ON BOUNDEDNESS

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Abstract. A subset A of $\Re$, the set of real numbers, is bounded if $|a| \leq M$ for all $a \in A$ where M is a positive real constant number. This is equivalent to the statement that any sequence of points in A has a Cauchy subsequence. It is proved that a subset A of $\Re$ is bounded if and only if any sequence of points in A has a subsequence which is any type of the following, quasi-Cauchy, statistically quasi-Cauchy, lacunary statistically quasi-Cauchy, slowly oscillating. It turns out that a function on a subset A of $\Re$ is uniformly continuous if and only if it preserves either quasi-Cauchy sequences or slowly oscillating sequences.

1. Introduction

The concept of boundedness and continuity; and any concept involving boundedness and continuity play a very important role not only in pure mathematics but also in other branches of sciences involving mathematics especially in computer sciences, biological sciences, and dynamical systems. The Bolzano-Weierstrass theorem is a fundamental result about convergence in $\Re$, the set of real numbers. The theorem states that each bounded sequence in $\Re$ has a convergent subsequence. An equivalent statement is that a subset of $\Re$ is sequentially compact if and only if it is closed and bounded. If we omit the term "closed" and take only the term "bounded", it is the case that a subset of $\Re$ is bounded if and only if any sequence of points in A has a Cauchy subsequence.

The purpose of this paper is to present various characterizations of a bounded subset of $\Re$, and two characterizations of uniform continuity in terms of sequences.

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2. Definitions and Notation

A subset of $\Re$ is compact if and only if it is closed and bounded. A is called bounded if $|a| \leq M$ for all $a \in A$ where M is a positive real constant number. This is equivalent to the statement that any sequence of points in A has a Cauchy subsequence. The concept of a Cauchy sequence involves far more than that the distance between successive terms is tending to zero. Nevertheless, sequences which satisfy this weaker property are interesting in their own right. These sequences are named as quasi-Cauchy by Burton and Coleman [1], while those sequences were called as forward convergent to zero sequences in both [2], and [3]. As it seems more suitable, we also call them quasi-Cauchy. Explicitly, A sequence (a_n) is called quasi-Cauchy if $lim \Delta a_n = 0$ where Δa_n is either forward or backward difference operator, i.e. , either $\Delta a_n = a_{n+1} - a_n$ or $\Delta a_n = a_n - a_{n+1}$. Recently, some further results on Quasi-Cauchy sequences are obtained in [4], and [5].

A sequence (x_n) of points in $\Re$ is called slowly oscillating if for any given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ and $N = N(\varepsilon)$ such that $|a_m - a_n| < \varepsilon$ if $n \geq N(\varepsilon)$ and $n \leq m \leq (1+\delta)n$ ([6]). Any Cauchy sequence is slowly oscillating, and any slowly oscillating sequence is quasi-Cauchy. But the converses are not always true. For example, the sequences $(\sum_{k=1}^{\infty} \frac{1}{n})$, (ln n), (ln ln n), and combinations like that are slowly oscillating, but Cauchy (see also [7]). The sequence $(\sum_{k=1}^{k=n} (\frac{1}{k})(\sum_{j=1}^{j=k} \frac{1}{j}))$ is quasi-Cauchy, but slowly oscillating ([8], and [9]).

A sequence (a_k) of points in $\Re$ is called to be statistically convergent to an element ℓ of $\Re$ if for each ε

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |a_k - \ell| \geq \varepsilon\}| = 0,$$

and this is denoted by $st - \lim_{n\to\infty} a_n = \ell$ ([10]). We call a sequence (a_n) of points in $\Re$ statistically quasi-Cauchy if $st - \lim_{n\to\infty} \Delta a_n = 0$.

A sequence (a_k) of points in $\Re$ is called lacunary statistically convergent to an element ℓ of $\Re$ if

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |a_k - \ell| \geq \varepsilon\}| = 0,$$

for every $\varepsilon > 0$ where $I_r = (k_{r-1}, k_r]$ and $k_0 = 0$, $h_r : k_r - k_{r-1} \to \infty$ as $r \to \infty$ and $\theta = (k_r)$ is an increasing sequence of positive integers, and $\liminf_r \frac{k_r}{k_{r-1}} > 1$, and this is denoted by $S_\theta - \lim_{n\to\infty} a_n = \ell$ ([11]). We call a sequence (a_n) of points in $\Re$ lacunary statistically quasi-Cauchy if $S_\theta - \lim_{n\to\infty} \Delta a_n = 0$.

3. Characterizations of Boundedness

In this section, we give a further investigation of quasi-Cauchy sequences, and slowly oscillating sequences; and obtain some more characterizations of boundedness of a subset of $\Re$ by using the concepts of a quasi-Cauchy sequence, a slowly oscillating sequence, a statistically quasi-Cauchy sequence, and a lacunary statistically quasi-Cauchy sequence of points in $\Re$.

