Finsler spaces with infinite dimensional holonomy group

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Dedicated to Professor Joseph Grifone on the occasion of his 70th birthday

Abstract

Our paper is devoted to the study of the holonomy groups of Finsler surfaces using the methods of infinite dimensional Lie theory. The notion of infinitesimal holonomy algebra will be introduced, by the smallest Lie algebra of vector fields on an indicatrix, containing the curvature vector fields and their horizontal covariant derivatives with respect to the Berwald connection. We obtain that the topological closure of the holonomy group contains the exponential image of any tangent Lie algebra of the holonomy group. A class of Randers surfaces is determined, for which the infinitesimal holonomy algebra coincides with the curvature algebra. We prove that for all projectively flat Randers surfaces of non-zero constant flag curvature the infinitesimal holonomy algebra has infinite dimension and hence the holonomy group cannot be a Lie group of finite dimension. Finally, in the case of the Funk metric we prove that the infinitesimal holonomy algebra is a dense subalgebra of the Lie algebra of the full diffeomorphism group and hence the topological closure of the holonomy group is the orientation preserving diffeomorphism group of the circle.

1 Introduction

The holonomy group of a Finsler manifold is the subgroup of the diffeomorphism group of an indicatrix, generated by canonical homogeneous (nonlinear) parallel translations along closed loops. We showed in a previous paper [6] that the holonomy group of a Finsler manifold of non-zero constant flag curvature cannot be a compact Lie group, and in general, the study its holonomy theory needs the technique of infinite dimensional Lie groups. This paper is devoted to the investigation of holonomy groups of Finsler manifolds using the results of Omori's infinite dimensional Lie theory [8, 9].

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We prove that the topological closure of the holonomy group is contained the exponential image of any tangent Lie algebra of the holonomy group. After the introduction of the notion of *infinitesimal holonomy algebra* as the smallest Lie algebra of vector fields on an indicatrix, containing the curvature vector fields and their horizontal covariant derivatives with respect to the Berwald connection, we give a direct proof of the tangent property of this Lie algebra, which is a result of M. Crampin, D. J. Saunders [3] obtained in the course of the discussion of what would be the natural notion of holonomy algebra. We apply our results on the infinitesimal holonomy algebra to the study of holonomy properties of two-dimensional non-Riemannian Finsler manifolds of non-zero constant flag curvature. We find a class of Randers surfaces for which the infinitesimal holonomy algebra coincides with the curvature algebra. We prove that for all projectively flat Randers surfaces of non-zero constant flag curvature the infinitesimal holonomy algebra has infinite dimension and hence the holonomy group cannot be a Lie group of finite dimension. Finally, we prove that in the case of Funk metric the infinitesimal holonomy algebra is a dense subalgebra of the Lie algebra of the full diffeomorphism group of the indicatrix, containing the real Witt algebra of trigonometric polynomial vector fields. Hence the topological closure of this holonomy group is the orientation preserving diffeomorphism group of the circle.

2 Preliminaries

Throughout this article, M denotes a C^∞ manifold, $\mathfrak{X}^\infty(M)$ denotes the vector space of smooth vector fields on M and $\mathsf{Diff}^\infty(M)$ denotes the group of all C^∞-diffeomorphism of M with the C^∞-topology.

If M is a compact manifold then $\mathsf{Diff}^\infty(M)$ is a F-regular infinite dimensional Lie group modeled on the vector space $\mathfrak{X}^\infty(M)$. Particularly $\mathsf{Diff}^\infty(M)$ is a strong ILB-Lie group. In this category of group one can define the exponential mapping and the group structure is locally determined by the Lie algebra by the exponential mapping. The Lie algebra of $\mathsf{Diff}^\infty(M)$ is $\mathfrak{X}^\infty(M)$ equipped with the negative of the usual Lie bracket (cf. [8, 9]).

Finsler manifold, canonical connection, parallelism

A *Finsler manifold* is a pair $(M,\mathcal{F})$, where M is an n-dimensional smooth manifold and $\mathcal{F}\colon TM \to \mathbb{R}$ is a continuous function, smooth on $\hat{T}M := TM\setminus\{0\}$, its restriction $\mathcal{F}_x = \mathcal{F}\big|_{T_xM}$ is a positively homogeneous function of degree 1 and the symmetric bilinear form

$$g_{x,y}\colon (u,v) \ \mapsto \ g_{ij}(x,y)u^iv^j = \frac{1}{2}\frac{\partial^2\mathcal{F}_x^2(y+su+tv)}{\partial s\,\partial t}\Big|_{t=s=0}$$

is positive definite at every $y\in\hat{T}_xM$.

Geodesics of Finsler manifolds are determined by a system of 2nd order ordinary differential equation:

$$\ddot{x}^i + 2G^i(x,\dot{x}) = 0, \quad i = 1,...,n$$

where $G^i(x,\dot{x})$ are locally given by

$$G^i(x,y) := \frac{1}{4}g^{il}(x,y)\Big(2\frac{\partial g_{jl}}{\partial x^k}(x,y) - \frac{\partial g_{jk}}{\partial x^l}(x,y)\Big)y^j y^k. \tag{1}$$

The associated *homogeneous (nonlinear) parallel translation* can be defined as follows: a vector field $X(t) = X^i(t)\frac{\partial}{\partial x^i}$ along a curve $c(t)$ is said to be parallel if it satisfies

$$D_{\dot{c}}X(t) := \Big(\frac{dX^i(t)}{dt} + G^i_j(c(t),X(t))\dot{c}^j(t)\Big)\frac{\partial}{\partial x^i} = 0, \tag{2}$$

where $G^i_j = \frac{\partial G^i}{\partial y^j}$.

Horizontal distribution, Berwald connection, curvature

Let (TM,π,M) and (TTM,τ,TM) denote the first and the second tangent bundle of the manifold M, respectively. The horizontal distribution $\mathcal{H}TM \subset TTM$ associated to the Finsler manifold $(M,\mathcal{F})$ can be defined as the image of the horizontal lift which is at each $x \in M$ an isomorphism $X \to X^h$ between T_xM and $\mathcal{H}_xTM$ defined by the formula

$$\Big(X^i\frac{\partial}{\partial x^i}\Big)^h := X^i\left(\frac{\partial}{\partial x^i} - G^k_i(x,y)\frac{\partial}{\partial y^k}\right). \tag{3}$$

If $\mathcal{V}TM := \mathrm{Ker}\,\pi_* \subset TTM$ denotes the vertical distribution on TM, $\mathcal{V}_yTM := \mathrm{Ker}\,\pi_{*,y}$, then for any $y \in TM$ we have $T_yTM = \mathcal{H}_yTM \oplus \mathcal{V}_yTM$. The projectors corresponding to this decomposition will be denoted by $h : TTM \to \mathcal{H}TM$ and $v : TTM \to \mathcal{V}TM$. We note that the vertical distribution is integrable.

