CUBICAL APPROXIMATION FOR DIRECTED TOPOLOGY

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ABSTRACT. Topological spaces - such as classifying spaces, configuration spaces and spacetimes - often admit extra temporal structure. Qualitative invariants on such directed spaces often are more informative yet more difficult to calculate than classical homotopy invariants on underlying spaces because directed spaces rarely decompose as homotopy colimits of simpler directed spaces. Directed spaces often arise as geometric realizations of simplicial sets and cubical sets equipped with temporal structure encoding the orientations of simplices and 1-cubes. In an attempt to develop calculational tools for directed homotopy theory, we prove appropriate simplicial and cubical approximation theorems. We consequently show that geometric realization induces an equivalence between weak homotopy diagram categories of cubical sets and directed spaces and that its right adjoint satisfies an excision theorem. Along the way, we give criteria for two different homotopy relations on directed maps in the literature to coincide.

CONTENTS

1. Overview

Spaces in nature often come equipped with temporal structure. Examples of such structure include smooth choices of future-oriented tangent vectors on spacetimes, orientations of morphisms in higher categories, and partial orders describing causal relationships in a state space. Certain qualitative characteristics of computational processes, spacetimes, and higher categories often arise as properties of spaces invariant under homotopy equivalences (=deformations) respecting extra temporal structure [8, 9, 28]. Such spaces often admit combinatorial descriptions as cubical sets and topological descriptions as "directed topological spaces"; a "directed" geometric realization functor translates from the combinatorial to the topological. Beyond a 1-dimensional cubical approximation theorem [7, Theorem 4.1] and a Seifert-van Kampen theorem for *fundamental categories* [14, Theorem 3.6], there do not exist tools in the literature for extracting information about the *directed homotopy type* of a directed space X from the combinatorics of a well-behaved diagram having colimit X. We cannot hope for a general and useful theory of homotopy colimits; maps almost never satisfy homotopy extension properties in our directed setting. Nonetheless, we show that geometric realization defines an equivalence between appropriate weak homotopy diagram categories of cubical sets and directed spaces.

We fix a working category of "directed spaces" in §4. Just as nearness spaces [16], proximity spaces [5], and topological spaces all models points equipped with spatial structure, various formalisms [8, 14, 20] model topological spaces equipped with some compatible temporal structure. A category $\mathscr{S}$ of *streams*, spaces equipped with "cosheaves" preordering their open subsets [20], suffices for our purposes: the category $\mathscr{S}$ is Cartesian closed [20, Theorem 5.13], the forgetful functor from $\mathscr{S}$ to the category $\mathscr{T}$ of compactly generated spaces creates limits and colimits [20, Proposition 5.8], and there exists an intuitive full and faithful embedding $\mathscr{P} \hookrightarrow \mathscr{S}$ from the category $\mathscr{P}$ of connected and compact Hausdorff topological lattices [Theorem 3.9].

We regard cubical sets and simplicial sets as combinatorial models of streams in §5, §6, and §7. Canonical lattice operations on the vertices of combinatorial simplices and combinatorial hypercubes linearly extend to continuous lattice operations on topological simplices and topological hypercubes. Thus classical geometric realizations of our combinatorial models naturally admit cosheaves of preorders encoding orientations of simplices and 1-cubes [Lemma 5.12]. Our resulting *stream realizations* send finite products of simplicial sets to finite products of streams [Lemma 5.11] and inclusions of cubical sets to closed embeddings of streams [Proposition 6.28]. Just as barycentric subdivisions respect classical geometric realization, ordinal subdivision [6] *sd* and a cubical analogue *cd* respect stream realization. Just as double barycentric subdivisions factor through polyhedral complexes [4], quadruple cubical subdivisions "locally factor" through cubical complexes [Lemmas 6.15 and 6.16]. *Triangulation* [18] relates our cubical and simplicial constructions [Propositions 7.4, and 7.5].

Cubical sets equipped with extra structure can model higher categories [2, 15]. A modification *ex* of the right adjoint to *cd* naturally fills in certain cubical analogues of simplicial horns. We might think of natural inclusions $C \hookrightarrow ex\, C$ and retractions $ex\, C \to C$ to such inclusions as defining the identities and *semistrict compositions* of higher morphisms. However, we do not explore connections with related models of higher categories [2, 15] in this paper.

We introduce homotopy theories of cubical sets, simplicial sets, and streams in §8. Definitions of simplicial homotopies and cubical homotopies are standard [4, 18]. Different definitions of homotopy between maps in directed topology as continuous parametrizations of stream maps by trivially [8] and totally [14] ordered unit intervals induce identical homotopy relations in the setting of compact *quadrangulable* streams [Theorem 8.13]. In classical homotopy theory, homotopy extension properties for inclusions of spheres into disks allow us to construct classical homotopies, such as cellular approximations of continuous maps between CW complexes, one cell at a time. We must instead construct directed homotopies, such as simplicial and cubical approximations [Theorem 8.27 and Corollary 8.28] of maps between stream realizations, all at once.

Homotopy equivalences of finite cubical sets admitting general *cubical compositions* (=semistrict compositions up to cubical homotopies) and compact quadrangulable streams generalize to *weak equivalences* of cubical sets in §8.4 and streams in §8.3. The category $\mathscr{S}$ admits the additional structure of a category of fibrant objects [Proposition 8.17] and hence a localization $\bar{h}\mathscr{S}$ with respect to its weak equivalences. Our main point is that the category $\hat{\square}$ of cubical sets admits an equivalent localization $\bar{h}\hat{\square}$ with respect to its weak equivalences [Corollary 8.33]. We formulate and prove our main observations in the more general setting of $\mathscr{G}$-shaped diagrams of cubical sets and $\mathscr{G}$-shaped diagrams of streams, for all small categories $\mathscr{G}$.

Corollary 6.21. For each small $\mathscr{G}$, $\upharpoonleft - \downharpoonright \dashv \mathit{sing}$ induces an equivalence

$$\bar{h}\hat{\square}^{\mathscr{G}} \leftrightarrows \bar{h}\mathscr{S}^{\mathscr{G}}.$$

Corollary 6.24 (Excision)**.** Fix $\mathscr{G}$. The natural $\mathscr{G}$-cubical function

$$\mathit{sing}\,U \cup_{\mathit{sing}\,U\cap V} \mathit{sing}\,V \to \mathit{sing}\,X$$

is a weak equivalence for all $\mathscr{G}$-streams U, V, X such that $U(d), V(d)$ are substreams of $X(d)$ whose interors in $X(d)$ cover $X(d)$ for all $\mathscr{G}$-objects d and $U(\gamma), V(\gamma)$ are restrictions and corestrictions of $X(\gamma)$ for all $\mathscr{G}$-morphisms γ.

2. Category theory

We fix some conventions and make some observations. Throughout the paper, we allow general "categories" to admit hom-sets of inaccessible cardinalities [29], although all of our localizations of diagram categories turn out to be locally small [Proposition 8.17 and Corollary 8.34].

2.1. **Conventions.**

We first fix some notation. We let k, m, m', n, n' denote natural numbers and $\mathscr{G}$, $\mathscr{C}$, and $\mathscr{D}$ denote categories. We occassionally regard functors $d : \mathscr{G} \to \mathscr{C}$ as "$\mathscr{G}$-equivariant $\mathscr{C}$-objects" and thus sometimes write $F(d)$ for the composites of such d with functors $F : \mathscr{C} \to \mathscr{D}$. We write: $\eta^{GF} : id_{\mathscr{C}} \to GF$ and $\epsilon^{FG} : FG \to id_{\mathscr{D}}$ for the unit and counit of an adjunction $F : \mathscr{C} \leftrightarrows \mathscr{D} : G$;

$$\int_{\mathscr{G}}^{d} F(-, d, d) : \mathscr{C} \to \mathscr{D}$$

for the parametrized coend [23] of a functor $F : \mathscr{C} \times \mathscr{G}^{\mathrm{op}} \times \mathscr{G} \to \mathscr{D}$ for cocomplete $\mathscr{D}$ and small $\mathscr{G}$. We write $\hat{\mathscr{G}}$ for the category of functors from $\mathscr{G}^{\mathrm{op}}$ to the category of sets and functions, for each small $\mathscr{G}$.

Example 2.1. Fix $\mathscr{G}$. The presheaf category

$$\hat{\mathscr{G}}$$

is complete, cocomplete, and Cartesian closed. Limits, colimits, subobjects, quotients, monos, and epis are just object-wise limits, colimits, subsets, quotient sets, injections, and surjections. All monos are regular. Filtered colimits commute with finite limits.

We write $\mathscr{G}[-] : \mathscr{G} \to \hat{\mathscr{G}}$ for the Yoneda embedding sending a $\mathscr{G}$-object to the corresponding representable presheaf, for each small $\mathscr{G}$.

Example 2.2. The Yoneda embedding defines a natural isomorphism

$$id_{\hat{\mathscr{G}}} \cong \int_{\mathscr{G}}^{d} -(d) \cdot \mathscr{G}[d] : \hat{\mathscr{G}} \to \hat{\mathscr{G}}.$$

We write $\mathscr{G} \downarrow C$ for the category whose objects are $\hat{\mathscr{G}}$-morphisms of the form $\mathscr{G}[*] \to C$ and whose morphisms are commutative triangles of the form

$$\begin{array}{ccc} \mathscr{G}[d] & \longrightarrow & \mathscr{G}[e] \\ & \searrow \quad \swarrow & \\ & C, & \end{array}$$

for each small $\mathscr{G}$ and $\hat{\mathscr{G}}$-object C.

Example 2.3. For each simplicial set C, $\Delta \downarrow C$ is its category of simplices.

We write $colim\ \eta$ for the morphism $colim\ X \to colim\ Y$ induced from a natural transformation $X \to Y$; $\varnothing$ for the initial object in a given category; $X \cdot g$ for the copower, indexed by a set X, of an object g; $a \hookrightarrow b$ for a mono whose definition is clear from context; $a \subset b$ to indicate that we are abusing notation and identifying an object a with an implicitly defined mono $a \hookrightarrow b$; $\gamma_{\restriction a} : a \to c$ for a composite morphism of the form $a \hookrightarrow b \xrightarrow{\gamma} c$; ω for the ordinal of natural numbers; and Δ_c for the diagonal morphism $id_c \times id_c : c \to c^2$ of an object c.

2.2. **Supports.** We will often want to talk about the "support of a point in a geometric realization," the "carrier of a cube with respect to a subdivision operation," or the "carrier of a simplex under triangulation." We therefore identify a parsimonious setting under which we can generalize "supports" and "carriers." We call a category $\bigcap$-*complete* if it is closed under intersections of subobjects and a functor $\bigcap$-*continuous* if it preserves intersections of subobjects (and in particular preserves monos.)

Example 2.4. Right adjoints are $\bigcap$-continuous.

Example 2.5. The Yoneda embedding $\mathscr{G} \to \hat{\mathscr{G}}$ is $\bigcap$-continuous, for each small $\mathscr{G}$.

For each morphism γ in a given $\bigcap$-complete category, we write $\gamma(a)$ for the *image* of a under γ, the minimal subobject of g through which the composite of inclusion $a \hookrightarrow b$ followed by γ factors, for each morphism $\gamma : b \to c$ and $a \subset b$ in a given $\bigcap$-complete category.. We write $supp_F(a, g)$ for the unique minimal subobject of g such that $F\, supp_F(a, g)$ contains $a \subset Fg$ for each $\bigcap$-complete $\mathscr{C}$, $\bigcap$-continuous functor F from $\mathscr{C}$, $\mathscr{C}$-object g, and $a \subset g$.

Example 2.6. In the case F is geometric realization or barycentric subdivision,

$$supp_F(a, g)$$

describes the "support" of a point (forming a singleton a) or the "carrier" of a simplex (generating a subobject a) in a simplicial set g.

We formalize the observation that $supp_F(a, g)$ is often "atomic" in size when a is "atomic" in size. A category is (*infinitarily*) *extensive* [3] if its coproducts are stable under pullback, the natural maps from summands into coproducts are monic, and intersections of distinct such summands are empty (=initial object). In each extensive category, a morphism from an indecomposable object to a coproduct factors through a summand of the coproduct and every object decomposes into a coproduct of indecomposables [3]. An object p in a given category is *indecomposable* if p is not the coproduct of more than one non-initial object and *projective* if $\mathscr{C}(p, \epsilon)$ is surjective for each epi ϵ. A category *has enough projectives* if each of its objects is the codomain of an epi from a projective object. We call an object a in a given category *atomic* if a is the codomain of an epi from a projective indecomposable object and *finite* if it contains only finitely many atomic subobjects.

Example 2.7. Fix small $\mathscr{G}$. The presheaf category

$$\hat{\mathscr{G}}$$

is $\bigcap$-complete, extensive, and has enough projectives. The atomic projective objects, up to isomorphism, are the representable presheaves.

Lemma 2.8. *Fix extensive $\bigcap$-complete $\mathscr{C}$ having enough projectives. Then*

$$supp_F(a, g)$$

is atomic for all $\bigcap$-continuous functors F preserving epis and coproducts, $\mathscr{C}$-objects g, and atomic $a \subset Fg$.

Proof. We take $g = supp_F(a, g)$ without loss of generality. There exist epi $\gamma : \hat{g} \to g$ from a projective $\hat{g}$ because $\mathscr{C}$ has enough projectives and epi $\alpha : \hat{a} \to a$ from an indecomposable projective $\hat{a}$ because a is atomic. There exists a small family $\mathcal{C}$ of indecomposable projective objects such that $\coprod \mathcal{C} = \hat{g}$ because $\mathscr{C}$ is extensive. Let $\iota = (a \hookrightarrow Fg)$. There exists morphism $\hat{\iota} : \hat{a} \to F\hat{g}$ such that $F(\gamma)\hat{\iota} = \iota\alpha$ because $\hat{a}$ is projective and $F\gamma$ is epi. There exists $\hat{c} \in \mathcal{C}$ such that $\hat{\iota}(\hat{a}) \subset F\hat{c}$ because $\hat{a}$ is indecomposable, $Fg = \coprod_{c \in \mathcal{C}} Fc$, and $\mathscr{C}$ is extensive. Therefore $a \subset (F\gamma)(F\hat{c}) = F(\gamma(\hat{c}))$. Thus $supp_F(a, g) = \gamma(\hat{c})$ by the minimality of $supp_F(a, g)$. □

2.3. **Relations.** We will later want to exploit when "transitive-reflexive closures" of certain relations "commute with a given functor F." In order to even articulate such an observation [Lemma 2.9], we need to define transitive-closures of abstract relations internal to an abstract category $\mathscr{C}$; for convenience, we take our $\mathscr{C}$ to be complete. A $\mathscr{C}$*-relation* $\gamma : a \rightharpoonup b$ consists of the data of $\mathscr{C}$-objects a, b and $graph(\gamma) \subset a \times b$. We write $\beta \circ \alpha$ for the $\mathscr{C}$-relation $a \rightharpoonup c$ such that

$$graph(\beta \circ \alpha) = graph(\alpha) \times_b graph(\beta)$$

for all $\mathscr{C}$-relations $\alpha : a \rightharpoonup b$ and $\beta : b \rightharpoonup c$. A $\mathscr{C}$-relation $\gamma : c \rightharpoonup c$ is a *preorder* if $\Delta_c(c), graph(\gamma^{\circ 2}) \subset graph(\gamma)$. The *reflexive-transitive closure* $\gamma^{\circ\infty}$ of a $\mathscr{C}$-relation $\gamma : c \rightharpoonup c$ is the $\mathscr{C}$-relation $c \rightharpoonup c$ such that $graph(\gamma^{\circ\infty})$ is the supremum of $\Delta_c(c)$ and all subobjects in c^2 of the form $graph(\gamma \circ \cdots \circ \gamma)$. Transitive-reflexive closures are preorders.

