0	0	0	0	0	1	0
1001111	0	0	0	0	0	0	1	0
1011000	0	0	0	1	0	0	0	1
1011001	0	0	0	1	0	0	0	1
1011010	0	0	1	1	1	0	0	0
1011011	0	0	1	2	0	0	0	0
1011100	0	0	0	1	0	0	0	1
1011101	0	0	1	1	0	0	0	0
1011110	0	0	0	1	0	0	0	0
1011111	0	0	0	1	0	0	0	0
1101000	1	0	1	0	1	0	0	0
1101001	1	0	1	0	1	0	0	0
1101010	0	0	1	0	2	1	0	0
1101011	0	0	1	1	1	1	0	0
1101100	0	0	1	1	0	0	0	1
1101101	0	0	2	1	0	0	0	0
1101110	0	0	1	1	0	0	0	0
1101111	0	0	1	1	0	0	0	0
1111000	0	0	0	0	0	0	0	1
1111001	0	0	0	0	0	0	0	1
1111010	0	0	1	0	1	0	0	0
1111011	0	0	1	1	0	0	0	0
1111100	0	0	0	0	0	0	0	1
1111101	0	0	1	0	0	0	0	0

1111110	0	0	0	0	0	0	0	0
1111111	0	0	0	0	0	0	0	0

Table 4: Sequences adjoined, $d_n = 1$

Now, there is only to prove that the difference between the number of occurrences of the two terms for each periodic sequence is the same for every P' and P'' . For each case, there are the sequences lost to remove (computed in table 2) and the sequences adjoined (computed in table 3 when $d_n = 0$ and in table 4 when $d_n = 1$). The results are displayed in table 5 and 6. In order to understand better these tables, consider as example $P' = 1010101$ in table 5. By moving from $D = 101, d_3, d_4, \dots, d_{n-4}, 101$ to $D' = 101, d_3, d_4, \dots, d_{n-4}, 1010$ we lose one sequence 1101 and one 1011, as shown by table 2 ($P = 101101$). Nevertheless, we gain two sequences 1010 and two sequences 0101, as shown by table 5 ($P' = 1010101$). In this case, the difference between the sequences is the same for D and D' ; if there was a Δ Difference of +1, this means that if the sequence D has a difference E between the number of occurrences of the two terms for each periodic sequence, the sequence D' has a difference of $E + 1$.

P'	0100	0010	1101	1011	1010	0101	0011	1100	Δ Difference
0000000	0	0	0	0	0	0	0	0	0
0000001	0	0	0	0	0	0	0	0	0
0000010	0	1-1	0	0	0	0	0	0	0
0000011	0	0	0	0	0	0	1-1	0	0
0000100	1-1	1-1	0	0	0	0	0	0	0
0000101	0	1-1	0	0	0	1-1	0	0	0
0000110	0	0	0	0	0	0	1-1	0	0
0000111	0	0	0	0	0	0	1-1	0	0
0010000	1-1	1-1	0	0	0	0	0	0	0
0010001	1-1	1-1	0	0	0	0	0	0	0

0010010	1	2-1	0	0	-1	-1	0	0	0
0010011	1	1-1	0	-1	0	-1	1	0	+1
0010100	1	1	0	0	1	1	-1	-1	0
0010101	0	1	-1	0	1	2	-1	0	-1
0010110	0	1	0	1	0	1	-1	0	-1
0010111	0	1	0	1	0	1	-1	0	-1
0100000	1-1	0	0	0	0	0	0	0	0
0100001	1-1	0	0	0	0	0	0	0	0
0100010	1-1	1-1	0	0	0	0	0	0	0
0100011	1-1	0	0	0	0	0	1-1	0	0
0100100	2-1	1	0	0	-1	-1	0	0	0
0100101	1	1	0	0	-1	1-2	0	0	0
0100110	1	0	0	-1	0	-1	1	0	+1
0100111	1	0	0	-1	0	-1	1	0	+1
0110000	0	0	0	0	0	0	0	1-1	0
0110001	0	0	0	0	0	0	0	1-1	0
0110010	0	1	-1	0	-1	0	0	1	-1
0110011	0	0	-1	-1	0	0	1	1	0
0110100	1	0	1	0	1	0	0	-1	1
0110101	0	0	1-1	0	1	1	0	0	0
0110110	0	0	1	1	0	0	0	0	0
0110111	0	0	1	1	0	0	0	0	0
1000000	0	0	0	0	0	0	0	0	0
1000001	0	0	0	0	0	0	0	0	0
1000010	0	1-1	0	0	0	0	0	0	0
1000011	0	0	0	0	0	0	1-1	0	0
1000100	1-1	1-1	0	0	0	0	0	0	0
1000101	0	1-1	0	0	0	1-1	0	0	0
1000110	0	0	0	0	0	0	1-1	0	0
1000111	0	0	0	0	0	0	1-1	0	0