Let $(\hat{\mathcal{V}}TM,\tau,\hat{T}M)$ be the vertical bundle over $\hat{T}M := TM\setminus\{0\}$. We denote by $\mathfrak{X}^\infty(M)$, respectively by $\hat{\mathfrak{X}}^\infty(TM)$ the vector space of smooth vector fields on M and of smooth sections of the bundle $(\hat{\mathcal{V}}TM,\tau,\hat{T}M)$, respectively. The *horizontal Berwald covariant derivative* of a section $\xi \in \hat{\mathfrak{X}}^\infty(TM)$ by a vector field $X \in \mathfrak{X}^\infty(M)$ is defined by

$$\nabla_X\xi := [X^h,\xi]. \tag{4}$$

In an induced local coordinate system (x^i,y^i) on TM for the vector fields $\xi(x,y) = \xi^i(x,y)\frac{\partial}{\partial y^i}$ and $X(x) = X^i(x)\frac{\partial}{\partial x^i}$ we have (3) and hence

$$\nabla_X\xi = \left(\frac{\partial\xi^i(x,y)}{\partial x^j} - G^k_j(x,y)\frac{\partial\xi^i(x,y)}{\partial y^k} + G^i_{jk}(x,y)\xi^k(x,y)\right)X^j\frac{\partial}{\partial y^i}, \tag{5}$$

where

$$G^i_{jk}(x,y) := \frac{\partial G^i_j(x,y)}{\partial y^k}.$$

Let $(\pi^*TM, \bar{\pi}, \hat{T}M)$ be the pull-back bundle of $(\hat{T}M, \pi, M)$ by the map $\pi : TM \to M$. Clearly, the mapping

$$(x, y, \xi^i \frac{\partial}{\partial y^i}) \mapsto (x, y, \xi^i \frac{\partial}{\partial x^i}) : \ \hat{\mathcal{V}}TM \to \pi^*TM \tag{6}$$

is a canonical bundle isomorphism. In the following we will use the isomorphism (6) for the identification of these bundles. If we define

$$\nabla_X \phi = \left(\frac{\partial \phi}{\partial x^j} - G^k_j(x, y) \frac{\partial \phi(x, y)}{\partial y^k} \right) X^j$$

for a smooth function $\phi : \hat{T}M \to \mathbb{R}$, the horizontal Berwald covariant derivation (5) can be extended to the tensor bundle over $(\pi^*TM, \bar{\pi}, \hat{T}M)$.

The *Riemannian curvature tensor* field characterizes the integrability of the horizontal distribution:

$$R_{(x,y)}(X, Y) := v\big[X^h, Y^h\big], \qquad X, Y \in T_xM. \tag{7}$$

If the horizontal distribution $\mathcal{H}TM$ is integrable, then the Riemannian curvature is identically zero. Using a local coordinate system the expression of the Riemannian curvature tensor $R_{(x,y)} = R^i_{jk}(x, y) dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^i}$ on the pull-back bundle $(\pi^*TM, \bar{\pi}, \hat{T}M)$ is

$$R^i_{jk}(x, y) = \frac{\partial G^i_j(x, y)}{\partial x^k} - \frac{\partial G^i_k(x, y)}{\partial x^j} + G^m_j(x, y) G^i_{km}(x, y) - G^m_k(x, y) G^i_{jm}(x, y).$$

The manifold is called of constant flag curvature $\lambda \in \mathbb{R}$, if for any $x \in M$ the local expression of the Riemannian curvature is

$$R^i_{jk}(x, y) = \lambda \left(\delta^i_k g_{jm}(x, y) y^m - \delta^i_j g_{km}(x, y) y^m \right). \tag{8}$$

In this case the flag curvature of the Finsler manifold (cf. [2], Section 2.1 pp. 43-46) does not depend either on the point or on the 2-flag.

The *Berwald curvature tensor* field $B_{(x,y)} = B^i_{jkl}(x, y) dx^j \otimes dx^k \otimes dx^l \otimes \frac{\partial}{\partial x^i}$ is

$$B^i_{jkl}(x, y) = \frac{\partial G^i_{jk}(x, y)}{\partial y^l} = \frac{\partial^3 G^i(x, y)}{\partial y^j \partial y^k \partial y^l}. \tag{9}$$

The *mean Berwald curvature tensor* field $E_{(x,y)} = E_{jk}(x, y) dx^j \otimes dx^k$ is the trace

$$E_{jk}(x, y) = B^l_{jkl}(x, y) = \frac{\partial^3 G^l(x, y)}{\partial y^j \partial y^k \partial y^l}. \tag{10}$$

The *Landsberg curvature tensor* field $L_{(x,y)} = L^i_{jkl}(x, y) dx^j \otimes dx^k \otimes dx^l \otimes \frac{\partial}{\partial x^i}$ is

$$L_{(x,y)}(u, v, w) = g_{(x,y)} \left(\nabla_w B_{(x,y)}(u, v, w), y \right), \quad u, v, w \in T_xM.$$

According to Lemma 6.2.2, equation (6.30), p. 85 in [10], one has for $u, v, w \in T_xM$

$$\nabla_w g_{(x,y)}(u,v) = -2L(u,v,w).$$

Lemma 2.1 *The horizontal Berwald covariant derivative of the tensor field*

$$Q_{(x,y)} = \big(\delta^i_j g_{km}(x,y)y^m - \delta^i_k g_{jm}(x,y)y^m\big)\, dx^j \otimes dx^k \otimes dx^l \otimes \frac{\partial}{\partial x^i}$$

vanishes.

Proof. For any vector field $W \in \mathfrak{X}^\infty(M)$ we have $\nabla_W y = 0$ and $\nabla_W \mathsf{Id}_{TM} = 0$. Moreover, since $L_{(x,y)}(y, v, w) = 0$ (cf. equation 6.28, p. 85 in [10]) we get the assertion. ■

3 Holonomy of Finsler spaces

Let $(M, \mathcal{F})$ be an n-dimensional Finsler manifold. We denote by $(\mathcal{I}M, \pi, M)$ the *indicatrix bundle* of $(M, \mathcal{F})$, the *indicatrix* $\mathcal{I}_xM$ at $x \in M$ is the compact hypersurface

$$\mathcal{I}_xM := \{y \in T_xM;\ \mathcal{F}(x,y) = 1\},$$

of T_xM diffeomorphic to the standard $(n-1)$-sphere.