Lemma 2.9. *Fix complete $\mathscr{C}$. For each $\mathscr{C}$-relation $\gamma : c \rightharpoonup c$,*

$$F\, graph(\gamma^{\circ\infty}) = graph((\gamma^F)^{\circ\infty}),$$

where γ^F is the relation $Fc \rightharpoonup Fc$ defined by $graph(\gamma^F) = F\, graph(\gamma)$, for each functor F from $\mathscr{C}$ preserving monos, finite limits, and suprema of subobjects.

A proof is straightforward and therefore left to the reader.

3. Order theory

The temporal structures on both discrete and topological spaces takes the form of various preorders. We therefore review some definitions and fix some notation. We often take "preorder" in the sense of §2.3 to mean "preorder in the category of sets and functions." A *preordered set* P is a set P equipped with preorder $\leqslant_P$ on it. A *minimum* in a preordered set P is an element $m \in P$ such that $m \leqslant_P p$ for all $p \in P$; we similarly define a *maximum*. A *monotone function*

$$\phi : P \to Q$$

is a functor from a preordered set P to a preordered set Q. We call a monotone function $\phi : P \to Q$: *extrema-preserving* if it sends minima to minima and maxima to maxima; *convex* if $q \in \phi(P)$ whenever $\phi(p) \leqslant_Q q \leqslant_Q \phi(r)$; and *full* if $p \leqslant_P q$ whenever $\phi(p) \leqslant_Q \phi(q)$. We write $\mathscr{Q}$ for the category of preordered sets and monotone functions. The isomorphisms in $\mathscr{Q}$ are the full monotone bijections.

Example 3.1. Every set P admits the trivial preorder $=_P$ such that

$$\begin{aligned} graph(=_P) &= \Delta_P(P) \\ &= \{(p,p) \mid p \in P\}. \end{aligned}$$

A *lattice* is a set L equipped with a pair of commutative, associative, and idempotent multiplications $L^2 \to L$, which we write as $\vee_L, \wedge_L$, such that

$$p \vee_L (q \wedge_L p) = p, \quad p \wedge_L (q \vee_L p) = p, \quad p, q \in L.$$

We regard lattices L as preordered sets such that

$$\begin{aligned} graph(\leqslant_L) &= L \times_{\Delta_L, (((p,q)\mapsto q)\times \vee_L)} L^2 \\ &= \{(p,q) \mid p \vee_L q = q\}. \end{aligned}$$

For each lattice L having a minimum (maximum), the minimum (maximum) is unique and we write this unique minimum of L as $\min L$.

Example 3.2. A minimum of a lattice L is a (unique) point $m \in L$ satisfying

$$m \vee_L p = p, \quad p \in L.$$

A *lattice homomorphism* is a function between lattices preserving the lattice multiplications. Lattice homomorphisms are monotone functions, full if injective. We write $\mathscr{L}$ for the category of lattices and lattice homomorphisms. The *finite ordinals*, the preordered sets

$$[-1]=\varnothing, \quad [n]=\{0 \leqslant_{[n]} 1 \leqslant_{[n]} \cdots \leqslant_{[n]} n\}, \;\; n=0,1,\ldots,$$

are lattices and monotone functions between them are lattice homomorphisms. A *finite chain* in a preordered set P is a preordered set M isomorphic to a finite ordinal in $\mathscr{Q}$ such that $M \subset P$ and $graph(\leqslant_M)=graph(\leqslant_P)\cap M^2$. Every preordered set containing a minimum and maximum naturally is a colimit in $\mathscr{Q}$ of all of its maximal finite chains and inclusions between them. Products in $\mathscr{L}$ are products in $\mathscr{Q}$.

4. Directed topology

A single preorder does not often suffice in describing the local structure of time in a topological state space. For example, we might write $x \leqslant_{\mathbb{S}} y$ to indicate that a looping process can evolve from a state x to a state y, but the graph of the resulting preorder is $\mathbb{S}\times\mathbb{S}$ and hence cannot distinguish between clockwise and counterclockwise travels of the circular state space $\mathbb{S}$. We therefore recall a definition of temporal structure as a coherent preordering of all open subsets in a topological space [20].

Definition 4.1. A *circulation* $\leqslant$ on a space X is a function assigning each open subset $U \subset X$ a preorder $\leqslant_U$ on U such that for each collection $\mathscr{O}$ of open subsets of X, $\leqslant_{\bigcup \mathscr{O}}$ is the preorder with smallest graph containing

$$\bigcup_{U\in\mathscr{O}} graph(\leqslant_U).$$

A *stream* X is a space equipped with a circulation on it, which we always write as $\leqslant$.

We write $\ddot{X}$ for a topological space X equipped with the trivial circulation assigning to each open subset $U \subset X$ the trivial preorder on U.

Definition 4.2. Consider streams X and Y. A *stream map*

$$f : X \to Y$$

is a continuous function $X \to Y$ satisfying $f(x) \leqslant_U f(y)$ whenever $x \leqslant_{f^{-1}U} y$, for each open subset $U \subset Y$.

We write $\ddot{f}$ for the stream map $\ddot{X} \to \ddot{Y}$ defined by a continous map $f : X \to Y$.

The category of streams and stream maps is not Cartesian closed. Mimicking [26], we wrinkle the definition of a stream in order to obtain categorically convenient foundations for a homotopy theory. A *k-circulation* is a circulation which is “compactly generated.”

Definition 4.3. The circulation $\leqslant$ of a stream X is a *k-circulation* if for each open subset $U \subset X$ and pair $x \leqslant_U y$, there exist compact Hausdorff stream K, pair $\tilde{x} \leqslant_K \tilde{y}$, and stream map $k : K \to X$ satisfying

$$k(K) \subset U, \quad k(\tilde{x})=x,\; k(\tilde{y})=y.$$

Proposition 4.4. [20, Proposition 5.4] *All circulations on locally compact Hausdorff spaces are k-circulations.*

We henceforth redefine “space” to mean “weak Hausdorff k-space” and “stream” to mean “weak Hausdorff k-space equipped with a k-circulation on it.” We write $\mathscr{S}$ for the category of streams and stream maps and $\mathscr{T}$ for the category of spaces and continuous functions.

Example 4.5. The indecomposable projective streams are precisely the singletons.

Consider a space X. Continuous functions from X into streams induce “initial” circulations on X and continuous functions from streams into X induce “final” circulations on X, in a sense made precise in the language of categorical topology [1].

Proposition 4.6 ([20, Proposition 5.8])**.** *The forgetful functor*

$$\mathscr{S} \to \mathscr{T} \tag{1}$$

is topological.

In particular, the forgetful functor (1) creates limits and colimits [1, Proposition 7.3.8] and $\mathscr{S}$ is hence complete and cocomplete. We sometimes implicitly equip $\mathscr{S}$ and $\mathscr{T}$ with the structures of Cartesian monoidal categories. The following proposition describes how to construct the circulations of colimits and finite products as “point-wise” colimits and products of preordered sets.

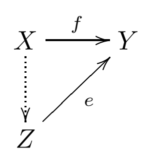
Proposition 4.7 ([20, Lemma 5.5, Proposition 5.11])**.** *The forgetful functor*

$$\Gamma : \mathscr{S} \to \mathscr{Q},$$

sending each stream X to its underlying set equipped with $\leqslant_X$, preserves colimits and finite products.

An equalizer in $\mathscr{S}$ of a pair $X \rightrightarrows Y$ of stream maps is a stream map $e : E \to X$ such that e defines an equalizer in $\mathscr{T}$ and e is a *stream embedding*.

Definition 4.8. A *stream embedding* e is a stream map $Y \to Z$ such that for all stream maps $f : X \to Z$ satisfying $f(X) \subset Y$, there exists dotted stream map making the following diagram commute.



A stream map i from a stream X is a stream embedding precisely when its composites ik with stream embeddings k from compact Hausdorff streams are stream embeddings because the circulations on X are k-circulations. Open subspaces of a stream X equipped with suitable restrictions of the circulation on X form substreams. General stream embeddings are difficult to explicitly characterize. However, the following lemma gives us practical criteria for an inclusion of spaces to define a stream embedding.

Lemma 4.9. *Consider stream Y having topology $\mathscr{O}_Y$. A stream map $i : X \to Y$ is a stream embedding if i is an inclusion of spaces and for all open $U \subset X$,*

$$graph(\leqslant_U) \supset U^2 \cap \bigcap_{U \subset V \in \mathscr{O}_Y} graph(\leqslant_U).$$

We leave the proof as an exercise for the reader.

We can also form "mapping streams": the k-ification of the compact-open topology and a natural k-circulation turn each hom-set $\mathscr{S}(X,Y)$ into a stream Y^X such that for all streams X,Y,Z, there exist natural isomorphisms

$$Z^{(X\times Y)} \cong (Z^X)^Y.$$

Theorem 4.10 ([20, Theorem 5.13])**.** *The category $\mathscr{S}$ is Cartesian closed.*

The direct verification of the axioms for streams and stream maps can be tedious. A *topological lattice* is a lattice topologized so that its lattice operations are jointly continuous. Let $\mathscr{P}$ be the category of compact and connected Hausdorff topological lattices and all continuous lattice homomorphisms between them. We can regard such topological lattices as streams [20, Propositions 4.7, 5.4, 5.11], [27, Proposition 1, Proposition 2, and Theorem 5], [11, Proposition VI-5.12 (i)], [11, Proposition VI-5.15].

Theorem 4.11. *There exists a full, concrete, and product-preserving embedding*

$$\mathscr{P} \hookrightarrow \mathscr{S}$$

sending each topological lattice L to a unique stream having the same underlying space and having a circulation sending the entire space to the preorder on L.

Thus we have enlarged the category of compact Hausdorff connected topological lattices to a category exhibiting categorical structure convenient for homotopy theory. We henceforth regard such topological lattices as streams and lattice homomorphisms between such topological lattices as stream maps.

We generalize streams to "streams equipped with actions of categories."

Definition 4.12. Fix $\mathscr{C}$. A *$\mathscr{C}$-stream* is a functor of the form

$$\mathscr{C} \to \mathscr{S}$$

and a *$\mathscr{C}$-stream map* is a natural transformation between $\mathscr{C}$-streams. A $\mathscr{C}$-stream X is *compact* if its colimit is compact and $X(c)$ is compact for all $\mathscr{C}$-objects c.

5. Simplicial theory

Simplicial sets share some of the flexibility of streams and some of the rigidity of cubical sets. Thus simplicial sets provide an intermediate setting for cubical approximation: we will later construct simplicial approximations of stream maps by direct geometric constructions and then subsequently bootstrap a cubical approximation theorem by combinatorics. We recall definitions in §5.1, recall a construction [6] of ordinal subdivision in §5.2, and interpret simplicial sets as combinatorial models of streams in §5.3.

5.1. **Simplicial sets.**

We recall basic definitions of the del category and simplicial sets, referring the reader elsewhere [19] for details. We write Δ for the category of finite non-empty ordinals and monotone functions between them. We sometimes implicitly equip $\hat{\Delta}$ with the structure of a Cartesian monoidal category. *Simplicial sets* are objects of $\hat{\Delta}$ and *simplicial functions* are $\hat{\Delta}$-morphisms. The *dimension* of a simplicial set C is the minimal n such that the natural simplicial function $C[n]\cdot\Delta[n]\to C$ is epi or ∞ if no such n exists.

Example 5.1. For each atomic simplicial set A, $\dim A < \infty$.

An n-*simplex* in a simpliciall set C is an element of $C[n]$ for each n. Every atomic simplicial set has a natural "minimum" vertex. We write $\min A$ for the image of the 0-simplex $[0] \to [n]$ sending 0 to 0 in $\Delta[\dim A]$ under the $[0]$-component of the natural simplicial function $\Delta[\dim A] \to A$, for each atomic simplicial set A. For each simplicial function ψ from an atomic simplicial set A,

$$\psi_{[0]} \min A = \min \psi(A).$$

Example 5.2. The m-simplices of $\Delta[n]$ are the monotone functions

$$[m] \to [n].$$

We write sn for the nerve functor $\mathscr{Q} \to \hat{\Delta}$, defined on morphisms $\phi : P \to Q$ by

$$\mathscr{Q}(-,\phi)_{\restriction \Delta^{\mathrm{op}}} : \mathscr{Q}(-,P)_{\restriction \Delta^{\mathrm{op}}} \to \mathscr{Q}(-,Q)_{\restriction \Delta^{\mathrm{op}}}.$$

Simplicial sets naturally admit intrinsic "simplicial preorders." We write $\leqslant_C$ for the reflexive-transitive closure of the relation on C having graph

$$\int_{\Delta}^{[n]} C[n] \cdot \left(\Delta[n] \times_{sn\,\Delta_{[n]}, sn((p,q)\mapsto q)) \times \vee_{[n]}} sn[n]^2\right)$$

natural in simplicial sets C.

Example 5.3. For each n, *graph*$(\leqslant_{\Delta[n]}) = \Delta[n] \times_{sn\,\Delta_{[n]}, sn((p,q)\mapsto q)) \times \vee_{[n]}} sn[n]^2$.

We generalize simplicial sets to "simplicial sets equipped with actions of categories".