Trivially, Cauchy sequences are slowly oscillating. It is easy to see that any slowly oscillating sequence is quasi-Cauchy. Therefore Cauchy sequences are quasi-Cauchy. There are quasi-Cauchy sequences which are not Cauchy. For example, the sequence $(\sqrt{n})$ is quasi-Cauchy, but Cauchy. Any subsequence of a Cauchy sequence is Cauchy. The analogous property fails for not only quasi-Cauchy sequences, but also slowly oscillating sequences. A counterexample for the case, quasi-Cauchy, is again the sequence $(a_n) = (\sqrt{n})$ with the subsequence $(a_{n^2}) = (n)$. A counterexample for the case slowly oscillating is the sequence (lnn) with the subsequence (n).

Now we introduce the definitions of statistically ward compactness, lacunary statistically ward compactness of a subset of $\Re$, and recall the definitions of ward compactness, and slowly oscillating compactness.

Definition 1. (i) A subset A of $\Re$ is called ward compact if whenever $\mathbf{a} = (a_n)$ is a sequence of points in A there is a quasi-Cauchy subsequence $\mathbf{z} = (z_k) = (a_{n_k})$ of $\mathbf{a}$.
(ii) A subset A of $\Re$ is called statistically ward compact if whenever $\mathbf{a} = (a_n)$ is a sequence of points in A there is a statistically quasi-Cauchy subsequence $\mathbf{z} = (z_k) = (a_{n_k})$ of $\mathbf{a}$.
(iii) A subset A of $\Re$ is called lacunary statistically ward compact if whenever $\mathbf{a} = (a_n)$ is a sequence of points in A there is a lacunary statistically quasi-Cauchy subsequence $\mathbf{z} = (z_k) = (a_{n_k})$ of $\mathbf{a}$.
(iv) A subset A of $\Re$ is called slowly oscillating compact if whenever $\mathbf{a} = (a_n)$ is a sequence of points in A there is a slowly oscillating subsequence $\mathbf{z} = (z_k) = (a_{n_k})$ of $\mathbf{a}$.

Now we give an important result, which enables us to have certain characterizations of boundedness of a subset of $\Re$, in the following.

Theorem 1. Let A be a subset of $\Re$. The following statements are equivalent.
(a) A is bounded.
(b) A is ward compact.
(c) A is slowly oscillating compact.
(d) A is statistically ward compact.
(e) A is lacunary statistically ward compact.

Proof. It is an easy exercise to check that bounded subsets of $\Re$ are ward compact. Thus (a) implies (b). To prove that (b) implies (a), suppose that a subset A of $\Re$ is unbounded. If it is unbounded above, then one can construct a sequence (a_n) of terms in A such that $a_{n+1} > 1 + a_n$ for each positive integer n. Then the set of the terms of the sequence (a_n) is not ward compact. If A is unbounded below, then write $F = -E$ and apply the above result. Hence the proof that (b) implies (a) is completed. If A is slowly oscillating compact, then any sequence (a_n) of terms in A has a slowly oscillating subsequence which is quasi-Cauchy, since any slowly oscillating sequence is quasi-Cauchy. Therefore A is ward compact. If A is bounded, then any sequence has a Cauchy subsequence which is also slowly oscillating. Hence (a) implies (c). Since any slowly oscillating sequence is quasi-Cauchy, it follows that (c) implies (b). If A is ward compact, then any sequence (a_n) of terms in A has a quasi-Cauchy subsequence, which is also statistically quasi-Cauchy. Thus (b) implies (d). If A is statistically ward compact, then any sequence (a_n) of terms in A has a statistically quasi-Cauchy subsequence, (a_{k_n}) i.e. $st - \lim_{n\to\infty} \Delta a_{k_n} = 0$. Since any statistically convergent sequence with limit ℓ has a convergent subsequence with the same limit ℓ in the ordinary sense, there exists a subsequence (z_j) of the sequence (Δa_{k_n}) such that $\lim_{j\to\infty} z_j = \lim_{n\to\infty} \Delta a_{k_{n_j}} = 0$. This means that the subsequence $(a_{k_{n_j}})$ is quasi-Cauchy. So we get that (d) implies (b). Since any convergent sequence is lacunary statistically convergent with tha same limit, it follows that (b) implies (e). Now let A be lacunary statistically ward compact. Then any sequence (a_n) of terms in A has a lacunary statistically quasi-Cauchy subsequence, i.e. $S_\theta - \lim_{n\to\infty} \Delta a_{k_n} = 0$. Since any lacunary statistically convergent sequence with limit ℓ has a convergent subsequence with the same limit ℓ in the ordinary sense, there exists a subsequence (z_m) of the sequence (Δa_{k_n}) such that $\lim_{n\to\infty} z_m = 0$. This means that the subsequence we have obtained,$(a_{k_{n_m}})$, is quasi-Cauchy. So we get that (e) implies (b). Finally, if A is ward compact, then any sequence (a_n) of terms in A has a quasi-Cauchy subsequence. Since any convergent sequence is also lacunary statistically convergent, it is lacunary statistically quasi-Cauchy. This completes the proof of the theorem.