The homogeneous (nonlinear) parallel translation $\tau_c : T_{c(0)}M \to T_{c(1)}M$ along a curve $c : [0,1] \to M$, defined by (2) preserves the value of the Finsler function, hence it induces a map

$$\tau^{\mathcal{I}}_c : \mathcal{I}_{c(0)}M \longrightarrow \mathcal{I}_{c(1)}M \tag{11}$$

between the indicatrices.

The notion of the holonomy group of a Riemannian manifold can be generalized very naturally for a Finsler manifold:

Definition 3.1 The *holonomy group* $\mathsf{Hol}(x)$ of a Finsler space $(M, \mathcal{F})$ at a point $x \in M$ is the subgroup of the group of diffeomorphisms $\mathsf{Diff}^\infty(\mathcal{I}_xM)$ of the indicatrix $\mathcal{I}_xM$ generated by (nonlinear) parallel translations of $\mathcal{I}_xM$ along piece-wise differentiable closed curves initiated at the point $x \in M$.

Clearly, the holonomy groups at different points of M are isomorphic.

Definition 3.2 A vector field $\xi \in \mathfrak{X}^\infty(\mathcal{I}M)$ on the indicatrix bundle $\mathcal{I}M$ is a *curvature vector field* of the Finsler manifold $(M, \mathcal{F})$, if there exist vector fields $X, Y \in \mathfrak{X}^\infty(M)$ on the manifold M such that $\xi = r(X, Y)$, where for every $x \in M$ and $y \in \mathcal{I}_xM$ we have

$$r(X,Y)(x,y) := R_{(x,y)}(X_x, Y_x). \tag{12}$$

If $x \in M$ is fixed and $X, Y \in T_xM$, then the vector field $y \to r(X,Y)(x,y)$ on $\mathcal{I}_xM$ is a *curvature vector field at* x (see [6]).

The Lie algebra $\mathfrak{R}(M)$ of vector fields generated by the curvature vector fields of $(M, \mathcal{F})$ is called the *curvature algebra* of the Finsler manifold $(M, \mathcal{F})$. For a fixed $x \in M$ the Lie algebra $\mathfrak{R}_x$ of vector fields generated by the curvature vector fields at x is called the *curvature algebra at the point* x.

Definition 3.3 The *infinitesimal holonomy algebra* of the Finsler manifold $(M, \mathcal{F})$ is the smallest Lie algebra $\mathfrak{hol}^*(M)$ of vector fields on the indicatrix bundle $\mathcal{I}M$ satisfying the properties

(i) any curvature vector field ξ belongs to $\mathfrak{hol}^*(M)$,

(ii) if $\xi, \eta \in \mathfrak{hol}^*(M)$ then $[\xi, \eta] \in \mathfrak{hol}^*(M)$,

(iii) if $\xi \in \mathfrak{hol}^*(M)$ and $X \in \mathfrak{X}^\infty(M)$ then the horizontal Berwald covariant derivative $\nabla_X \xi$ also belongs to $\mathfrak{hol}^*(M)$.

The *infinitesimal holonomy algebra at a point* $x \in M$ is the Lie algebra

$$\mathfrak{hol}^*(x) := \big\{ \, \xi(x) \; ; \; \xi \in \mathfrak{hol}^*(M) \, \big\} \subset \mathfrak{X}^\infty(\mathcal{I}_xM)$$

of vector fields on the indicatrix $\mathcal{I}_xM$.

Clearly, $\mathfrak{R}(M) \subset \mathfrak{hol}^*(M)$ and $\mathfrak{R}_x \subset \mathfrak{hol}^*(x)$ for any $x \in M$.

4 Tangent Lie algebras to the holonomy group

Before formulating the next theorem, we recall the notion of tangent vector fields to a subgroup of the diffeomorphism group, introduced and discussed in [6]. Let H be a subgroup of the diffeomorphism group $\mathsf{Diff}^\infty(M)$ of a differentiable manifold M and let $\mathfrak{X}^\infty(M)$ be the Lie algebra of smooth vector fields on M.

Definition 4.1 A vector field $X \in \mathfrak{X}^\infty(M)$ is called *tangent to* $H \subset \mathsf{Diff}^\infty(M)$ if there exists a $\mathcal{C}^1$-differentiable 1-parameter family $\{\Phi(t) \in H\}_{t \in \mathbb{R}}$ of diffeomorphisms of M such that $\Phi(0) = \mathsf{Id}$ and $\frac{\partial \Phi(t)}{\partial t}\big|_{t=0} = X$.

A Lie subalgebra $\mathfrak{g}$ of $\mathfrak{X}^\infty(M)$ is called *tangent to* H, if all elements of $\mathfrak{g}$ are tangent vector fields to H.

Proposition 4.2 *If the Lie subalgebra* $\mathfrak{g}$ *of* $\mathfrak{X}^\infty(M)$ *is tangent to a closed subgroup* H *of* $\mathsf{Diff}^\infty(M)$*, then the exponential image* $\exp(\mathfrak{g})$ *of* $\mathfrak{g}$ *is contained in* H*.*

Proof. The diffeomorphism group $\mathsf{Diff}^\infty(M)$ is a strong ILB-Lie group ([9], Theorem 2.1 in Ch. VI, p. 137) and hence it is a regular F-Lie group ([9], Corollary 5.4, p. 84). According to the arguments used in the proof of this corollary, if $\{\Phi(s)\in H\}_{s\in\mathbb{R}}$ is a $\mathcal{C}^1$-differentiable 1-parameter family of diffeomorphisms of M such that

$$\Phi(0)=\mathsf{Id}\quad\text{and}\quad \frac{\partial\Phi(s)}{\partial s}\Big|_{s=0}=X,$$

then the 1-parameter families

$$\left\{\Phi\left(\frac{s}{n}\right)^n\in H\right\}_{s\in\mathbb{R}},\quad n=1,2\dots$$

of diffeomorphisms converge uniformly on each compact interval to the 1-parameter group $\{\exp sX\}_{s\in\mathbb{R}}$, where $X\in\mathfrak{g}$. It follows that $\{\exp sX\}_{s\in\mathbb{R}}\subset H$ for any $X\in\mathfrak{g}$. ∎

Theorem 4.3 *The infinitesimal holonomy algebra $\mathfrak{hol}^*(x)$ at a point $x\in M$ has the following properties:*

(i) $\mathfrak{hol}^*(x)$ *is tangent to the holonomy group* $\mathsf{Hol}(x)$,

(ii) *the group generated by the exponential image* $\exp\big(\mathfrak{hol}^*(x)\big)$ *is a subgroup of the topological closure of the holonomy group* $\mathsf{Hol}(x)$.