Definition 5.4. Fix $\mathscr{C}$. A $\mathscr{C}$-*simplicial set* is a functor of the form

$$\mathscr{C} \to \hat{\Delta}$$

and a $\mathscr{C}$-*simplicial function* is a natural transformation between $\mathscr{C}$-simplicial sets. A $\mathscr{C}$-simplicial set C is *finite* if its colimit is finite and $C(c)$ is finite for each $\mathscr{C}$-object c.

5.2. **Subdivisions.** Ordinal subdivision plays a role in directed topology analogous to the role barycentric subdivision plays in topology. We recall a construction [6] in terms of *ordinal subdivision*, the tensor $\oplus$ on the category $\Delta_\star$ of finite ordinals and monotone functions between them sending pairs $[m]$, $[n]$ of finite ordinals to $[m+n+1]$ and pairs $\phi : [m] \to [n]$, $\phi' : [m'] \to [n']$ of monotone functions to

$$(\phi \oplus \phi')(k) = \begin{cases} \phi(k), & k = 0,1,\ldots,m \\ \phi'(k-m-1), & k = m+1, m+2, \ldots, m+m' \end{cases}$$

We write γ, $\bar{\gamma}$ for the natural transformations $id_\Delta \to (-)^{\oplus 2}$ respectively defined by $id_\Delta \oplus ([-1] \to id_\Delta)$ and $([-1] \to id_\Delta) \oplus id_\Delta$.

Example 5.5. The functions $\gamma_{[1]}, \bar{\gamma}_{[1]} : [1] \to [3]$ are defined by

$$\gamma_{[1]}(0) = 0, \quad \gamma_{[1]}(1) = 1, \quad \bar{\gamma}_{[1]}(0) = 2, \quad \bar{\gamma}_{[1]}(1) = 3.$$

We write sd for the functor $\hat{\Delta} \to \hat{\Delta}$ induced from $(-)^{\oplus 2} : \Delta \to \Delta$. Ordinal subdivision sd admits left and right adjoints [6] and hence is continuous, cocontinuous, and $\bigcap$-continuous. We abuse notation and write $\gamma, \bar{\gamma}$ for the natural transformations $sd \to id_{\hat{\Delta}}$ induced from the respective natural transformations $\gamma, \bar{\gamma} : id_\Delta \to id_\Delta \oplus id_\Delta$.

Example 5.6. The composite sd^2 is the functor induced from $(-)^{\oplus 4} : \Delta \to \Delta$.

Double ordinal subidivison and double barycentric subdivision share certain similar convenient properties in the restricted setting of 1-dimensional simplicial sets. For example, the simplicial function $\gamma\bar{\gamma}_{\Delta[1]}$ pushes simplices carried near $\partial\Delta[1]$ onto $\partial\Delta[1]$.

Lemma 5.7. *For the cases* $\delta = \delta_-, \delta_+$,

$$\gamma\bar{\gamma}_{\Delta[1]}\, Star_{sd^2\Delta[1]}\{[3] \to [0] \xrightarrow{\delta} [1]\} = \Delta[\delta](\Delta[0]).$$

Proof. Consider a 1-simplex $\phi : [7] \to [1]$ in $sd^2\Delta[1]$ generating an atomic subobject having as a 0-simplex the constant function $[3] \to [1]$ at $\delta(0)$. There exists $k \in \{0, 1\}$ such that $\phi(k) = \phi(k+2) = \phi(k+4) = \phi(k+6) = \delta(0)$. For either case $\delta(0) = 0$ or $\delta(0) = 1$, $\phi(4) = \phi(5) = \delta(0)$ by monotonicity and hence

$$\begin{aligned} \gamma\bar{\gamma}_{sd^2\Delta[1]}\phi &= \phi\bar{\gamma}\gamma \\ &= \phi(- + 4) \\ &\equiv \delta(0). \end{aligned}$$

□

Consequently, the behavior of $\gamma\bar{\gamma}_{\Delta[1]}$ on each atomic $B \subset sd^2C$ reduces to the behavior of $\gamma\bar{\gamma}_{\Delta[1]}$ on a projective retraction of B for all 1-dimensional simplicial sets C. We write $ret_C B$ for the subobject

$$B \cap sd^2\left(\bigcap_{\varnothing \neq A \subset B} supp_{sd^2}(A, C)\right),$$

of a $B \subset sd^2C$, for each simplicial set C. It suffices to record the particular behavior of $\gamma\bar{\gamma}_C$ for just the case $C = \Delta[1]$.

Lemma 5.8. *There exists a retraction*

$$\pi_{B\Delta[1]} : B \to ret_{\Delta[1]}B$$

and $(\gamma\bar{\gamma}_{\Delta[1]})_{\restriction B} = (\gamma\bar{\gamma}_{\Delta[1]})_{\restriction ret_{\Delta[1]}B}\pi_{B\Delta[1]}$, *for each atomic* $B \subset sd^2\Delta[1]$.

Proof. In the case $B = ret_{\Delta[1]}B$, id_B is our desired retraction π. It therefore suffices to consider the case $B \cap sd^2\Delta[0]$ non-empty and hence equal to $sd^2\Delta[0]$, the other case $B \cap sd^2\Delta[\delta_+](\Delta[0]) \neq \varnothing$ following similarly. Then $\gamma\bar{\gamma}_{\Delta[1]}B = sd^2\Delta[0]$ by Lemma 5.7, hence $\gamma\bar{\gamma}_{\Delta[1]}B \neq sd^2\Delta[\delta_+](\Delta[0])$, hence $B \cap sd^2\Delta[\delta_+](\Delta[0]) = \varnothing$ by Lemma 5.7, hence $ret_{\Delta[1]}B = sd^2\Delta[0]$, hence the object-wise constant simplicial function $B \to sd^2\Delta[0]$ is our unique desired retraction by Lemma 5.7. □

5.3. **Stream realizations.** Extra structure lurks behind classical geometric realizations of simplicial sets. The *standard cosimplicial space* is the functor $\nabla : \Delta \to \mathscr{T}$ assigning to each $[n]$ the topological n-simplex and assigning to each monotone function $\phi : [m] \to [n]$ the linear map sending each point with kth barycentric coordinate 1 to the point with $\phi(k)$th barycentric coordinate 1. Ceometric realization is the cocontinuous and finitely continuous functor

$$|-| = \int_{\Delta}^{[n]} -([n]) \cdot \nabla[n] : \hat{\Delta} \to \mathscr{T}.$$

sending monos to closed embeddings. The functor $|\,sn(-)|$ preserves finite products and hence sends the lattice operations on $[n]$ to lattice operations turning $\nabla[n]$ into the stream [Theorem 4.11] $\vec{\nabla}[n]$, natural in $[n]$. We write $\upharpoonleft - \downharpoonright$ for

$$\int_{\Delta}^{[n]} -([n]) \cdot \vec{\nabla}[n] : \hat{\Delta} \to \mathscr{S}.$$

We henceforth identify $|C|$ with the underlying space of $\upharpoonleft C \downharpoonright$ for each simplicial set C by Proposition 4.6. The graph of the global preorder $\leqslant_{\upharpoonleft C \downharpoonright}$ on a stream of the form $\upharpoonleft C \downharpoonright$ is simplicial in the following sense.

Lemma 5.9. *For each simplicial set C, $graph(\leqslant_{\upharpoonleft C \downharpoonright}) = |\,graph(\leqslant_C)|$.*

Proof. The case C representable follows because $|-|$ preserves finite pullbacks and

$$graph(\leqslant_{\upharpoonleft \Delta[n] \downharpoonright}) = graph(\leqslant_{\vec{\nabla}[n]}) = \nabla[n] \times_{\Delta_{\nabla[n]},((p,q)\mapsto q, \vee_{\vec{\nabla}[n]})} \nabla[n]^2.$$

The case for general simplicial sets C follows from Lemma 2.9 because $\leqslant_{\upharpoonleft C \downharpoonright}$ is the reflexive-transitive closure of the relation on $|C|$ having as its graph

$$\int_{\Delta}^{[n]} C[n] \cdot |\,graph(\leqslant_{\Delta[n]})| = \left| \int_{\Delta}^{[n]} C[n] \cdot graph(\leqslant_{\Delta[n]}) \right| \subset |C^2| = |C|^2$$

by the previous case and Proposition 4.7. $\square$

Thus topological limits commute with inequalities in stream realizations.

Lemma 5.10. *For all simplicial sets C, $graph(\leqslant_{\upharpoonleft C \downharpoonright})$ is closed in $|C| \times_{\mathscr{T}} |C|$.*

Proof. Lemma 5.9 implies the claim because $|\,graph(\leqslant_C)|$ is closed in $|C^2|$. $\square$

Stream realizations preserve finite products.

Lemma 5.11. *The functor $\upharpoonleft - \downharpoonright : \hat{\Delta} \to \mathscr{S}$ preserves finite products.*

Proof. Let m_{AB} be the universal stream map $\upharpoonleft A \times B \downharpoonright \to \upharpoonleft A \downharpoonright \times \upharpoonleft B \downharpoonright$ natural in simplicial sets A and B. Let M denote a finite chain in $[m] \times [n]$. Consider m, n. All pairs $\Gamma \upharpoonleft sn(M \to [m] \times [n]) \downharpoonright$, $\Gamma \upharpoonleft sn(M \to [m] \times [n]) \downharpoonright$ of lattice homomorphisms induce all injective monotone functions, and hence all injective lattice homomorphisms, of the form $(\Gamma m_{\Delta[m]\Delta[n]})_{\restriction \Gamma \upharpoonleft sn\, M \downharpoonright}$ because Γ preserves finite products by Proposition 4.7 and $\mathscr{L}$-products are $\mathscr{Q}$-products. Thus $\Gamma m_{\Delta[m]\Delta[n]}$ is full because it is the universal monotone function induced from injective lattice homomorphisms, and hence full monotone functions, of the form $(\Gamma m_{\Delta[m]\Delta[n]})_{\restriction \Gamma \upharpoonleft sn\, M \downharpoonright}$. Thus m_{AB}, a homeomorphism of underlying spaces, is a full monotone bijection of underlying preordered sets, hence an isomorphism of topological lattices, and hence a stream isomorphism for the case $A = \Delta[m]$ and $B = \Delta[n]$ by Theorem 4.11 and hence the general case because finite products preserve colimits in $\mathscr{S}$ by Theorem 4.10 and in $\hat{\Delta}$. $\square$

Stream realizations remember simplicial orientations.

Lemma 5.12. *For all preordered sets P and pairs $x \leqslant_{\upharpoonleft sn\, P \downharpoonright} y$,*

$$\min \Gamma \upharpoonleft supp_{|-|}(\{x\}, sn\, P) \downharpoonright \leqslant_P \min \Gamma \upharpoonleft supp_{|-|}(\{y\}, sn\, P) \downharpoonright .$$

Proof. It suffices to consider the case $P = [n]$ because $\leqslant_{\upharpoonleft sn\,P \downharpoonright}$ is transitive and $sn(P)$ is the colimit of all inclusions between simplicial sets of the form $sn\,M$ for finite chains M in P. Let t_k be the kth barycentric coordinate of t for all $t \in \nabla[n]$ and $k \in [n]$. Then

$$\begin{aligned} \min \Gamma \upharpoonleft supp_{|-|}(\{x\}, sn\,P) \downharpoonright \quad &= \quad \min\{k \in [n] \mid x_k = 1\} \\ &\leqslant_{[n]} \quad \min\{k \in [n] \mid y_k = 1\} \\ &= \quad \min \Gamma \upharpoonleft supp_{|-|}(\{y\}, sn\,P) \downharpoonright \end{aligned}$$

because $x_k \vee_{\vec{\nabla}[n]} y_k = (x \vee_{\vec{\nabla}[n]} y)_k$ for each $k \in [n]$ by linearity of $\vee_{\vec{\nabla}[n]}$. ☐

Prism decompositions [24] define piecewise linear homeomorphisms

$$\varphi_{\Delta[n]} : |\, sd\,\Delta[n]| \cong |\Delta[n]| = \nabla[n],$$

natural in non-empty finite ordinals $[n]$, characterized by the rule

$$|\phi| \mapsto {}^1\!/\!{}_2|\phi(0)| + {}^1\!/\!{}_2|\phi(1)|, \quad \phi \in (sd\,\Delta[n])[0] = \Delta([0] \oplus [0], [n]).$$

These homeomorphisms define lattice isomorphisms $\Gamma \upharpoonleft sd\,\Delta[n] \downharpoonright \cong \Gamma \upharpoonleft \Delta[n] \downharpoonright$ by linearity of $\vee_{\vec{\nabla}[n]}, \wedge_{\vec{\nabla}[n]}$ and hence stream isomorphisms $\upharpoonleft sd\,\Delta[n] \downharpoonright \cong \upharpoonleft \Delta[n] \downharpoonright$ [Theorem 4.11]. We write φ for the extension of $\{\varphi_{\Delta[n]}\}_{n \in \mathbb{N}}$ to a natural isomorphism

$$\varphi : \upharpoonleft sd(-) \downharpoonright \cong \upharpoonleft - \downharpoonright : \hat{\Delta} \to \mathscr{S}.$$

6. Cubical theory

Cubical sets are rigid and economical descriptions of state spaces [8, 12]. We recall basic definitions in §6.1; investigate a cubical analogue of ordinal subdivision in §6.2; introduce cubical models for higher categories in §6.3; and interpret cubical sets as combinatorial models of streams in §6.4.

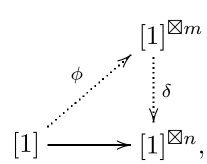
6.1. Cubical sets.

We recall basic definitions of the box category and cubical sets, referring the reader elsewhere [13, 18] for details. Let $\square_1$ be the subcategory of $\mathscr{Q}$ generated by the function $[1] \to [0]$ and the functions $\delta_- : [0] \to [1]$ and $\delta_+ : [0] \to [1]$ sending 0 to the respective points 0 and 1. Let $\square$ be the monoidal subcategory of the Cartesian monoidal category $\mathscr{Q}$ generated by $\square_1$. We write $\boxtimes$ for the tensor on $\square$. The category $\square$ admits the following convenient characterization [13].

Lemma 6.1 ([13, Theorem 4.2])**.** *The free monoidal category over $\square_1$ is $\square$.*

In particular, retractions to injective $\square$-morphisms are unique because δ_-, δ_+ have unique retractions. A characterization of injective $\square$-morphisms as inclusions of intervals [18, §2] implies the following lemma.

Lemma 6.2. *For every solid monotone function given in the diagram*



there exist unique choices of minimal m, monotone function ϕ, and injective $\square$-morphism δ making the entire diagram commute. The function ϕ preserves extrema.