1010000	1-1	0	0	0	1-1	0	0	0	0
1010001	1-1	0	0	0	1-1	0	0	0	0
1010010	1	1	0	0	1-2	-1	0	0	0
1010011	1	0	0	-1	1-1	-1	1	0	+1
1010100	1	0	0	-1	2	1	0	-1	+1
1010101	0	0	-1	-1	2	2	0	0	0
1010110	0	0	0	1-1	1	1	0	0	0
1010111	0	0	0	1-1	1	1	0	0	0
1100000	0	0	0	0	0	0	0	1-1	0
1100001	0	0	0	0	0	0	0	1-1	0
1100010	0	1-1	0	0	0	0	0	1-1	0
1100011	0	0	0	0	0	0	1-1	1-1	0
1100100	1-1	1	-1	0	-1	0	0	1	-1
1100101	0	1	-1	0	-1	1-1	0	1	-1
1100110	0	0	-1	-1	0	0	1	1	0
1100111	0	0	-1	-1	0	0	0	0	0
1110000	0	0	0	0	0	0	0	1-1	0
1110001	0	0	0	0	0	0	0	1-1	0
1110010	0	1	-1	0	-1	0	0	1	0
1110011	0	0	-1	-1	0	0	1	1	0
1110100	1	0	1	0	1	0	0	-1	+1
1110101	0	0	1-1	0	1	1	0	0	0
1110110	0	0	1	1	0	0	0	0	0
1110111	0	0	1	1	0	0	0	0	0

Table 5: Δ Difference, $d_n = 0$

P''	0100	0010	1101	1011	1010	0101	0011	1100	Δ Difference
0001000	1	1	0	0	0	0	0	0	0
0001001	1	1	0	0	0	0	0	0	0

0001010	0	1-1	0	0	1	1	0	0	0
0001011	0	1	0	1	0	1	-1	0	-1
0001100	-1	-1	0	0	0	0	1	1	0
0001101	0	-1	1	0	0	-1	1	0	1
0001110	0	0	0	0	0	0	1-1	0	0
0001111	0	0	0	0	0	0	1-1	0	0
0011000	-1	-1	0	0	0	0	1	1	0
0011001	-1	-1	0	0	0	0	1	1	0
0011010	0	-1	1	0	1-1	-1	1	0	+1
0011011	0	-1	1	1-1	0	-1	1	0	+1
0011100	0	0	0	0	0	0	1-1	1-1	0
0011101	0	0	1-1	0	0	0	1-1	0	0
0011110	0	0	0	0	0	0	1-1	0	0
0011111	0	0	0	0	0	0	1-1	0	0
0101000	1-1	0	0	0	1	1	0	0	0
0101001	1-1	0	0	0	1	1	0	0	0
0101010	-1	-1	0	0	2	2	0	0	0
0101011	-1	0	0	1	1	2	-1	0	-1
0101100	-1	0	0	1	-1	1-1	0	1	-1
0101101	0	0	1	1	-1	1-2	0	0	0
0101110	0	0	0	1-1	0	1-1	0	0	0
0101111	0	0	0	1-1	0	1-1	0	0	0
0111000	0	0	0	0	0	0	0	1-1	0
0111001	0	0	0	0	0	0	0	1-1	0
0111010	0	0	1-1	0	1-1	0	0	0	0
0111011	0	0	1-1	1-1	0	0	0	0	0
0111100	0	0	0	0	0	0	0	1-1	0
0111101	0	0	1-1	0	0	0	0	0	0
0111110	0	0	0	0	0	0	0	0	0
0111111	0	0	0	0	0	0	0	0	0

1001000	1	1	0	0	0	0	0	0	0
1001001	1	1	0	0	0	0	0	0	0
1001010	0	1-1	0	0	1	1	0	0	0
1001011	0	1	0	1	0	1	-1	0	-1
1001100	1-1	1-1	0	0	0	0	1	1	0
1001101	0	-1	1	0	0	-1	1	0	+1
1001110	0	0	0	0	0	0	1-1	0	0
1001111	0	0	0	0	0	0	1-1	0	0
1011000	-1	0	0	1	-1	0	0	1	-1
1011001	-1	0	0	1	-1	0	0	1	-1
1011010	0	0	1	1	1-2	-1	0	0	0
1011011	0	0	1	2-1	-1	-1	0	0	0
1011100	0	0	0	1-1	0	0	0	1-1	0
1011101	0	0	1-1	1-1	0	0	0	0	0
1011110	0	0	0	1-1	0	0	0	0	0
1011111	0	0	0	1-1	0	0	0	0	0
1101000	1	0	1	0	1	0	0	-1	+1
1101001	1	0	1	0	1	0	0	-1	+1
1101010	0	-1	1	0	2	1	0	-1	+1
1101011	0	0	1	1	1	1	-1	-1	0
1101100	-1	0	1-1	1	-1	0	0	1	-1
1101101	0	0	2-1	1	-1	-1	0	0	0
1101110	0	0	1-1	1-1	0	0	0	0	0
1101111	0	0	1-1	1-1	0	0	0	0	0
1111000	0	0	0	0	0	0	0	1-1	0
1111001	0	0	0	0	0	0	0	1-1	0
1111010	0	0	1-1	0	1-1	0	0	0	0
1111011	0	0	1-1	1-1	0	0	0	0	0
1111100	0	0	0	0	0	0	0	1-1	0
1111101	0	0	1-1	0	0	0	0	0	0

1111110	0	0	0	0	0	0	0	0	0
1111111	0	0	0	0	0	0	0	0	0

Table 6: Δ Difference, $d_n = 1$

3 Conclusion

The theorem allows us to answer the question by illustrating that for each binary sequence D the property is verified. This theorem can be generalized in order to prove (or to refuse) the same properties for inverse sequences of length bigger than 4.

References

- [1] Xavier Grandsart, Concours de Mathématiques pour résoudre une énigme au jeu de pile ou face (in French), *Organized by Pixis RCS Paris A 434 463 055*, <http://xaviergrandsart.com/concours-de-mathematiques.html>.