To prove the theorem we have to introduce some notion. We say that a vector field $\xi\in\mathfrak{X}^\infty(M)$ is *strongly tangent* to $H\subset\mathsf{Diff}^\infty(M)$ if there exist $k\in\mathbb{N}$ and a $\mathcal{C}^\infty$-differentiable k-parameter family $\{\Phi_{(t_1,\dots,t_k)}\in H\}_{t_i\in(-\varepsilon,\varepsilon)}$ of diffeomorphisms associated to ξ such that

1. $\Phi_{(t_1,\dots,t_k)}=\mathsf{Id}$, if $t_j=0$ for some $1\le j\le k$;
2. $\frac{\partial^k\Phi_{(t_1,\dots,t_k)}}{\partial t_1\cdots\partial t_k}\Big|_{(t_1,\dots,t_k)=(0,\dots,0)}=\xi$.

In [6] it was proved that if $\xi,\eta\in\mathfrak{X}^\infty(M)$ are strongly tangent vector field to the group H, then $[\xi,\eta]\in\mathfrak{X}^\infty(M)$ is also strongly tangent to H. Moreover, if $\mathcal{V}$ is a set of strongly tangent vector fields to the group H, then the Lie subalgebra $\mathfrak{v}$ of $\mathfrak{X}^\infty(M)$ generated by $\mathcal{V}$ is tangent to H.
Now let U be an open neighbourhood in the manifold M and let $\mathsf{Hol}_U(x)$ denote the holonomy group at $x\in U$ of the Finsler submanifold $(U,\mathcal{F}|_U)$. The group $\mathsf{Hol}_\mathsf{f}(U)$ of fibre preserving diffeomorphisms of the indicatrix bundle $(\mathcal{I}U,\pi|_U,U)$, inducing elements of the holonomy group $\mathsf{Hol}_U(x)$ on any indicatrix $\mathcal{I}_x$ will be called the *fibred holonomy group* of the submanifold $(U,\mathcal{F}|_U)$.
It is clear that any strongly tangent vector field $\xi\in\mathfrak{X}^\infty(\mathcal{I}U)$ to the fibred holonomy group $\mathsf{Hol}_\mathsf{f}(U)$ is a vertical vector field and for any $x\in U$ its restriction $\xi_x:=\xi\big|_{\mathcal{I}_x}$ to the indicatrix $\mathcal{I}_x$ is strongly tangent to the holonomy group $\mathsf{Hol}_U(x)$.

Lemma 4.4 *If $U \subset M$ is diffeomorphic to $\mathbb{R}^n$, then any curvature vector field on U is strongly tangent to the fibred holonomy group $\mathsf{Hol}_\mathsf{f}(U)$.*

Proof. Since U is diffeomorphic to $\mathbb{R}^n$, we can identify U with the vector space $\mathbb{R}^n$. Let $\xi = r(X,Y) \in \mathfrak{X}^\infty(\mathcal{I}\mathbb{R}^n)$ be a curvature vector field, where $X, Y \in \mathfrak{X}^\infty(\mathbb{R}^n)$. We show that there exists a family $\big\{\Phi_{(s,t)}\big|_{\mathcal{I}\mathbb{R}^n}\big\}_{s,t\in(-\varepsilon,\varepsilon)}$ of fibre preserving diffeomorphisms of the indicatrix bundle $(\mathcal{I}\mathbb{R}^n, \pi, \mathbb{R}^n)$ such that for any $x \in \mathbb{R}^n$ the induced family of diffeomorphisms of the indicatrix $\mathcal{I}_x$ is contained in $\mathsf{Hol}_U(x)$ and $\xi_x = \xi\big|_{\mathcal{I}_x\mathbb{R}^n}$ is the corresponding strongly tangent vector field to $\mathsf{Hol}_U(x)$.
For any $x \in \mathbb{R}^n$ and $0 \leq s, t \leq 1$ let $\Pi(sX_x, tY_x)$ be the parallelogram in $\mathbb{R}^n$ determined by the vertices x, $x+sX_x$, $x+sX_x+tY_x$, $x+tY_x \in \mathbb{R}^n$ and let $\tau_{\Pi(sX_x,tY_x)} \colon \mathcal{I}_xM \to \mathcal{I}_xM$ denote the (nonlinear) parallel translation of the indicatrix $\mathcal{I}_x$ along the parallelogram $\Pi(sX_x, tY_x)$. Clearly we have $\tau_{\Pi(sX_x,tY_x)} = \mathsf{Id}_{\mathcal{I}\mathbb{R}^n}$, if $s = 0$ or $t = 0$ and

$$\frac{\partial^2 \tau_{\Pi(sX_x,tY_x)}}{\partial s \partial t}\bigg|_{(s,t)=(0,0)} = \xi_x, \quad \text{for every} \quad x \in \mathbb{R}^n.$$

Since $\Pi(sX_x, tY_x)$ is a differentiable field of parallelograms in $\mathbb{R}^n$, the maps $\tau_{\Pi(sX_x,tY_x)}$ are fibre preserving diffeomorphisms of the indicatrix bundle $\mathcal{I}\mathbb{R}^n$ for any $0 \leq s, t \leq 1$, and for any $x \in \mathbb{R}^n$ the induced family of diffeomorphisms of the indicatrix $\mathcal{I}_x$ is contained in $\mathsf{Hol}_U(x)$. Hence the vector field $\xi \in \mathfrak{X}^\infty(\mathbb{R}^n)$ is strongly tangent to the fibred holonomy group $\mathsf{Hol}_\mathsf{f}(U)$. ■

Corollary 4.5 *If $U \subset M$ is diffeomorphic to $\mathbb{R}^n$, then the curvature algebra $\mathfrak{R}(U)$ is tangent to the fibred holonomy group $\mathsf{Hol}_\mathsf{f}(U)$.*