We regard $\hat{\square}$ as a monoidal category with tensor $\boxtimes$ defined by

$$-_1 \boxtimes -_2 = \int_{\square\times\square}^{([1]^{\boxtimes m},[1]^{\boxtimes n})} -_1([1]^{\boxtimes m}) \cdot -_2([1]^{\boxtimes n}) \cdot \square[1]^{\boxtimes m+n} : \hat{\square} \times \hat{\square} \to \hat{\square}.$$

Projections $B\boxtimes C \to B$ and $B\boxtimes C \to C$ induce monos $B\boxtimes C \hookrightarrow B\times C$ natural in cubical sets B and C allowing us to henceforth regard tensor products as subobjects of categorical products.

Example 6.3. For all cubical sets C, $C \boxtimes \square[0] = C \times \square[0]$.

Cubical sets are $\hat{\square}$-objects, *oriented cubical complexes* are cubical sets whose atomic subobjects are projective, and *cubical functions* are $\hat{\square}$-morphisms. The *dimension* of a cuboicall set C is the minimal n such that the natural cubical function $C[1]^{\boxtimes n} \cdot \square[1]^{\boxtimes n} \to C$ is epi or ∞ if no such n exists.

Example 6.4. For each atomic cubical set A, $\dim A < \infty$.

An inclusion between atomic cubical sets admits at most one retraction because retractions in $\square$ are unique. We write $\partial\square[1]^{\boxtimes n}$ for the unique maximal proper subobject of $\square[1]^{\boxtimes n}$ for each n. An *n-cube* in a cubical set C is an element of $C[1]^{\boxtimes n}$ for each n.

Example 6.5. The m-cubes of $\square[1]^{\boxtimes n}$ are the $\square$-morphisms

$$[1]^{\boxtimes m} \to [1]^{\boxtimes n}.$$

A cubical set C is *connected* if its underlying reflexive graph, the reflexive graph having vertices $C[0]$, edges $C[1]$, and structure maps $v \mapsto C([1] \to [0])(v)$ and $e \mapsto \{C(\delta_-)(e), C(\delta_+)(e)\}$, is connected. Every cubical function induces a map of underlying reflexive graphs and hence maps connected subobjects onto connected subobjects.

Example 6.6. Every non-empty cubical set of the form $\mathit{Star}_C V$ is connected.

Example 6.7. Every atomic cubical set C is connected because $C = \mathit{Star}_C C[0]$.

Example 6.8. The connected subobjects of $\square[1]$ are atomic.

Fix an atomic cubical set A. We write ϱ_A for the unique epi of the form $\square[1]^{\boxtimes n} \to A$ such that n is minimal and $\dim A$ for this minimal n. The epi ϱ_A does not identify an n-cube θ which does not inhabit $\partial\square[1]^{\boxtimes \dim A}$ with another distinct n-cube by minimality of $\dim A$. The cubical set A is projective if and only if ϱ_A is monic. For each cubical function $\psi : A \to B$ and epi cubical function $\epsilon : E \to B$, there exists a dotted cubical function, monic if ψ is monic and E is an oriented cubical complex, making the following diagram commute by projectivity of $\square[1]^{\boxtimes \dim A}$.

$$\begin{array}{ccc} \square[1]^{\boxtimes \dim A} & \dashrightarrow & E \\ {\scriptstyle \varrho_A}\downarrow & & \downarrow{\scriptstyle \epsilon} \\ A & \xrightarrow{\psi} & B. \end{array}$$

We generalize cubical sets to "cubical sets equipped with actions of categories."

Definition 6.9. Fix $\mathscr{C}$. A *$\mathscr{C}$-cubical set* is a functor of the form

$$\mathscr{C} \to \hat{\square}$$

and a *$\mathscr{C}$-cubical function* is a natural transformation between $\mathscr{C}$-cubical sets. A $\mathscr{C}$-cubical set C is *finite* if its colimit is finite and $C(c)$ is finite for each $\mathscr{C}$-object c.

We write cn for the functor $\mathscr{Q} \to \hat{\square}$ defined on morphisms $\phi : P \to Q$ by

$$\mathscr{Q}(-,\phi)_{\restriction \square^{\mathrm{op}}} : \mathscr{Q}(-,P)_{\restriction \square^{\mathrm{op}}} \to \mathscr{Q}(-,Q)_{\restriction \square^{\mathrm{op}}}.$$

6.2. **Subdivisions.** We define a cubical analogue to ordinal subdivision in terms of the combinatorics of subdivided hypercubes instead of an operation on the box category itself. Just as $\square$ models abstract hypercubes, a larger category models abstract subdivided hypercubes. We write $\boxplus$ for the smallest monoidal subcategory of the Cartesian monoidal category $\mathscr{Q}$ containing the non-empty finite ordinals and convex monotone functions between them. We abuse notation and write $\boxtimes$ for the tensor on $\boxplus$.

Example 6.10. All $\square$-morphisms are $\boxplus$-morphisms.

We can model cubical subdivision of abstract hypercubes as a monoidal functor $\square \to \boxplus$. We write $[2] \otimes -$ for the unique monoidal functor $\square \to \boxplus$ sending δ to the monotone functions $[0] \to [2]$ defined by the rule $0 \mapsto 2\delta(0)$ for $\delta = \delta_-, \delta_+$. We abuse notation and write $\gamma, \bar{\gamma}$ for the monoidal natural transformations $[2] \otimes - \to [1] \otimes -$ having as their $[1]$-components the respective convex monotone functions $\max(-,1) - 1, \min(-,1) : [2] \to [1]$. The following lemma justifies our abuse in notation.

Lemma 6.11. *For all $n = 0, 1$ and monotone functions $\phi : [n] \oplus [n] \to [n]$,*

$$\phi = \gamma_{[n]}(\phi\gamma_{[n]} + \phi\bar{\gamma}_{[n]}) \oplus \bar{\gamma}_{[n]}(\phi\gamma_{[n]} + \phi\bar{\gamma}_{[n]}).$$

Lemma 6.12. *For all $n = 0, 1$ and convex monotone functions $\phi : [n] \to [2n]$,*

$$\phi = (\gamma_{[n]}\phi + \bar{\gamma}_{[n]}\phi)\gamma_{[n]} + (\gamma_{[n]}\phi + \bar{\gamma}_{[n]}\phi)\bar{\gamma}_{[n]}.$$

Proofs are straightforward verifications of function values and are therefore left to the reader.

We abuse notation and also write $\square[-]$ for the $\bigcap$-continuous composite of the Yoneda embedding $\boxplus \to \hat{\boxplus}$ with the functor $\hat{\boxplus} \to \hat{\square}$ induced from inclusion $\square \hookrightarrow \boxplus$.

Example 6.13. For all $\boxplus$-objects $\mathfrak{p}$, $\square[\mathfrak{p}]$ is an oriented cubical complex.

We write cd for the cocontinuous and $\bigcap$-continuous functor

$$\int_{\square}^{[1]^{\boxtimes n}} -([1]^{\boxtimes n}) \cdot \square[2]^{\boxtimes n} : \hat{\square} \to \hat{\square}$$

and cx for its right adjoint. We abuse notation and also write $\gamma, \bar{\gamma}$ for the monoidal natural transformations $cd \to id_{\hat{\square}}$ induced from the monoidal natural transformations $\gamma, \bar{\gamma} : [2] \otimes - \to - : \square \to \boxplus$.

Example 6.14. There exists natural isomorphism $cd\square[-] \cong \square[[2] \otimes -]$.

The behavior of $\gamma\bar{\gamma}_C : cd^2 C \to C$ on each atomic $B \subset cd^2 C$ reduces to the behavior of a projective retract $ret_C B$. We write $ret_C B$ for

$$B \cap cd^2 \left(\bigcap_{\varnothing \neq A \subset B} supp_{cd^2}(A, C) \right),$$

for each cubical set C and $B \subset cd^2 C$. We define a cubical function π_{BC} for each cubical set C and atomic $B \subset cd^2 C$ by the following lemma.

Lemma 6.15. *Fix cubical set C. There exists a retraction*

$$\pi_{BC} : B \to ret_C B$$

and $(\gamma\bar{\gamma}_C)_{\restriction B} = (\gamma\bar{\gamma}_C)_{\restriction ret_C B}\pi_{BC}$, for each atomic $B \subset cd^2 C$.

We postpone a proof until §7, when we can bootstrap a proof for a simplicial analogue of the lemma. We continue and list some consequences.

Lemma 6.16. *For each cubical set C and atomic $B \subset cd^2 C$,*

$$ret_C B$$

is isomorphic to a representable cubical set.

Proof. We take C to be atomic. There exists dotted mono making the diagram

$$\begin{array}{ccc} \square[1]^{\boxtimes \dim ret_C B} & \overset{\iota}{\dashrightarrow} & cd^2 \square[1]^{\boxtimes \dim C} \\ \downarrow{\scriptstyle \varrho_{ret_C B}} & & \downarrow{\scriptstyle cd^2 \varrho_C} \\ ret_C B & \hookrightarrow & cd^2 C \end{array}$$

commute because $\square[1]^{\boxtimes \dim ret_C B}$ is projective. Then

$$\iota(\square[1]^{\boxtimes \dim ret_C B}) \cap cd^2 \partial \square[1]^{\boxtimes \dim C} = \varnothing$$

by minimality of $ret_C B$, hence $cd^2 \varrho_C \iota = (ret_C B \hookrightarrow cd^2 C)\varrho_{ret_C B}$ is monic, hence $\varrho_{ret_C B}$ is monic, and hence $ret_C B$ is projective. $\square$

The retracts $ret_C B$ are natural in in the sense of the following two lemmas.

Lemma 6.17. *Fix cubical set D. For all atomic $A \subset B \subset C \subset cd^2 D$,*

$$(\pi_{CD})_{\restriction ret_C A} = (\pi_{CD})_{\restriction ret_C B}(\pi_{BD})_{\restriction ret_C A}.$$

Proof. Retractions to inclusions of atomic cubical sets are unique. $\square$

Lemma 6.18. *For each cubical function $\gamma : C \to C'$,*

$$cd^2 \gamma : cd^2 C \to cd^2 C'$$

restricts and corestricts to a cubical function $ret_C B \to ret_{C'}(cd^2\gamma)(B)$ for each atomic $B \subset cd^2 C$.

Proof. Let $B' = (cd^2\gamma)(B)$. For each non-empty atomic $A' \subset B'$, there exists non-empty $A \subset B$ such that $(cd^2\gamma)A = A'$ and $\gamma\, supp_{cd^2}(A, B) \subset supp_{cd^2}((cd^2\gamma)(A), \gamma B) = supp_{cd^2}(A', B')$ by minimality of $supp_{cd^2}(A', B')$. $\square$

The cubical function $\gamma\bar{\gamma}_C$ converts combinatorial neighhborhoods into atomic subobjects of C.

Lemma 6.19. *Fix a cubical set C. For each $V \subset (cd^2C)[0]$,*

$$\gamma\bar{\gamma}_C \, Star_{\,cd^2C}V$$

is $\varnothing$ or atomic.

Proof. The cases $C = \square[0], \square[1]$ follow because $Star_{\,cd^2C}V$, and hence its image under $\gamma\bar{\gamma}_C$, are connected and the only connected subobjects of $\square[0], \square[1]$ are atomic.

The case C representable thence follows from Lemma 6.1.

Consider the general case. For a minimal atomic $A \subset C$ such that $V \subset (cd^2A)[0]$,

$$\begin{aligned}
(2)\qquad \gamma\bar{\gamma}_C \, Star_{\,cd^2C}V &= \gamma\bar{\gamma}_C \, Star_{\,cd^2A}V \\
(3)\qquad &= \gamma\bar{\gamma}_A (cd^2\varrho_A)(Star_{\,cd^2\square[1]^{\boxtimes \dim A}}(\varrho_A)^{-1}_{[0]}V) \\
(4)\qquad &= \varrho_A\gamma\bar{\gamma}_A \, Star_{\,cd^2\square[1]^{\boxtimes \dim A}}(\varrho_A)^{-1}_{[0]}V,
\end{aligned}$$

(2) by Lemma 6.15, is atomic by the previous case. $\square$

6.3. **Extensions.** An operation somewhat dual to cubical subdivision is the cubical extension of a cubical set C to a cubical model of a higher category "presented by C." We define structure maps turning a cubical set into such a higher categorical structure. Just as $\square$ models abstract hypercubes, a larger category models adjacent abstract hypercubes. We write $\boxdot$ for the smallest monoidal subcategory of $\mathscr{Q}$ having $[0]$ as a terminal object and containing the pushout square

$$\begin{array}{ccc}
[1] & \longrightarrow & [2] \\
{\scriptstyle \delta_+}\uparrow & & \uparrow \\
[0] & \xrightarrow[\delta_-]{} & [1]
\end{array}$$

in $\mathscr{Q}$; in particular, $\square \subset \boxdot \subset \boxplus$. We write ex for the endofunctor

$$\int_{\boxdot}^{\mathfrak{p}} \hat{\square}(\square[\mathfrak{p}], -) \cdot cn\,\mathfrak{p} : \hat{\square} \to \hat{\square}.$$

Example 6.20. The cubical function

$$cn\,\mathfrak{p} \to ex\,\square[\mathfrak{p}],$$

natural in $\boxdot$-objects $\mathfrak{p}$, is an isomorphism.

We regard certain natural cubical functions $C \hookrightarrow ex\,C$ as the identity structure maps for cubical sets C presenting cubical models of higher categories. We define the natural transformation $\kappa : id_{\hat{\square}} \to ex$ by the following lemma.

Lemma 6.21. *There exists a unique natural transformation*

$$id_{\hat{\square}} \to ex : \hat{\square} \to \hat{\square}$$

Proof. For each preordered set $\mathfrak{p}$, there exists unique cubical function $\square[\mathfrak{p}] \to cn\,\mathfrak{p}$ whose $[0]$-component is the identity function $\boxplus([0], \mathfrak{p}) = \mathscr{Q}([0], \mathfrak{p})$ because n-cubes in $cn\,\mathfrak{p}$ are determined by their composites with functions $[0] \to [1]^{\boxtimes n}$. Thus the unique cubical function $\square[0] \cong cn[0]$ of terminal cubical sets uniquely extends to a natural isomorphism $\square[-] \cong cn[-] : \boxdot \to \hat{\square}$. The lemma follows by naturality. $\square$

We regard the cubical set $ex\,C$ as encoding the possible "pasting diagrams" of a cubical set C. We thus define composition operators for cubical models of higher categories.