A sequence $\mathbf{a} = (a_n)$ is δ-quasi-Cauchy if $\lim_{k\to\infty} \Delta^2 a_n = 0$ where $\Delta^2 a_n = a_{n+2} - 2a_{n+1} + a_n$ ([12]). A subset A of $\Re$ is called δ-ward compact if whenever $\mathbf{a} = (a_n)$ is a sequence of points in A there is a subsequence $\mathbf{z} = (z_k) = (a_{n_k})$ of $\mathbf{a}$ with $\lim_{k\to\infty} \Delta^2 z_k = 0$. We note that any ward compact subset of $\Re$ is δ-ward compact.

We see that for any regular subsequential method G defined on $\Re$, if a subset A of $\Re$ is G-sequentially compact, then any one of the conditions of Theorem 1 is satisfied (see [13] for the definition of G-sequentially compactness). But the converse is not always true.

4. Characterizations of uniform continuity

A real function f is continuous if and only if, for each point x_0 in the domain, $\lim_{n\to\infty} f(x_n) = f(x_0)$ whenever $\lim_{n\to\infty} x_n = x_0$. This is equivalent to the statement that $(f(x_n))$ is a convergent sequence whenever (x_n) is. This is also equivalent to the statement that $(f(x_n))$ is a Cauchy sequence whenever (x_n) is Cauchy provided that domain of the function is either whole $\Re$ or a bounded and closed subset of $\Re$. These well known results for continuity for real functions in terms of sequences suggested us to give a new type continuity, namely, ward continuity in [3].

The functions preserving quasi-Cauchy sequences are called forward continuous in [2]. In [3], it is proved that any ward continuous function on a ward compact subset A of $\Re$ is uniformly continuous. From Theorem 1 we see that a function defined on a bounded subset of $\Re$ is uniformly continuous if and only if it preserves quasi-Cauchy sequences of points in A. On the other hand, in [1], Burton and Coleman proved that a function defined on an interval is uniformly continuous if and only if it preserves quasi-Cauchy sequences. The functions preserving slowly oscillating sequences are called slowly oscillating continuous in [9]. In [8], it is proved that any slowly oscillating continuous function on a slowly oscillating compact subset A of $\Re$ is uniformly continuous. It follows from Theorem 1 that a function defined on a bounded subset of $\Re$ is uniformly continuous if and only if it preserves slowly oscillating sequences of points in A. On the other hand, in [7], Vallin also proved that a function is uniformly continuous if and only if it preserves slowly oscillating sequences for functions defined on a subset of $\Re$.

Theorem 2. Let A be a subset of $\Re$, and f be a function defined on A. Then the following statements are equivalent.

(i) f is uniformly continuous.

(ii) f is ward continuous.

(iii) f is slowly oscillating continuous.

Proof. It is Theorem 6 in [3], and [2] that any uniformly continuous function preserves quasi-Cauchy sequences. The converse was proved in [1] for the case that A is an interval and we see that the proof is valid if A is any subset of $\Re$. This completes the proof that (i) is equivalent to (ii). Since the sequence constructed for the contradiction in the proof that (ii) implies that (i) is not slowly oscillating as well, it follows that (iii) implies (i). This is also proved by Vallin in [7]. The implication (ii) implies (iii) is Theorem 2.4 in [9]. Thus the proof of the theorem is completed.

Corollary 1. If f preserves δ-quasi-Cauchy sequences of points in a subset A of $\Re$, then it is slowly oscillating continuous, and ward continuous on A.

Proof. The proof follows from Theorem 7 in [12] and Theorem 2 above.
We note that the converse is not true. □

Corollary 2. Let G be a regular subsequential method. If a function is uniformly continuous, then it is G-sequentially continuous (see [14]).

In this paper, two new concepts, namely a statistically quasi-Cauchy, and a lacunary statistically quasi-Cauchy sequence are introduced. We give four characterizations of a bounded subset of $\Re$, which seem to be very useful, and fruitful for further investigations. We also present two characterizations of uniform continuity of a function defined on a subset of $\Re$.

For further study, we suggest to investigate quasi Cauchy sequences of fuzzy points, and characterizations of ward continuity for the fuzzy functions. However due to the change in settings, the definitions and methods of proofs will not always be analogous to those of the present work (for example see [15]).

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