Lemma 4.6 *If $\xi \in \mathfrak{X}^\infty(\mathcal{I}U)$ is strongly tangent to the fibred holonomy group $\mathsf{Hol}_\mathsf{f}(U)$ of $(U, \mathcal{F}|_U)$ then its horizontal covariant derivative $\nabla_X\xi$ with respect to any vector field $X \in \mathfrak{X}^\infty(U)$ is also strongly tangent to $\mathsf{Hol}_\mathsf{f}(U)$. Moreover, if U is diffeomorphic to $\mathbb{R}^n$ then its infinitesimal holonomy algebra $\mathfrak{hol}^*(U)$ is tangent to the fibred holonomy group $\mathsf{Hol}_\mathsf{f}(U)$.*

Proof. Let τ be the (nonlinear) parallel translation along the flow φ of the vector field X, i.e. for every $x \in U$ and $t \in (-\varepsilon_x, \varepsilon_x)$ the map $\tau_t(x) \colon T_xU \to T_{\varphi_t(x)}U$ is the (nonlinear) parallel translation along the integral curve of X. If $\{\Phi_{(t_1,\dots,t_k)}\}_{t_i\in(-\varepsilon,\varepsilon)}$ is a $\mathcal{C}^\infty$-differentiable k-parameter family $\{\Phi_{(t_1,\dots,t_k)}\}_{t_i\in(-\varepsilon,\varepsilon)}$ of fibre preserving diffeomorphisms of the indicatrix bundle $(\mathcal{I}U, \pi|_U, U)$ associated to the strongly tangent vector fields ξ satisfying the conditions 1. and 2. on page 7, then the commutator

$$[\Phi_{(t_1,\dots,t_k)}, \tau_{t_{k+1}}] := \Phi^{-1}_{(t_1,\dots,t_k)} \circ \tau^{-1}_{t_{k+1}} \circ \Phi_{(t_1,\dots,t_k)} \circ \tau_{t_{k+1}}$$

in the group $\mathsf{Diff}^\infty\big(\mathcal{I}U\big)$ fulfills $[\Phi_{(t_1,\dots,t_k)}, \tau_{t_{k+1}}] = \mathsf{Id}$, if some of its variables equals 0. Moreover

$$\frac{\partial^{k+1}[\Phi_{(t_1\dots t_k)}, \tau_{(t_{k+1})}]}{\partial t_1 \dots \partial t_{k+1}}\Bigg|_{(0\dots0)} = \big[X^h, \xi\big] \tag{13}$$

at any point of U, which shows that the vector field $\left[X^h, \xi\right]$ is strongly tangent to $\mathsf{Hol}_{\mathfrak{f}}(U)$. Moreover, since the vector field ξ is vertical, we have $h[X^h, \xi] = 0$, and using (4) we obtain

$$[X^h, \xi] = v[X^h, \xi] = \nabla_X \xi$$

which yields to the first part of the assertion. Moreover, with Lemma 4.4 we obtain that the generating elements of $\mathfrak{hol}^*(U)$ are strongly tangent to $\mathsf{Hol}_{\mathfrak{f}}(U)$, therefore $\mathfrak{hol}^*(U)$ is tangent $\mathsf{Hol}_{\mathfrak{f}}(U)$. ∎

Proof of Theorem 4.3. Assertion (i) follows from the previous lemma. Assertion (ii) is a consequence of Proposition 4.2. ∎

5 Holonomy algebras of Finsler surfaces

We call a Finsler manifold $(M, \mathcal{F})$ Fisler surface if $\dim M = 2$. Let M be diffeomorphic to $\mathbb{R}^2$ and let $X, Y \in \mathfrak{X}^\infty(M)$ be nonvanishing vector fields such that for any $x \in M$ the values $X_x, Y_x \in T_xM$ are linarly independent. We denote by $\xi \in \mathfrak{X}(\mathcal{I}M)$ the curvature vector field $\xi = r(X, Y)(x, y) = R_{(x,y)}(X_x, Y_x)$. Since the indicatrix is 1-dimensional, any curvature vector field η can be written as $\eta_x = f(x)\xi_x$, where the factor $f(x)$ is arbitrary smooth function on M. Therefore the commutators of curvature vector fields are trivial and the curvature algebra $\mathfrak{R}(M)$ is commutative. Particularly, the curvature algebra $\mathfrak{R}_x(M)$ at any point $x \in M$ is generated by the vector field $\xi_x \in \mathfrak{X}(\mathcal{I}_xM)$ and hence it is at most 1-dimensional.
Even in this case, the infinitesimal holonomy algebra $\mathfrak{hol}^*(x)$ at a point $x \in M$ (and hence the corresponding holonomy group $\mathsf{Hol}_x(M)$) can be higher – even infinite – dimensional. To show this we use a classical result of S. Lie on the classification of Lie group actions on one-manifolds (cf. [1] or [7], pp. 58-62):

If a finite-dimensional connected Lie group acts on a 1-dimensional manifold without fixed points, than its dimension is less than 4.

Proposition 5.1 *If the infinitesimal holonomy algebra $\mathfrak{hol}^*(x)$ contains 4 simultanously non-vanishing $\mathbb{R}$-linearly independent vector fields, then the holonomy group $\mathsf{Hol}_M(x)$ is not a finite-dimensional Lie group.*

Proof. Indeed, in this case the holonomy group acts on the 1-dimensional indicatrix without fixed points. If it would be finite-dimensional then its dimension, and hence the dimension of its Lie algebra should be less than 4. This is a contradiction. ∎

5.1 Finsler surfaces of constant flag curvature

The relation between the infinitesimal holonomy algebra and the curvature algebra is enlightened by the following

Theorem 5.2 *Let $(M,\mathcal{F})$ be a Fisler surface with non-zero constant flag curvature. The infinitesimal holonomy algebra $\mathfrak{hol}^*(x)$ at a point $x \in M$ coincides with the curvature algebra $\mathfrak{R}_x$ at x if and only if the mean Berwald curvature $E_{(x,y)}$ of $(M,\mathcal{F})$ vanishes for any $y \in \mathcal{I}_x M$.*

Proof. Let $U \subset M$ be a neighbourhood of $x \in M$ diffeomorphic to $\mathbb{R}^2$. Identifying U with $\mathbb{R}^2$ and considering a coordinate system (x_1, x_2) in $\mathbb{R}^2$ we can write

$$R^i_{jk}(x,y) = \lambda \left(\delta^i_j g_{km}(x,y) y^m - \delta^i_k g_{jm}(x,y) y^m\right), \quad \text{with} \quad \lambda \neq 0.$$