Definition 6.22. Fix $\mathscr{G}$. For each $\mathscr{G}$-cubical set C, a *strict composition*

$$ex\,C \to C$$

on C is a retraction to κ_C.

Example 6.23. The *cubical nerve* of a small category $\mathscr{C}$, the cubical set

$$\mathrm{Cat}(-, \mathscr{C})_{\restriction \square^{\mathrm{op}}},$$

where we regard $[1]^{\boxtimes n}$ as the n-fold categorical product of the free category generated by an arrow $0 \to 1$ for each n, admits a strict composition.

Example 6.24. A *Kan cubical set* [18] is a cubical set satisfying the right lifting property with respect to all inclusions into $\square[1]^{\boxtimes n}$ of its largest subobject $\sqcup^i[1]^{\boxtimes n}$ not having $[1]^{\boxtimes i-1} \boxtimes \delta \boxtimes [1]^{\boxtimes n-i}$ as its $(n-1)$-cube for each n, $\delta = \delta_+, \delta_-$, and $i = 1, 2, \ldots, n$. Kan cubical sets, the fibrant objects with respect to a Quillen model structure [18] whose acyclic cofibrations include κ_C [Corollary 8.25], admit strict compositions.

Our extension operator ex - not cx - serves a role in cubical directed homotopy theory analogous to the role that the right adjoint to barycentric subdivision plays in classical simplicial homotopy theory [4]. For example, ex - and not cx - turns out to preserve weak homotopy types. We write ν for the natural transformation $cx \to ex$ induced from the functor $[2] \otimes - : \square \to \boxplus$.

6.4. **Stream realizations.** Extra structure lurks behind classical geometric realizations of cubical sets. The *standard cocubical space* $\square : \square \to \mathscr{T}$ is the unique monoidal functor composing δ with $\{0,1\} \hookrightarrow [0,1]$ for each $\delta = \delta_-, \delta_+$. Geometric realization is the monoidal, cocontinuous, and $\bigcap$-continuous functor

$$|-| = \int_{\square}^{[1]^{\boxtimes n}} -([1]^{\boxtimes n}) \cdot \square[1]^{\boxtimes n} : \hat{\square} \to \mathscr{T}.$$

sending monos to closed embeddings. We write $star_C V$ for the topological interior of $|\,Star_C V|$ in $|C|$ for each $V \subset C[0]$ and $|c|$ for the unique point in $|C|$ for which $c \in supp_{|-|}(\{|c|\}, C)[0]$ for each $c \in C[0]$.

Example 6.25. Consider a cubical set C. For each $V \subset C[0]$,

$$star_C V = \bigcap_{v \in V} star_C\{v\}.$$

Thus for each cubical set C, the family of subsets $\{star_C V \mid V \subset C[0]\}$ forms an open cover of $|C|$ closed under finite intersections.

Example 6.26. For each cubical set C and 0-cube c of C,

$$|c| \in star_C(\{c\}).$$

We write $\vec{\Box}$ for the unique monoidal functor $\Box \to \mathscr{S}$ sending δ to the stream map $0 \mapsto \delta(0)$ from $\{0\}$ to $\vec{\Box}[1]$ for $\delta = \delta_-, \delta_+$. We abuse notation and also write $\rceil - \lfloor$ for the cocontinuous and monoidal functor

$$\int_{\Box}^{[1]^{\boxtimes n}} -([1]^{\boxtimes n}) \cdot \vec{\Box}[1]^{\boxtimes n} : \hat{\Box} \to \mathscr{S}.$$

Topological limits commute with inequalities in stream realizations.

Lemma 6.27. *For all cubical sets C, graph$(\leqslant_{\rceil C \lfloor})$ is closed in $|C|^2$.*

Proposition 6.28. *The functor $\rceil - \lfloor : \hat{\Box} \to \mathscr{S}$ sends monos to embeddings.*

We postpone proofs until the end of the next section.

Definition 6.29. For each $\mathscr{C}$, a $\mathscr{C}$-stream X is *quadrangulable* if X is a composite

$$\mathscr{C} \longrightarrow \hat{\Box} \xrightarrow{\rceil - \lfloor} \mathscr{S}$$

up to natural isomorphism.

7. Triangulations

We write *tri* for the cocontinuous and $\bigcap$-continuous functor

$$\int_{\Box}^{[1]^{\boxtimes n}} -([1]^{\boxtimes n}) \cdot sn[1]^{\boxtimes n} : \hat{\Box} \to \hat{\Delta}$$

and *qua* for its right adjoint. Triangulation behaves somewhat like classical geometric realization; both functors convert models of spaces into more flexible models of spaces, induce equivalences of associated classical weak homotopy categories, and admit right adjoints. The adjunction $tri \dashv qua$ also exhibits the following convenient property.

Lemma 7.1. *The composite $qua \circ tri : \hat{\Box} \to \hat{\Box}$ is cocontinuous.*

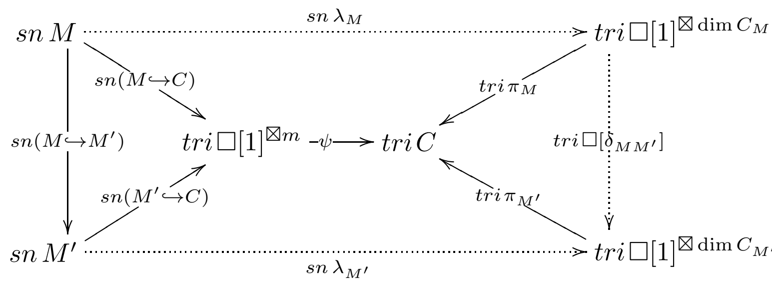
Proof. Fix cubical set C. It suffices to show that the natural cubical function

$$\int_{\Box}^{[1]^{\boxtimes n}} C[1]^{\boxtimes n} \cdot qua\, tri\, \Box[1]^{\boxtimes n} \to qua\, tri\, C,$$

monic because *qua*, *tri*, and hence the composite $qua \circ tri$ are $\bigcap$-continuous, is also epi.

Fix m and m-cube $\psi : tri\, \Box[1]^{\boxtimes m} \to tri\, C$ in $qua\, tri\, C$. Let $\mathscr{G}$ be the subcategory of $\mathscr{Q}$ consisting of all maximal chains of $[1]^{\boxtimes m}$ and inclusions between them.

Consider $\mathscr{G}$-object M. Let σ_M and k_M be the unique isomorphism and natural number such that $\sigma_M : [k_M] \cong M$. The cubical set $C_M = supp_{\,tri}(\psi(sn\, M))$ is atomic by Lemma 2.8. Thus we can let $\pi_M = (C_M \hookrightarrow C)\varrho_{C_M}$. There exists monotone function λ_M, extrema-preserving by minimality of C_M and Lemma 6.2, making the top trapezoid in the diagram below commute because $sn\, M$ is projective and sn is full.



Consider another $\mathscr{G}$-object $M' \supset M$. There exists injective $\square$-morphism $\delta_{MM'}$ such that the right triangle commute because $\square[1]^{\boxtimes \dim C_M}$ is projective and $C_M \subset C_{M'}$. The monotone function $\lambda_{M'}\sigma_M$ preserves extrema and hence is not a k_M-simplex of $tri\,\partial\square[1]^{\boxtimes \dim M'}$ by Lemma 6.2. Thus $(tri\,\pi_{M'})_{[k_M]}(\lambda_{M'}\sigma_M)$ has unique preimage under $(tri\,\pi_{M'})_{[k_M]}$. Thus $\delta_{MM'}\lambda_M\sigma = \lambda_{M'}\sigma_M$ because

$$(tri\,\pi_{M'})_{[k_M]}(\delta_{MM'}\lambda_M\sigma_M) = (tri\,\pi_{M'})_{[k_M]}(\lambda_{M'}\sigma_M)$$

by the commutativity of the trapezoids. We conclude $\delta_{MM'}$ preserves extrema because $\lambda_M, \lambda_{M'}$, $M \hookrightarrow M'$ preserve extrema. Thus $\delta_{MM'} = id_{\square[1]^{\boxtimes \dim C_M}}$ by Lemma 6.2 and we can therefore let

$$n = \dim C_M = \dim C_{M'}, \quad \theta = \pi_M = \pi_{M'}.$$

Our constructions of the form $sn\,\lambda_*$ define a cocone $\Lambda : sn_{\restriction\mathscr{G}} \to tri\;\square[1]^{\boxtimes n}$. The cocone $\mathrm{I} : sn_{\restriction\mathscr{G}} \to tri\;\square[1]^{\boxtimes m}$ defined by inclusions is universal. Thus there exists simplicial function $\sigma : tri\;\square[1]^{\boxtimes m} \to tri\;\square[1]^{\boxtimes n}$ such that $\Lambda = \sigma\mathrm{I}$. Thus $\psi\mathrm{I} = (tri\;\theta)\Lambda = (tri\;\theta)\sigma\mathrm{I}$ by the commutativity of the diagram. We conclude $\psi = (tri\,\theta)\sigma$. $\square$

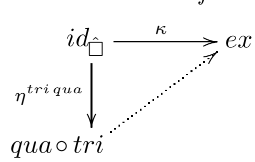
Triangulation and quadrangulation relate our various cubical and simplicial constructions with one another.

Lemma 7.2 ([18, Example 5]). *There exists a natural isomorphism*

$$cn \cong qua \circ sn : \mathscr{Q} \to \hat{\square}.$$

Consequently, we can regard $qua(tri\,C)$ as a particular stage in the extension from a cubical set C to $ex\,C$.

Lemma 7.3. *There exists dotted natural transformation making the diagram*



commute.

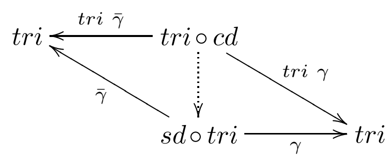
Proof. We can make the identification

$$qua \circ tri \cong \int_{\square}^{[1]^{\boxtimes n}} -[1]^{\boxtimes n} \cdot cn[1]^{\boxtimes n}$$

by Lemmas 7.1 and 7.2. Our desired dotted natural transformation is then induced from inclusion $\square \hookrightarrow \boxplus$. $\square$

Triangulation relates our different subdivision operators.

Proposition 7.4. *There exists a dotted natural isomorphism making*



commute.

Proof. The simplicial function $\hat{\tau}_{[m][n]} : \boxplus([m],[2]\otimes[n])\cdot\Delta[m] \to sd\,\Delta[n]$ defined by

$$(\hat{\tau}_{[m][n]}(\alpha : [m] \to [2n] \cdot \beta : [k] \to [m]))_{[k]} = \gamma\alpha\beta \oplus \bar{\gamma}\alpha\beta, \quad k = 0, 1, \ldots$$

is dinatural in $\square_1$-objects $[m]$ and hence induces a simplicial function

$$\tau_{[n]} : tri\,cd\square[n] \to sd\,\Delta[n],$$

epi by Lemma 6.11, monic by Lemma 6.12, and natural in $\square_1$-objects $[n]$. Plugging in γ and $\bar{\gamma}$ for the symbol η, we see that $\eta_{tri\square[n]}\tau_{[n]} = tri\,\eta_{\square[n]}$ because

$$\begin{aligned}(\eta_{\Delta[n]}(\hat{\tau}_{[m][n]}(\alpha : [m] \to [2n] \cdot \beta : [k] \to [m])))_{[k]} &= (\eta_{\Delta[n]})_{[k]}(\gamma_{[n]}\alpha\beta \oplus \bar{\gamma}_{[n]}\alpha\beta) \\ &= (\gamma_{[n]}\alpha\beta \oplus \bar{\gamma}_{[n]}\alpha\beta)\eta_{[n]} \\ &= \eta_{[n]}\alpha\beta.\end{aligned}$$

for $n = 0, 1$. Thus the desired natural isomorphism exists because *tri*, *cd*, γ, and $\bar{\gamma}$ are monoidal. □

Triangulation restricts and corestricts to an isomorphism between 1-dimensional cubical sets and 1-dimensional simplicial sets. Thus we can make the identification

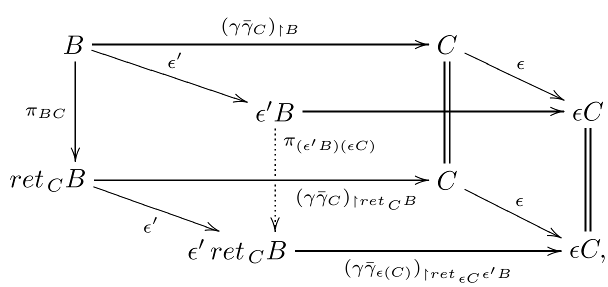
$$tri\ ret_C B = ret_{tri\,C}\, tri\, B$$

for all 1-dimensional cubical sets C and $B \subset cd^2 C$. We can now bootstrap Lemma 5.8 to give a proof for Lemma 6.15.

Proof of Lemma 6.15. We assume C is atomic without loss of generality.

The case $C = \square[1]$ and hence also the case $C = \square[0]$ follow from Lemma 5.8 and Proposition 7.4.

Consider epi ϵ from a representable cubical set C and atomic $B \subset cd^2 C$. Let $\epsilon' = cd^2\epsilon$; we abuse notation and also write ϵ' for its restrictions and corestrictions. There exists unique retraction $\pi_{BC} : B \to ret_C B$, and the back face of



commutes by the previous cases and Lemma 6.1. Moreover,

$$\epsilon'\, ret_C B = ret_{\epsilon C}\epsilon' B$$

because ϵ' is epi and $\epsilon\, supp_{\,cd^2}(A,C) = supp_{\,cd^2}(\epsilon' A, epsilonC)$ for all $A \subset B$. It therefore suffices to show the existence of a dotted cubical function making the left face commute; it would follow that the front face commutes because ϵ' is a quotient cubical function and the rest of the diagram commutes.

We induct on $\dim C$. The base case $C \cong \square[0]$ follows from the previous case. Fix k. Assume the claim holds for the case $\dim C < k$, and consider the case $\dim C = k$. Consider n and n-cube θ in B. It suffices to show $\epsilon'\pi_{BC}\theta$ only depends on $\epsilon'\theta$ and therefore it suffices to consider the case θ not the unique preimage of $\epsilon'_{[1]^{\boxtimes n}}\theta$ under $\epsilon'_{[1]^{\boxtimes n}}$; in particular, there exists atomic $B' \subset B$ and proper atomic $C' \subset C$ such that θ is an n-cube of $B' \cap cd^2C'$. Then

$$
\begin{aligned}
(5)\qquad (\epsilon'\pi_{BC})_{[1]^{\boxtimes n}}(\theta) &= ((\epsilon'\pi_{BC})_{\restriction B'})_{[1]^{\boxtimes n}}(\theta) \\
(6)\qquad &= (\epsilon'\pi_{B'C'})_{[1]^{\boxtimes n}}(\theta) \\
(7)\qquad &= (\pi_{\epsilon' B' \epsilon C'})_{[1]^{\boxtimes n}}\epsilon'_{[1]^{\boxtimes n}}\theta,
\end{aligned}
$$

(5) by our choice of B', (6) by minimality and hence equality $\pi_{B'C'} = (\pi_{BC})_{\restriction B'}$, and (7) by the inductive hypothesis. $\square$

Triangulation relates our different stream realization functors.