Since the curvature tensor field is skew-symmetric, $R_{(x,y)}$ acts on the one-dimensional wedge product $T_xM \wedge T_xM$. According to Lemma 2.1 the covariant derivative of the curvature vector field $\xi = R(X,Y) = \frac{1}{2}R(X \otimes Y - Y \otimes X) = R(X \wedge Y)$ can be written in the form

$$\nabla_Z \xi = \nabla_Z \left(r(X,Y)\right) = R\left(\nabla_Z (X \wedge Y)\right) = R(\nabla_Z X \wedge Y + X \wedge \nabla_Z Y),$$

where $X, Y, Z \in \mathfrak{X}(U)$. If $X = X^i \frac{\partial}{\partial x^i}$, $Y = Y^i \frac{\partial}{\partial x^i}$ and $Z = Z^i \frac{\partial}{\partial x^i}$ then we have $X \wedge Y = \frac{1}{2}\left(X^1 Y^2 - X^2 Y^1\right) \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}$ and hence we obtain

$$\nabla_Z \xi = R\left(\nabla_k \left((X^1Y^2 - Y^1X^2)\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}\right) Z^k\right) = \tag{14}$$

$$= R\left(\frac{\partial (X^1Y^2 - Y^1X^2)}{\partial x^k} Z^k \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}\right) + (X^1Y^2 - Y^1X^2) R\left(\nabla_k \left(\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}\right)\right) Z^k,$$

where we denote the covariant derivative ∇_Z by ∇_k if $Z = \frac{\partial}{\partial x^k}$, $k = 1,2$. For given vector fields $X, Y, Z \in \mathfrak{X}^\infty(U)$ the expression $\frac{\partial(X^1Y^2 - Y^1X^2)}{\partial x^k} Z^k$ is a function on U. Hence there exists a function ψ on U such that

$$R\left(\frac{\partial (X^jY^h - Y^jX^h)}{\partial x^k} Z^k \frac{\partial}{\partial x^j} \wedge \frac{\partial}{\partial x^h}\right) = \psi\, R(X \wedge Y) = \psi\, R(X,Y),$$

and $\psi R(X,Y)$ is an element of the curvature algebra $\mathfrak{R}(U)$ of the submanifold $(U, \mathcal{F}|_U)$.

Now, we investigate the second term of the right hand side of (14).

$$\nabla_k \Big(\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}\Big) = \Big(\nabla_k \frac{\partial}{\partial x^1}\Big) \wedge \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^1} \wedge \Big(\nabla_k \frac{\partial}{\partial x^2}\Big) =$$

$$= G^l_{k1} \frac{\partial}{\partial x^l} \wedge \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^1} \wedge G^m_{k2} \frac{\partial}{\partial x^m} = \left(G^1_{k1} + G^2_{k2}\right) \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}.$$

Hence

$$(X^1Y^2 - Y^1X^2) R\Big(\nabla_k \Big(\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}\Big)\Big) Z^k = \left(G^1_{k1} + G^2_{k2}\right) Z^k R(X,Y) = \left(G^1_{k1} + G^2_{k2}\right) Z^k \xi$$

This expression belongs to the curvature algebra if and only if the function $G^1_{k1} + G^2_{k2}$ does not depend on the variable y, i.e. if and only if

$$E_{kh} = \frac{\partial \left(G^1_{k1} + G^2_{k2}\right)}{\partial y^h} = 0, \quad h, k = 1, 2,$$

identically. ■

Remark 5.3 Let $\xi = R(X, Y)$ be a curvature vector field. Assume that the vector fields $X, Y \in \mathfrak{X}^\infty(M)$ have constant coordinate functions in a local coordinate system $(x^1, ..., x^n)$ of the Finsler surface $(M, \mathcal{F})$. Then we have in this coordinate system

$$\nabla_Z \xi = \left(G^1_{k1} + G^2_{k2}\right) Z^k \xi.$$

5.2 Randers surfaces with $\mathfrak{hol}^*(x) = \mathfrak{R}_x$

A Fisler manifold $(M, \mathcal{F})$ is called *Randers manifold* if its Finsler function has the form $\mathcal{F} = \alpha + \beta$, where $\alpha = \sqrt{\alpha_{jk}(x) y^j y^k}$ is a Riemannian metric and $\beta = \beta_j(x) y^j$ is a linear form. Z. Shen constructed in [11] families of Randers surfaces depending on the real parameter ϵ, which are of constant flag curvature 1 on the unit sphere $S^2 \subset \mathbb{R}^3$ and of constant flag curvature -1 on a disk $\mathbb{D}^2 \subset \mathbb{R}^2$. These Finsler surfaces are not projectively flat and have vanishing S-curvature (c.f. [11], Theorems 1.1 and 1.2). Their Finsler function is defined by

$$\alpha = \frac{\sqrt{\epsilon^2 h(v, y)^2 + h(y, y)\left(1 - \epsilon^2 h(v, v)\right)}}{1 - \epsilon^2 h(v, v)}, \quad \beta = \frac{\epsilon h(v, y)}{1 - \epsilon^2 h(v, v)}, \tag{15}$$

where $h(v, y)$ is the standard metric of the sphere S^2, respectively $h(v, y)$ is the standard Klein metric on the unit disk $\mathbb{D}^2$ and v denotes the vector field defined by $(-x_2, x_1, 0)$ at $(x_1, x_2, x_3) \in S^2$, respectively by $(-x_2, x_1)$ at $(x_1, x_2) \in \mathbb{D}^2$.

Theorem 5.4 *For any Randers surface defined by (15) the infinitesimal holonomy algebra $\mathfrak{hol}^*(x)$ at a point $x \in M$ coincides with the curvature algebra $\mathfrak{R}_x$.*

Proof. According to Theorem 1.1 and 1.2 in [11], the above classes of not locally projectively flat Randers surfaces with non-zero constant flag curvature have vanishing S-curvature. Moreover, Proposition 6.1.3 in [10], p. 80, states that the mean Berwald curvature vanishes if and only if the S-curvature is a linear form on the surface. Hence the assertion follows from Corollary 5.2. ■

5.3 Randers surfaces with infinite dimensional $\mathfrak{hol}^*(x)$

Projectively flat Randers manifolds with constant flag curvature were classified by Z. Shen in [12]. He proved that any projectively flat Randers manifold with non-zero constant flag curvature has negative curvature. This metric can be normalized by a constant factor so that the curvature is $-\frac{1}{4}$. In this case it is isometric to the Randers manifold $(M, \mathcal{F})$ defined by $\mathcal{F} = \alpha + \beta$ on the unit ball $\mathbb{D}^2 \subset \mathbb{R}^2$, where

$$\alpha = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 - |x|^2}, \quad \beta = \frac{\langle x, y\rangle}{1 - |x|^2} + \frac{\langle a, y\rangle}{1 - \langle a, x\rangle} \tag{16}$$

and $a \in \mathbb{R}^2$ is any constant vector with $|a| < 1$.