Proposition 7.5. *The following commutes up to natural isomorphism.*

$$
\begin{array}{ccc}
\hat{\square} & \xrightarrow{\ \upharpoonleft - \downharpoonleft\ } & \mathscr{S} \\
 & \searrow^{tri} & \uparrow{\scriptstyle \upharpoonleft - \downharpoonleft} \\
 & & \hat{\Delta}
\end{array}
$$

Proof. It suffices to show that there exists a natural isomorphism

$$(8)\qquad \vec{\nabla}_{\restriction \square_1} \cong \vec{\square}_{\restriction \square_1} : \square_1 \to \mathscr{S}$$

because $\vec{\square}$, tri, $\upharpoonleft - \downharpoonleft : \hat{\square} \to \mathscr{S}$, and $\upharpoonleft - \downharpoonleft : \hat{\Delta} \to \mathscr{S}$ are monoidal [Lemma 5.11] and colimits commute with tensor products in $\hat{\square}$ and finite products in $\mathscr{S}$ [Theorem 4.10].

Both $\Gamma\vec{\square}[\delta]$ and $\Gamma\vec{\nabla}[\delta]$ send the unique point to the minimum for $\delta = \delta_-$ because

$$0 = \vec{\square}[\delta_-](0), \quad |0| \vee_{\vec{\nabla}[1]} t = t$$

for $t = |0|, |1|$ and hence all $t \in \mathbb{I}$ by linearity of $\vee_{\vec{\nabla}[1]}$ and similarly send the unique point to the maximum for the case $\delta = \delta_+$. The linear homeomorphism $\nabla[1] \to \mathbb{I}$ sending $|0|$ to 0 and $|1|$ to 1 hence defines the [1]-component of our desired natural isomorphism (8) by linearity of $\vee_{\vec{\nabla}[1]}$, $\vee_{\vec{\square}[1]}$ and Theorem 4.11. $\square$

We can now prove that the global preorders of stream realizations of cubical sets have closed graphs.

Proof of Lemma 6.27. Lemma 5.10 and Proposition 7.5 give the result. $\square$

We abuse notation and also write φ for a natural isomorphism

$$\varphi : \upharpoonleft cd(-) \downharpoonleft \cong \upharpoonleft - \downharpoonleft : \hat{\square} \to \mathscr{S}.$$

induced from $\varphi : \upharpoonleft sd(-) \downharpoonleft \cong \upharpoonleft - \downharpoonleft$ and natural isomorphisms claimed in Propositions 7.4 and 7.5. We can now prove that stream realization sends monic cubical functions to stream embeddings.

Proof of Proposition 6.28. Fix cubical sets $B \subset C$. Consider finite $A \subset B$. Let

$$V_n = \bigcup_{a \in A[0]} star_{\,cd^n B}\{a\}, \quad n = 0, 1, ...$$

Consider an open subset $U \subset |A|$. It suffices to show that

$$graph(\leqslant_U) \supset U^2 \cap \bigcap_{n=0}^{\infty} graph(\leqslant_{U \cap \varphi^n V_n}).$$

We could then conclude that for each stream embedding $k : K \hookrightarrow \upharpoonleft B \downharpoonleft$ such that K is compact Hausdorff, $\upharpoonleft supp_{|-|}(k(K), B) \hookrightarrow B \downharpoonleft$ is a stream embedding by Lemma 4.9 and thus conclude that $\upharpoonleft B \hookrightarrow C \downharpoonleft_{\upharpoonright K}$ is a stream embedding.

Consider $x, y \in |A|$ such that $x \nleqslant_U y$. For $n \gg 0$, $\varphi_A^{-n} x \nleqslant_{\varphi_A^{-n} U} \varphi^{-n} y$, hence

$$|\gamma\bar{\gamma}_{cd^n A}|(\varphi_A^{-n-2} x) \nleqslant_{\varphi_A^{-n} U} |\gamma\bar{\gamma}_{cd^n A}|(\varphi_A^{-n-2} y)$$

by Lemma 6.27 because $\varphi_A^n |\gamma\bar{\gamma}_{cd^n A}| \varphi_A^{-n-2}(*)$ is close to $*$ for each $*$, hence

$$\varphi_A^{-n-2} x \nleqslant_{(\varphi_A^{-n-2} U) \cap V_{n+2}} \varphi_A^{-n-2} y$$

because $\gamma\bar{\gamma}_{cd^n C}(\varphi_A^{-n-2} U) \cap V_{n+2} \subset |\, cd^n A|$ by Lemma 6.19, and hence

$$x \nleqslant_{U \cap \varphi_A^{n+2} V_{n+2}} y.$$

$\square$

8. Homotopy theories

We present categorical definitions of homotopy theory in §8.1; construct weak homotopy categories of streams in §8.3 and cubical sets in §8.4; and state and prove our main results in §8.5.

8.1. **Homotopies.** We recall categorical axiomatizations [18] of cylinder objects, equip categories of streams, simplicial sets, and cubical sets with such axiomatic structure, and explore the strengths of the resulting homotopy theories. Cylinder objects in practice amount to monoidal actions of $\square$. We regard functor categories of the form $\mathscr{C}^{\mathscr{C}}$ throughout as monoidal categories whose tensors are defined by composition.

Definition 8.1. A $\square$-*module* is a category $\mathscr{C}$ implicitly equipped with functor

$$\otimes : \mathscr{C} \times \square \to \mathscr{C}$$

whose adjoint $\square \to \mathscr{C}^{\mathscr{C}}$ is monoidal.

We spell out resulting definitions of homotopies and fibrations.

Definition 8.2. Fix a $\square$-module $\mathscr{C}$. Consider $\mathscr{C}$-morphisms $\alpha, \beta : x \to y$. A *homotopy* from α to β is a morphism η in $\mathscr{C}$ making the following commute.

$$\begin{array}{ccc} x \coprod_{\mathscr{C}} x \cong x \otimes [0] \coprod_{\mathscr{C}} [0] x \otimes [0] & \xrightarrow{\alpha, \beta} & y \\ \Big\downarrow {\scriptstyle -\otimes\delta_- \coprod -\otimes\delta_+} & \overset{\eta}{\nearrow} & \\ x \otimes [1] & & \end{array} \tag{9}$$

For each $\mathscr{C}$-object g, a g-*fibration* is a $\mathscr{C}$-morphism satisfying the right lifting property with respect to $g \otimes \delta_-$ and $g \otimes \delta_+$.

Fix $\square$-module $\mathscr{C}$. We write $\alpha \rightsquigarrow \beta$ if there exists a homotopy from α to β for each pair α, β of parallel $\mathscr{C}$-morphisms. We write $\leftrightsquigarrow$ for the equivalence, and hence congruence, on $\mathscr{C}$, generated by $\rightsquigarrow$ and $h\mathscr{C}$ for the quotient category $\mathscr{C}/\leftrightsquigarrow$.

Example 8.3. Regarding $\mathscr{T}$ as a $\square$-module such that

$$\otimes = - \times \square[-] : \mathscr{T} \times \square \to \mathscr{T},$$

$h\mathscr{C}$ is the standard homotopy category of $\mathscr{C}$.

We highlight a couple of straightforward properties of fibrations, useful for later showing that diagram categories of streams are categories of fibrant objects.

Lemma 8.4. *Fix $\square$-module $\mathscr{C}$. For each $\mathscr{C}$-object g and g-fibration γ,*

$$\mathscr{C}(g, \gamma)$$

is surjective if $h\mathscr{C}(g, \gamma)$ is surjective.

Lemma 8.5. *Fix $\square$-module $\mathscr{C}$. For each $\mathscr{C}$-object g and g-fibration γ,*

$$h\mathscr{C}(g, \gamma)$$

is injective.

We axiomatize homotopical triviality as a generalization of convex structure on topological vector spaces.

Definition 8.6. Fix $\square$-module $\mathscr{C}$. A $\mathscr{C}$-object c is *convex* if both projections

$$c^2 \to c$$

exist and are $\leftrightsquigarrow$-equivalent to one another.

Standard facts about classical convex objects straighforwardly generalize.

Lemma 8.7. *Parallel morphisms to convex objects in $\square$-modules are $\leftrightsquigarrow$-equivalent.*

A functor suitably respecting monoidal actions of $\square$ respects the associated homotopy theories.

Definition 8.8. For all $\square$-modules $\mathscr{C}$ and $\mathscr{D}$, a *lax $\square$-module map*

$$F : \mathscr{C} \to \mathscr{D}$$

is a functor $F : \mathscr{C} \to \mathscr{D}$ equipped with natural transformation $\otimes \circ (F \times \square) \to F \circ \otimes$, which we write as η, such that $\eta_{(c,[0])}$ is an isomorphism for each $\mathscr{C}$-object c.

Lemma 8.9. *Consider lax $\square$-module map F of the form*

$$F : \mathscr{C} \to \mathscr{D}$$

preserving cosquares. The functor F passes to a functor $h\mathscr{C} \to h\mathscr{D}$. If F is lax Cartesian monoidal, then F preserves convex objects.

A proof is straightforward and therefore left to the reader.

8.2. **Homotopy for simplicial sets.** Our starting point in investigating homotopy theories of "directed structures" is the observation that "simplicial lattices" are convex. We henceforth regard functor categories of the form $\hat{\Delta}^{\mathscr{C}}$ as $\square$-modules equipped with $(-\times sn)_{\upharpoonright\hat{\Delta}^{\mathscr{C}}\times\square}$; the resulting (strong) homotopy theory is standard [4, **?**]. All homotopical constructions in this paper originate from the following observation.

Lemma 8.10. *The $\mathscr{L}$-simplicial set $sn_{\upharpoonright\mathscr{L}}$ is convex.*

Proof. The unique functions $\eta_L,\eta'_L:[1]\times L^2\to L$ defined so that $\eta_L(0,-)$, $\eta'_L(0,-)$ respectively are projections onto first and second factors and $\eta_L(1,-)=\eta'_L(1,-)=\wedge_L$ are monotone for each lattice L. Then $sn\,\eta_L$ and $sn\,\eta'_L$ define homotopies from projections $(sn\,L)^2\to sn\,L$ to $sn\wedge_L$. $\square$

8.3. **Homotopy for streams.** We introduce a homotopy theory that intuitively classifies streams up to deformation. We henceforth regard functor categories of the form $\mathscr{S}^{\mathscr{C}}$ as $\square$-modules equipped with $-\times_{\mathscr{S}}\vec{\square}[-]$. The convexity of "simplicial lattices" [Lemma 8.10] implies the convexity of topological lattices.

Lemma 8.11. *The Δ-stream $\vec{\nabla}[-]$ and the $\square$-stream $\vec{\square}$ are convex.*

Proof. There exist natural isomorphisms Δ-stream $\vec{\nabla}[-]\cong\upharpoonleft sn(-)\downharpoonleft_{\upharpoonright\Delta}$ and $\vec{\square}[-]\cong\upharpoonleft sn(-)\downharpoonleft_{\upharpoonright\square}$ by Proposition 7.5. It therefore suffices to note that the $\mathscr{L}$-stream $\upharpoonleft sn(-)\downharpoonleft_{\upharpoonright\mathscr{L}}$, and hence all restrictions of it, are convex by Lemma 8.9 because $\upharpoonleft-\downharpoonleft:\hat{\Delta}\to\mathscr{S}$ is lax Cartesian monoidal by Lemma 5.11. $\square$

We can now give a special case of simplicial approximation, for the natural isomorphisms $\varphi_C:\upharpoonleft cd\,C\downharpoonleft\cong\upharpoonleft C\downharpoonleft$.

Proposition 8.12. *The following $\hat{\Delta}$-stream maps are $\leftrightsquigarrow$-equivalent.*

$$\varphi,\ \upharpoonleft\gamma\downharpoonleft,\ \upharpoonleft\bar{\gamma}\downharpoonleft:\upharpoonleft sd(-)\downharpoonleft\to\upharpoonleft-\downharpoonleft\,.$$

Proof. The $(\hat{\Delta}\times\Delta^{\mathrm{op}}\times\Delta)$-stream maps

$$\hat{\Delta}(\Delta[-_2],-_1)\cdot\varphi_{\Delta[-_3]},\quad\hat{\Delta}(\Delta[-_2],-_1)\cdot\upharpoonleft\gamma_{\Delta[-_3]}\downharpoonleft,\quad\hat{\Delta}(\Delta[-_2],-_1)\cdot\upharpoonleft\bar{\gamma}_{\Delta[-_3]}\downharpoonleft$$

are $\leftrightsquigarrow$-equivalent by Lemma 8.7 because $\vec{\nabla}[-]$ is convex by Lemma 8.11. We take parametrized coends to conclude the claim. $\square$

A topological enrichment [17] on diagram categories of topological catgories suggests an alternative homotopy relation [8] on $\mathscr{S}^{\mathscr{G}}$ generally weaker than $\leftrightsquigarrow$. We identify criteria for these homotopy relations to coincide.

Theorem 8.13. *Fix small $\mathscr{G}$. A pair of $\mathscr{G}$-stream maps*

$$f,g:X\to Y$$

are $\leftrightsquigarrow$-equivalent if the colimit of X is compact, Y is quadrangulable, and there exists $\mathscr{G}$-stream map $h:X\times\ddot{\square}[1]\to Y$ such that $h(-,0)=f$ and $h(-,1)=g$,

Proof. Let C be a cubical set; it suffices to suppose $Y=\upharpoonleft cd^4C\downharpoonleft$. Let $\varphi'=\upharpoonleft(\gamma\bar{\gamma})^2\downharpoonleft$. We write $X_U(c)$ for the open substream of $X(c)$ consisting of the preimage of an open subset U of $colim\,X$ under the natural stream map $X(c)\to colim\,X$ for each $\mathscr{G}$-object c.