Theorem 5.5 *Let $(M, \mathcal{F})$ be a locally projectively flat Randers surface of non-zero constant flag curvature. There exists a point $x \in M$ such that the infinitesimal holonomy algebra $\mathfrak{hol}^*(x)$ is an infinite dimensional Lie algebra and hence the holonomy group of $(M, \mathcal{F})$ is not a finite-dimensional Lie group.*

Proof. Since the metrics $\mathcal{F} = \alpha + \beta$ defined on the unit ball $\mathbb{D}^2 \subset \mathbb{R}^2$ by (16) are projectively flat, the geodesic coefficients (1) are of the form $G^i(x, y) = P(x, y)y^i$, and hence

$$G^i_k = \frac{\partial P}{\partial y^k} y^i + P\delta^i_k, \quad G^i_{kl} = \frac{\partial^2 P}{\partial y^k \partial y^l} y^i + \frac{\partial P}{\partial y^k}\delta^i_l + \frac{\partial P}{\partial y^l}\delta^i_k, \quad G^m_{km} = (n+1)\frac{\partial P}{\partial y^k}.$$

Let us choose $x = 0 \in \mathbb{D}^2 \subset \mathbb{R}^2$. According to [13], eq. (41) and (42), pp. 1722-1723, the function $P(x, y)$ has the form

$$P(x, y) = \frac{1}{2}\left\{\frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)} + \langle x, y\rangle}{1 - |x|^2} - \frac{\langle a, y\rangle}{1 - \langle a, x\rangle}\right\}. \tag{17}$$

Using Remark 5.3 we obtain

$$\nabla_Y \xi = G^m_{km} Y^k \xi = 3\frac{\partial P}{\partial y^k} Y^k \xi. \tag{18}$$

Hence

$$\nabla_X (\nabla_Y \xi) = 3\nabla_X \left(\frac{\partial P}{\partial y^k} Y^k \xi\right) = 3\left\{\nabla_X \left(\frac{\partial P}{\partial y^k} Y^k\right)\xi + \left(\frac{\partial P}{\partial y^k} Y^k\right)\left(\frac{\partial P}{\partial y^l} X^l\right)\right\}\xi.$$

Assume that the vector field Y has constant coordinate functions. Then we can write

$$\nabla_X \left(\frac{\partial P}{\partial y^k} Y^k\right) = \left(\frac{\partial^2 P}{\partial x^j \partial y^k} - G^k_j \frac{\partial^2 P}{\partial y^k \partial y^k}\right) Y^k X^j =$$

$$= \left(\frac{\partial^2 P}{\partial x^j \partial y^k} - \left(\frac{\partial P}{\partial y^j} y^m + P\delta^m_j\right)\frac{\partial^2 P}{\partial y^k \partial y^m}\right) Y^k X^j = \left(\frac{\partial^2 P}{\partial x^j \partial y^k} - P\frac{\partial^2 P}{\partial y^k \partial y^j}\right) Y^k X^j.$$

It follows that

$$\nabla_X(\nabla_Y\xi) = 3\left\{\frac{\partial^2 P}{\partial x^j\partial y^k} - P\frac{\partial^2 P}{\partial y^k\partial y^j} + \frac{\partial P}{\partial y^k}\frac{\partial P}{\partial y^l}\right\} Y^k X^l \xi. \tag{19}$$

We want to prove that the vector fields $\xi\big|_{x=0}$, $\nabla_1\xi|_{x=0}$, $\nabla_2\xi|_{x=0}$ and $\nabla_1\left(\nabla_2\xi\right)|_{x=0}$ are linearly independent. We obtain from equations (18) and (19) that it is sufficient to show that the functions

$$1, \quad \frac{\partial P}{\partial y^1}\Big|_{x=0}, \quad \frac{\partial P}{\partial y^2}\Big|_{x=0} \quad \text{and} \quad \left(\frac{\partial^2 P}{\partial x^1\partial y^2} - P\frac{\partial^2 P}{\partial y^1\partial y^2} + \frac{\partial P}{\partial y^1}\frac{\partial P}{\partial y^2}\right)\Big|_{x=0} \tag{20}$$

are linearly independent. We obtain from (17) that

$$P\Big|_{x=0} = \frac{1}{2}\left(|y| - \langle a, y\rangle\right), \quad \frac{\partial P}{\partial y^k}\Big|_{x=0} = \frac{1}{2}\left(\frac{y^k}{|y|} - a^k\right), \quad \frac{\partial^2 P}{\partial y^j\partial y^k}\Big|_{x=0} = \frac{1}{2|y|}\left(\delta^j_k - \frac{y^j y^k}{|y|^2}\right)$$

and

$$\frac{\partial P}{\partial x^j}\Big|_{x=0} = \frac{1}{2}\left(y^j - \langle a, y\rangle a^j\right), \qquad \frac{\partial^2 P}{\partial x^j\partial y^k}\Big|_{x=0} = \frac{1}{2}\left(\delta^j_k - a^j a^k\right).$$

Putting

$$\cos t = \frac{y^1}{|y|}, \quad \sin t = \frac{y^2}{|y|}$$

and omitting the constant terms from the last three functions we obtain that the functions (20) are independent if and only if the functions

$$1, \quad \cos t, \quad \sin t, \quad \cos t \sin t\left(1 - a^1\cos t - a^2\sin t\right) + \left(\cos t - a^1\right)\left(\sin t - a^2\right),$$

or equivalently, the functions 1, $\cos t$, $\sin t$, $\sin 2t(2 - a^1\cos t - a^2\sin t)$ are linearly independent. Clearly, this is the case and hence the infinitesimal holonomy algebra contains 4 linearly independent vector fields. It follows from Proposition 5.1 that $\mathfrak{hol}^*(x)$ is infinite-dimensional and hence the holonomy group $\mathsf{Hol}_M(x)$ is not a finite-dimensional Lie group. ■

6 The holonomy group of the Funk surface

Definition 6.1 A Randers surface $(M, \mathcal{F})$ is called *Funk surface* if its Finsler function is defined by

$$\mathcal{F}(x,y) = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)} + \langle x, y\rangle}{1 - |x|^2} \tag{21}$$

on the unit disk $\mathbb{D}^2 = \{x \in \mathbb{R}^2, |x| < 1\}$.