There exists finite category $\mathscr{O}$ of open subsets covering $colim\ X$ and all inclusions betwen them, k, and real numbers $0=t_0<t_1<t_2<\cdots<t_{k-1}<t_k=1$ such

that $(colim\ h)(U \times [t_i, t_{i+1}])$ inhabits a set of the form $star_{\ colim\ Y}(*)$ for each $\mathscr{O}$-object U and $i = 0, 1, 2, \ldots, k-1$ because $colim\, X \times \ddot{\square}[1]$ is compact. It suffices to consider $k = 1$ because $\leftrightsquigarrow$ is transitive.

There exists minimal set of the form $star_{\ C(c)}(*)$ containing $(colim\ h)(U)$ because sets of the form $star_{\ C(c)}(*)$ are closed under intersections, hence minimal atomic subobject of $C(c)$ whose geometric realization contains the image of such a minimal set under $|\gamma\bar{\gamma}_{C(c)}|$ by Lemma 6.19, hence factorizations of

$$\varphi'_{C(c)}(f_c)_{\upharpoonright X_U(c)} \coprod \varphi'_{C(c)}(g_c)_{\upharpoonright X_U(c)},$$

as the composite of a stream map to a stream of the form $\upharpoonleft\square[*]\downharpoonleft$ followed by a stream map of the form $\upharpoonleft * \to C(c)\downharpoonleft$ by Lemmas 6.15 and 6.16, natural in $\mathscr{O}$-object U by Lemma 6.17 and $\mathscr{G}$-object c by Lemma 6.18. Thus $\varphi'_C f \leftrightsquigarrow \varphi'_C g$ by naturality and convexity of $\vec{\square}$. Hence $f \leftrightsquigarrow g$ by Proposition 8.12. $\square$

We generalize homotopy equivalences between compact quadrangulable streams to *weak equivalences* between general streams as follows.

Definition 8.14. Fix small $\mathscr{G}$. A $\mathscr{G}$-stream map f is a *weak equivalence* if

$$h\mathscr{S}^{\mathscr{G}}(Q, f)$$

is a bijection and a *fibration* if it is an Q-fibration, for each compact quadrangulable $\mathscr{G}$-stream Q.

Example 8.15. Maps representing isomorphisms in $h\mathscr{S}^{\mathscr{G}}$ are weak equivalences.

We write $\mathscr{W}_{\mathscr{S}^{\mathscr{G}}}$ for the subcategory of $\mathscr{S}^{\mathscr{G}}$ consisting of the weak equivalences. A standard compactness argument yields the following observation.

Proposition 8.16. *For each $\mathscr{G}$, $\mathscr{W}_{\mathscr{S}^{\mathscr{G}}}$ is closed under transfinite pushouts in $\mathscr{S}^{\mathscr{G}}$.*

For each small $\mathscr{G}$, we write $\bar{h}\mathscr{S}^{\mathscr{G}}$ for the localization of $\mathscr{S}^{\mathscr{G}}$ with respect to the weak equivalences.

Proposition 8.17. *For each small $\mathscr{G}$, $\mathscr{S}^{\mathscr{G}}$ is a category of fibrant objects.*

Proof. The diagram category $\mathscr{S}^{\mathscr{G}}$ has finite products and a terminal object because $\mathscr{S}$ is complete by Proposition 4.6. Pullbacks of (acyclic) fibrations are (acyclic) fibrations by (Lemma 8.5 and) the preservation of right lifting properties by pullbacks. Every $\mathscr{G}$-stream is fibrant because δ_-, δ_+, and hence $\mathscr{G}$-stream maps of the form $Q \otimes \delta_-$, $Q \otimes \delta_+$, have retractions. Path-objects exist by Proposition 8.16 and a Quillen small object argument. $\square$

Corollary 8.18. *For each small $\mathscr{G}$, $\bar{h}\mathscr{S}^{\mathscr{G}}$ is locally small.*

Corollary 8.19. *For each small $\mathscr{G}$, there exists a bijection*

$$\bar{h}\mathscr{S}^{\mathscr{G}}(X, Y) = h\mathscr{S}^{\mathscr{G}}(X, Y)$$

natural in $\mathscr{G}$-streams X having compact colimits and quadrangulable $\mathscr{G}$-streams Y.

8.4. **Homotopy for cubical sets.** We henceforth regard functor categories of the form $\hat{\square}^{\mathscr{C}}$ as $\square$-modules equipped with $- \times \vec{\square}$; the resulting (strong) homotopy theory is standard [18]. A couple of our cubical constructions preserve homotopy types.

Lemma 8.20. *The $\hat{\square}$-simplicial functions*

$$tri\,\eta^{tri\,qua} : tri \leftrightarrows tri \circ qua \circ tri : \epsilon^{tri\,qua}\,tri$$

represent mutually inverse morphisms in $h\hat{\Delta}^{\hat{\square}}$.

Proof. The $(\hat{\square} \times \square^{\mathrm{op}} \times \square)$-simplicial functions

$$\hat{\square}(\square[-_2], -_1) \cdot tri\,\eta^{tri\,qua}\square[-_3], \quad \hat{\square}(\square[-_2], -_1) \cdot \epsilon^{tri\,qua}\,tri\,\square[-_3]$$

represent mutually inverse morphisms in $h\hat{\Delta}^{\hat{\square} \times \square^{\mathrm{op}} \times \square}$ by Lemma 8.7 because $tri\,\square[-] = sn_{\restriction \square}$ and $tri \circ qua \circ tri\,\square[-]$ are convex by Lemma 8.10 and Proposition 8.9. We take parametrized coends to conclude the lemma. $\square$

Lemma 8.21. *The following $\hat{\square}$-cubical functions are $\leftrightsquigarrow$-equivalent.*

$$\kappa_{cd(-)},\ \nu_{cd(-)}\eta^{cd\,ex}\gamma :\ cd \to ex \circ cd.$$

Proof. The $(\hat{\square} \times \boxbar^{\mathrm{op}} \times \boxbar)$-cubical functions

$$\hat{\square}(\square[-_2], -_1) \cdot \kappa_{cd\,\square[-_3]_{\restriction \boxbar}}, \quad \hat{\square}(\square[-_2], -_1) \cdot \nu_{cd(-)}\eta^{cd\,ex}\gamma_{\square[-_3]_{\restriction \boxbar}})$$

are $\leftrightsquigarrow$-equivalent by Lemma 8.7 because $cn_{\restriction \boxbar}$ is convex by Lemma 8.10 and Proposition 8.9. We take parametrized coends to conclude the lemma. $\square$

We generalize homotopy equivalences between finite cubical sets admitting cubical compositions to weak equivalences between general cubical sets.

Definition 8.22. Fix $\mathscr{G}$. For each $\mathscr{G}$-cubical set C, a *cubical composition*

$$ex\,C \to C$$

on C is a cubical function ϵ_C such that $id_C \leftrightsquigarrow \epsilon_C \kappa_C$.

Fix $\mathscr{G}$ and $\mathscr{G}$-cubical set E. We write $[C, E]$ for the limit of the diagram whose arrows are all functions $h\hat{\square}^{\mathscr{G}}(B, E) \to h\hat{\square}^{\mathscr{G}}(A, E)$ induced from all possible inclusions $A \hookrightarrow B$ between finite subobjects of C for each $\mathscr{G}$-cubical set C. We write $[\psi, E]$ for the induced function $[D, E] \to [C, E]$ for each cubical function $\psi : C \to D$.

Definition 8.23. Fix small $\mathscr{G}$. A $\mathscr{G}$-cubical function ψ is a *weak equivalence* if

$$[\psi, C]$$

is a bijection for all $\mathscr{G}$-cubical sets C admitting cubical compositions.

For each small $\mathscr{G}$, we write $\bar{h}\hat{\square}^{\mathscr{G}}$ for the localization of $\hat{\square}^{\mathscr{G}}$ with respect to the weak equivalences; we show in the next section that $\bar{h}\hat{\square}^{\mathscr{G}}$ is in fact locally small.

8.5. **An equivalence.** We establish a directed analogue of the classical equivalence between homotopy categories of cubical sets and topological spaces. We first prove that stream realizations of "directed anydone extensions" are homotopy equivalences [Proposition 8.24]. We then show that certain stream maps between stream realizations admit cubical approximations [Corollary 8.29]. We then conclude our main results.

Proposition 8.24. *The $\hat{\square}$-stream map $\rceil\kappa\lfloor$ represents an isomorphism in $h\mathscr{S}^{\hat{\square}}$.*

Proof. We identify cn with $qua \circ sn$ by Lemma 7.2. We regard $\rceil\square[\mathfrak{p}]\lfloor$ as a convex compact Hausdorff, connected sublattice of an ordered topological vector space because $\varphi_{\square[n]}$ and $id_{\rceil\square[n]\lfloor}$ induce isomorphisms of the form $\varphi'_{\mathfrak{p}} : \rceil\square[\mathfrak{p}]\lfloor \cong \vec{\square}[*]$, monoidal and natural in $\boxbar$-objects $\mathfrak{p}$, for $n = 0, 1$. We can thus let $\eta_{\mathfrak{p}} : \rceil sn\,\mathfrak{p}\lfloor \rightarrow \rceil\square[\mathfrak{p}]\lfloor$ be the piece-wise linear stream map, natural and monoidal in $\boxbar$-objects $\mathfrak{p}$ because all $\boxbar$-morphisms are composites of injections and projections, defined by $\eta_{\mathfrak{p}}|p| = |p|$ for all $p \in \mathfrak{p}$. The $\boxbar$-stream maps $\rceil\kappa_{\square[-]_{\restriction\boxbar}}\lfloor : \rceil\square[-]\lfloor_{\restriction\boxbar} \rightarrow \rceil cn(-)\lfloor_{\restriction\boxbar}$ and $\eta\,\rceil\epsilon^{tri\,qua}\lfloor$ represent mutually inverse morphisms in $h\mathscr{S}^{\boxbar}$ because $\rceil cn(-)\lfloor_{\restriction\boxbar}$ and $\rceil\square[-]\lfloor_{\restriction\boxbar}$ are convex by Lemma 8.10 and Proposition 8.9. The $\hat{\square} \times \boxbar^{\mathrm{op}} \times \boxbar$-stream map

$$\hat{\square}(\square[-_2], -_1) \cdot \rceil\kappa_{\square[-]}\lfloor$$

therefore represents an isomorphism in $h\mathscr{S}^{\hat{\square}\times\boxbar^{\mathrm{op}}\times\boxbar}$. We take parametrized coends to conclude the result. $\square$

Corollary 8.25. *For each cubical set C, the continuous function*

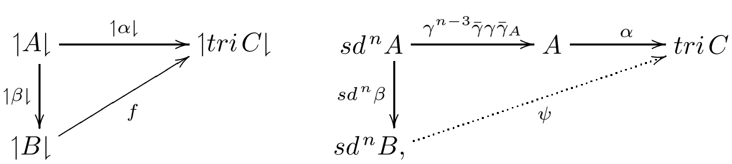
$$|\kappa_C| : |C| \rightarrow |\,ex\,C|$$

is a homotopy equivalence of spaces.

Corollary 8.26. *The $\mathscr{S}$-cubical set sing admits a $\mathscr{S}$-cubical composition.*

Theorem 8.27. *Consider small $\mathscr{G}$ and commutative diagram on the left side of*

(10)



where α, β are $\mathscr{G}$-simplicial functions, B is finite, and C is a $\mathscr{G}$-cubical set. For $n \gg 0$, there exist dotted $\mathscr{G}$-simplicial function ψ such that the right side commutes and $\rceil\psi\lfloor \rightsquigarrow f\varphi^n_B$.

Proof. We abbreviate $supp(*,*)$ for $supp_{\,|-|}(*,*)$. Let $\varphi' = \rceil(\gamma\bar{\gamma})^2\lfloor\,\varphi^{-4}$. Let d denote a $\mathscr{G}$-object d.

There exists minimal set of the form $star_{\,C(d)}(*)$ containing f_dU because sets of the form $star_{\,C(d)}(*)$ are closed under intersections, hence minimal atomic subobject of $C(d)$ whose geometric realization contains the image of such a minimal set under $|\gamma\bar{\gamma}_{C(d)}|$ by Lemma 6.19, hence stream map $g_d(\sigma) : \rceil\Delta[m]\lfloor \rightarrow \rceil tri\,\square[1]^{\boxtimes n}\lfloor$ and cubical function $\theta_d(\sigma) : \square[1]^{\boxtimes n} \rightarrow C$, natural in d by Lemma 6.18 and $(\Delta \downarrow sd^{\,n}B(d))$-objects σ by Lemma 6.17, such that

$$\varphi'_C f_d\,\rceil\sigma\lfloor = \rceil tri\,\theta_d(\sigma)\lfloor\,g_d(\sigma)$$

by Lemmas 6.15 and 6.16. The function $\phi_d(\sigma) : [m] \to [1]^{\boxtimes n}$ defined by

$$\phi_d(\sigma)(v) = \min supp_{|-|}(g_d(\sigma)|v|, tri\,\square[1]^{\boxtimes n}),$$

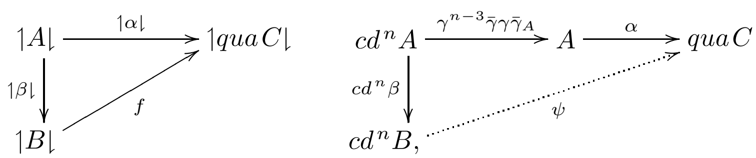
natural in d and $(\Delta \downarrow sd^n B(d))$-objects σ, satisfies

$$\begin{aligned}(tri\,\theta_d(\sigma))_{[0]}(\phi_d(\sigma)(v)) &= tri\,\theta_d(\sigma_{[0]} \min supp(g_d(\sigma)|v|, tri\,\square[1]^{\boxtimes n})\\ &= \min supp(\varphi'_C f \varphi^n_B |\sigma_{[0]} v|, tri\, C)\\ &= \min supp(f \varphi^{n-4}_B \upharpoonleft (\gamma\bar{\gamma})^2 \downharpoonleft |\sigma_{[0]} v|, tri\, C)\\ &= \min supp(f |\gamma^{n-4}\gamma\bar{\gamma}\gamma\bar{\gamma}||\sigma_{[0]} v|, tri\, C)\\ &= \min supp(|\alpha\gamma^{n-3}\bar{\gamma}\gamma\bar{\gamma}\sigma(v)|, tri\, C)\\ &= \alpha\gamma^{n-3}\bar{\gamma}\gamma\bar{\gamma}\sigma(v)\end{aligned}$$

for the case σ is of the form $\beta \circ (\Delta[n] \to A(d))$ and is monotone by Lemma 5.12. Thus the $sn\,\phi_d(\sigma)$'s induce $\mathscr{G}$-simplicial function $\psi : sd^n B \to tri\, C$ such that the right side of (10) commutes and $f\varphi^n_B \leftrightsquigarrow \upharpoonleft\psi\downharpoonleft$ by naturality and convexity of $\vec{\square}$. $\square$

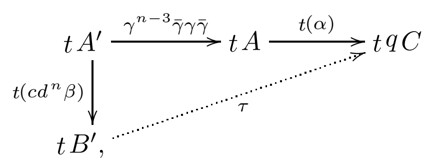
Corollary 8.28. *Consider small $\mathscr{G}$ and commutative diagram on the left side of*

(11)



where α, β are $\mathscr{G}$-cubical functions, B is finite, and C is a $\mathscr{G}$-simplicial set. For $n \gg 0$, there exist dotted $\mathscr{G}$-cubical function ψ such that the right side commutes and $\upharpoonleft\psi\downharpoonleft \leftrightsquigarrow f\varphi^n_B$.