We remark, that (21) can be obtained from (16) putting $a = 0$. The Funk surface is a projectively flat Finsler surface of constant flag curvature $-\frac{1}{4}$.

We recall that the Lie algebra $\mathfrak{X}^\infty(S^1)$ of smooth vector fields on the circe contains a dense subalgebra known as the *real Witt algebra* ([4], p.164). It consists of vector fields with finite Fourier series, and hence it is linearly generated by the vector fields

$$\cos nt\,\frac{\partial}{\partial t},\quad \sin nt\,\frac{\partial}{\partial t},\quad n=0,1,2,\dots$$

For the Funk surface the indicatrix $\mathcal{I}_0 M$ at $x=0\in\mathbb{D}^2$ is the unit circle $S^1\subset T_0\mathbb{D}^2$ and we have the following

Theorem 6.2 *The infinitesimal holonomy algebra of the Funk surface* $(\mathbb{D}^2,\mathcal{F})$ *at* $0\in\mathbb{D}^2$ *contains the real Witt algebra.*

Proof. Let us consider the curvature vector field

$$\xi = R\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2}\right)\Big|_{x=0} = -\frac{1}{4}\left(\delta^i_2 g_{1m}(0,y)y^m - \delta^i_1 g_{2m}(0,y)y^m\right)$$

Since $\mathcal{F}|_{x=0}=|y|$, we have $g_{jm}(0,y)y^m = y^j$ and hence $\xi=\frac{1}{4}\begin{bmatrix} y^2\\ -y^1\end{bmatrix}$. According to (18) the first covariant derivatives are:

$$\nabla_k\xi = 3\frac{\partial P}{\partial y^k}\Big|_{x=0}\xi = \frac{3}{2}\frac{y^k}{|y|}\xi.$$

Since $\nabla_1\xi$ and $\nabla_2\xi$ are not constant multiples of the vector field ξ, they are not elements of the one-dimensional curvature algebra $\mathfrak{R}_0=\{c\cdot\xi \mid c\in\mathbb{R}\}$. Let us introduce polar coordinates $y^1=r\cos t$, $y^2=r\sin t$ in the tangent space $T_0\mathbb{D}^2$, then we can express the curvature vector field and its first covariant derivatives at $x=0$ by

$$\xi=-\frac{1}{4}\frac{\partial}{\partial t},\qquad \nabla_1\xi=-\frac{3}{8}\cos t\frac{\partial}{\partial t},\qquad \nabla_2\xi=-\frac{3}{8}\sin t\frac{\partial}{\partial t}.$$

Hence the vector fields

$$\frac{\partial}{\partial t}=-4\xi,\quad \cos t\frac{\partial}{\partial t}=-\frac{8}{3}\nabla_1\xi\quad \sin t\frac{\partial}{\partial t}=-\frac{8}{3}\nabla_2\xi$$

are elements of the infinitesimal holonomy algebra $\mathfrak{hol}^*(0)$.

Similarly, the second covariant derivatives of the curvature vector field are

$$\nabla_1\nabla_1\xi=-\tfrac{3}{16}(4\cos^2 t+1)\frac{\partial}{\partial t},\quad \nabla_2\nabla_2\xi=-\tfrac{3}{16}(4\sin^2 t+1)\frac{\partial}{\partial t}\tag{22}$$

and

$$\nabla_1\nabla_2\xi=\nabla_2\nabla_1\xi=-\frac{3}{4}\cos t\sin t\frac{\partial}{\partial t}.$$

Since $\nabla_1\nabla_1\xi - \nabla_2\nabla_2\xi$ and $\nabla_1\nabla_2\xi$ belong to $\mathfrak{hol}^*(0)$, we obtain that $\cos 2t\frac{\partial}{\partial t}$ and $\sin 2t\frac{\partial}{\partial t}$ are also elements of $\mathfrak{hol}^*(0)$.

Let us suppose now that for $k\in\mathbb{Z}$ the vector fields $\cos kt\frac{\partial}{\partial t}$, $\sin kt\frac{\partial}{\partial t}$ belong to $\mathfrak{hol}^*(0)$. We compute

$$\Big[\cos t\frac{\partial}{\partial t}, \cos kt\frac{\partial}{\partial t}\Big] - \Big[\sin t\frac{\partial}{\partial t}, \sin kt\frac{\partial}{\partial t}\Big] = -(k-1)\sin(k+1)t\frac{\partial}{\partial t}$$

and

$$\Big[\cos t\frac{\partial}{\partial t}, \sin kt\frac{\partial}{\partial t}\Big] + \Big[\sin t\frac{\partial}{\partial t}, \cos kt\frac{\partial}{\partial t}\Big] = -(k+1)\cos(k+1)t\frac{\partial}{\partial t},$$

hence we obtain that $\cos(k+1)t\frac{\partial}{\partial t}$ and $\sin(k+1)t\frac{\partial}{\partial t}$ are also elements of $\mathfrak{hol}^*(0)$, which proves the assertion. ■

Theorem 6.3 *The topological closure of the holonomy group of the Funk surface is the orientation preserving diffeomorphism group* $\mathsf{Diff}^\infty_+(S^1)$.

Proof. Since the Funk surface is simply connected, the elements of the holonomy group are orientation preserving diffeomorphisms of the circle. Hence the topological closure of the holonomy group $\overline{\mathsf{Hol}(0)}$ is contained in $\mathsf{Diff}^\infty_+(S^1)$. From the other hand $\overline{\mathsf{Hol}(0)}$ contains the exponential image of the real Witt algebra. The exponential mapping is continuous (c.f. Lemma 4.1 in [9], p. 79) and the real Witt algebra is dense in the Lie algebra of $\mathsf{Diff}^\infty(S^1)$, hence $\overline{\mathsf{Hol}(0)}$ contains the normal subgroup generated the exponential image of the Lie algebra of $\mathsf{Diff}^\infty(S^1)$. Since $\mathsf{Diff}^\infty_+(S^1)$ is a simple group (cf. [5], Corollaire 2.) we get $\overline{\mathsf{Hol}(0)} = \mathsf{Diff}^\infty_+(S^1)$. ■

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