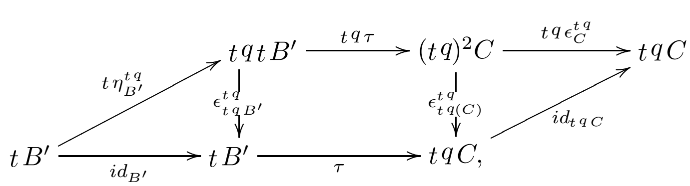
Proof. Let $t = tri$, $q = qua$, $A' = cd^n A$, $B' = cd^n B$. There exists τ such that



commutes and $\upharpoonleft\tau\downharpoonleft \leftrightsquigarrow f\varphi^n_B$ by Theorem 8.27 and Proposition 7.4. Then

$$\psi = q\,\epsilon^{t\,q}_C\, q\,\tau\eta^{t\,q}_{B'} : B' \to q\, C$$

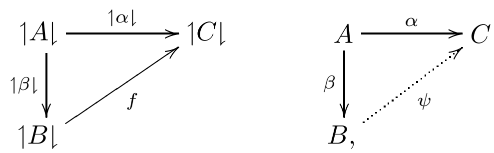
makes the right side of (13) commute by naturality. Inside



the left triangle commutes by zig-zag identities, the middle square commutes by naturality, and the right triangle commutes up to $\leftrightsquigarrow$ because $t\,q\,\epsilon^{t\,q}_C \leftrightsquigarrow t\,q\,\epsilon^{t\,q}_C\, t\,\eta^{t\,q}_{q\,C}\epsilon^{t\,q}_{t\,q\,C}$ by Lemma 8.20 and $t\,q\,\epsilon^{t\,q}_C\, t\,\eta^{t\,q}_{q\,C}\epsilon^{t\,q}_{t\,q\,C} = \epsilon^{t\,q}_{t\,q\,C}$ by the zig-zag identities. We conclude $t\,\psi \leftrightsquigarrow \tau$. Hence $\upharpoonleft t\,\psi\downharpoonleft \leftrightsquigarrow \upharpoonleft\tau\downharpoonleft \leftrightsquigarrow f\varphi^n_B$ because $\upharpoonleft - \downharpoonleft$ is a $\square$-module map by Lemma 8.9. $\square$

Corollary 8.29. *Consider small $\mathscr{G}$ and commutative diagram on the left side of*

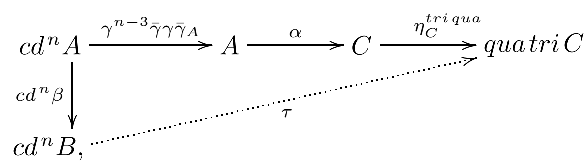
(12)



where α, β are $\mathscr{G}$-cubical functions, B is finite, and C admits a cubical composition. There exist dotted $\mathscr{G}$-cubical function ψ such that the right side commutes up to $\leftrightsquigarrow$ and $\upharpoonleft\psi\downharpoonleft \leftrightsquigarrow f \upharpoonleft\gamma^n\downharpoonleft$.

Proof. There exists $n \gg 0$ and dotted cubical function τ such that

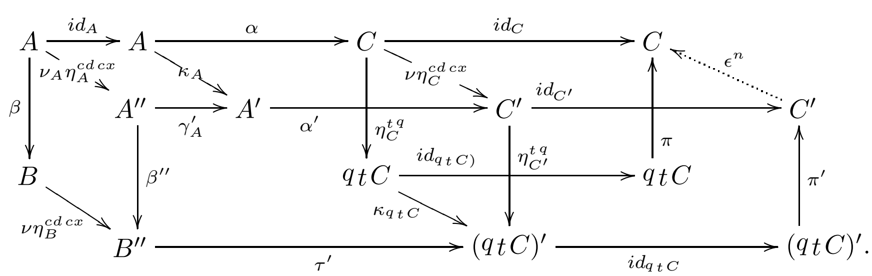
(13)



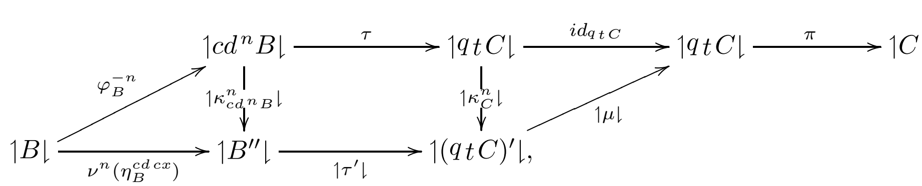
commutes and $\upharpoonleft\eta_C^{tq}\downharpoonleft f\varphi_B^n \leftrightsquigarrow \upharpoonleft\tau\downharpoonleft$ by Corollary 8.28. For brevity, we write

$$(*)'' = ex^n\, cd^n(*), \quad (*)' = ex^n(*), \quad qt = qua \circ tri, \quad \gamma' = \gamma^{n-3}\bar{\gamma}\gamma\bar{\gamma}.$$

Let ϵ be a retraction to C up to $\leftrightsquigarrow$. The cubical function $\eta_C^{qua\,tri}$ factors κ_C by Lemma 7.3 and thus admits a retraction π up to $\leftrightsquigarrow$. Let ψ be the appropriate composite $B \to C$ of the arrows in the diagram



The top left rectangle commutes by Lemma 6.21, the front left rectangle commutes by our choice of τ, the top right, back right, and front right rectangles commute up to $\leftrightsquigarrow$, the other solid rectangles commute by naturality, and therefore $\psi_{\upharpoonleft A} \leftrightsquigarrow \alpha$. In the diagram



the left triangle commutes up to $\leftrightsquigarrow$ by Lemma 8.21, the middle square commutes by naturality, the right triangle commutes up to $\leftrightsquigarrow$, and hence

$$\begin{aligned} \upharpoonleft\psi\downharpoonleft &\leftrightsquigarrow \upharpoonleft\pi\tau\downharpoonleft \varphi_B^{-n} \\ &\leftrightsquigarrow \upharpoonleft\pi\eta_C^{tri\,qua}\downharpoonleft f \\ &\leftrightsquigarrow f. \end{aligned}$$

$\square$

Corollary 8.30. *Fix small $\mathscr{G}$. The function $\upharpoonleft - \downharpoonleft_{B'C}$ passes to a function*

$$h\hat{\square}^{\mathscr{G}}(B', C) \to h\hat{\mathscr{S}}^{\mathscr{G}}(\upharpoonleft B' \downharpoonleft, \upharpoonleft C \downharpoonleft), \tag{14}$$

bijective if B' is finite and C admits a cubical composition, for all $\mathscr{G}$-cubical sets B' and C.

Proof. The function $\upharpoonleft - \downharpoonleft_{B'C}$ passes to our desired function (14) because $\upharpoonleft - \downharpoonleft$ is a lax $\square$-module map by Lemma 8.9. Surjectivity and injectivity follow from applying Corollary 8.30 to the respective cases $A = \varnothing$, $B = B'$ and $A = B' \coprod B'$, $B = B' \otimes [1]$. $\square$

Corollary 8.31. *Fix small $\mathscr{G}$. For all $\mathscr{G}$-streams X, the $\mathscr{G}$-stream map*

$$\epsilon_X^{\upharpoonleft - \downharpoonleft sing} : \upharpoonleft sing\, X \downharpoonleft \to X$$

is a weak equivalence.

Proof. For each finite $\mathscr{G}$-cubical set C and $\mathscr{G}$-stream X, the left vertical arrow is bijective [Corollaries 8.26 and 8.30] and hence the top horizontal arrow is bijective in the commutative diagram

$$\begin{array}{ccc}
h\mathscr{S}^{\mathscr{G}}(\upharpoonleft C \downharpoonleft, \upharpoonleft sing\, X \downharpoonleft) & \xrightarrow{h\mathscr{S}^{\mathscr{G}}(\upharpoonleft C \downharpoonleft, \epsilon_X^{\upharpoonleft - \downharpoonleft sing})} & h\mathscr{S}^{\mathscr{G}}(\upharpoonleft C \downharpoonleft, X) \\
\uparrow{\scriptstyle \upharpoonleft - \downharpoonleft_{C\, sing\, X}} & & \| \\
h\hat{\square}^{\mathscr{G}}(C, sing\, X) & \xrightarrow{\qquad} & h\mathscr{S}^{\mathscr{G}}(\upharpoonleft C \downharpoonleft, X)
\end{array}$$

whose bottom horizontal arrow is the bijection induced by the adjunction $\upharpoonleft - \downharpoonleft \vdash sing$. $\square$

Corollary 8.32. *Fix $\mathscr{G}$. For each $\mathscr{G}$-cubical set B, the $\mathscr{G}$-cubical function*

$$\eta_B^{\upharpoonleft - \downharpoonleft sing} : B \to sing \upharpoonleft B \downharpoonleft$$

is a weak equivalence.

Proof. For each $\mathscr{G}$-cubical set C admitting a cubical composition and finite $A \subset B$, the vertical arrows are bijective [Corollary 8.30], the bottom arrow is bijective [Corollary 8.31], and hence the top function is bijective in the following commutative diagram.

$$\begin{array}{ccc}
h\hat{\square}^{\mathscr{G}}(sing \upharpoonleft A \downharpoonleft, C) & \xrightarrow{h\hat{\square}^{\mathscr{G}}(\eta_A^{\upharpoonleft - \downharpoonleft sing}, C)} & h\hat{\square}^{\mathscr{G}}(A, C) \\
\downarrow{\scriptstyle \upharpoonleft\downharpoonleft_{sing \upharpoonleft A \downharpoonleft C}} & & \downarrow{\scriptstyle \upharpoonleft - \downharpoonleft_{AC}} \\
h\mathscr{S}^{\mathscr{G}}(\upharpoonleft sing \upharpoonleft A \downharpoonleft \downharpoonleft, \upharpoonleft C \downharpoonleft) & \xleftarrow[h\mathscr{S}^{\mathscr{G}}(\epsilon_A^{\upharpoonleft - \downharpoonleft sing}, \upharpoonleft C \downharpoonleft)]{} & h\mathscr{S}^{\mathscr{G}}(A, sing \upharpoonleft C \downharpoonleft)
\end{array}$$

$\square$

Corollary 8.33. *For each small $\mathscr{G}$, $\upharpoonleft - \downharpoonleft \dashv sing$ induces an equivalence*

$$\bar{h}\hat{\square}^{\mathscr{G}} \leftrightarrows \bar{h}\mathscr{S}^{\mathscr{G}}.$$

Corollary 8.34. *For each small $\mathscr{G}$, $\bar{h}\hat{\square}^{\mathscr{G}}$ is locally small.*

Corollary 8.35. *For each small $\mathscr{G}$, there exists a bijection*

$$\bar{h}\hat{\square}^{\mathscr{G}}(B,C)=[B,C]$$

natural in $\mathscr{G}$-cubical sets B having finite colimit and $\mathscr{G}$-cubical sets C that happen to admit cubical compositions.

Corollary 8.36 (Excision)**.** *Fix a small $\mathscr{G}$. The natural $\mathscr{G}$-cubical function*

$$sing\,U\cup_{sing\,U\cap V} sing\,V\to sing\,X$$

is a weak equivalence for all $\mathscr{G}$-streams U,V,X such that $U(d),V(d)$ are substreams of $X(d)$ whose interors in $X(d)$ cover $X(d)$ for all $\mathscr{G}$-objects d and $U(\gamma),V(\gamma)$ are restrictions and corestrictions of $X(\gamma)$ for all $\mathscr{G}$-morphisms γ.

Proof. Fix finite $\mathscr{G}$-cubical set C. In the commutative diagram

$$\begin{array}{ccc}
h\mathscr{S}^{\mathscr{G}}(\upharpoonleft C\downharpoonright,\upharpoonleft sing\,U\cup_{sing\,U\cap V} sing\,V\downharpoonright) & \longrightarrow & h\mathscr{S}^{\mathscr{G}}(\upharpoonleft C\downharpoonright,\upharpoonleft sing\,X\downharpoonright)\\
\uparrow & & \uparrow\\
\lim_n h\mathscr{S}^{\mathscr{G}}(cd^nC, sing\,U\cup_{sing\,U\cap V} sing\,V) & \longrightarrow & \lim_n h\mathscr{S}^{\mathscr{G}}(cd^nC, sing\,X),
\end{array}$$

where the inverse limits are taken over $\mathscr{G}$-cubical functions of the form

$$\cdots\xrightarrow{\gamma_{cd^{n+1}(*)}} cd^n(*)\xrightarrow{\gamma_{cd^n(*)}}\cdots\xrightarrow{\gamma_{cd^5(*)}} cd^4(*)\xrightarrow{(\gamma\bar{\gamma})^2_*} *,$$

the natural vertical arrows induced by the functor $\upharpoonleft-\downharpoonright$ are bijective by Corollary 8.28, the d-component of every $\mathscr{G}$-cubical function $cd^nC\to sing\,X$ maps cubes into either $sing\,U(d)$ or $sing\,V(d)$ for each $\mathscr{G}$-object d because $|C|$ is compact, hence the bottom horizontal function induced by inclusion is bijective, and hence the top horizontal function is bijective. □

9. Conclusion

We have thus established a formal equivalence between directed homotopy theories of streams and cubical sets, where quadrangulable streams and cubical sets admitting cubical compositions serve as directed analogues of CW complexes and Kan complexes, respectively. Thus we can study the directed homotopy types of streams in nature in terms of the combinatorics of their quadrangulations. In particular, the main results pave the way for the development of singular cubical (co)homology theory for directed topology.

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