Stabilizers of Subspaces under Similitudes of the Klein Quadric, and Automorphisms of Heisenberg Algebras

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Abstract

We determine the groups of automorphisms and their orbits for nilpotent Lie algebras of class 2 and small dimension, over arbitrary fields (including the characteristic 2 case).

Introduction

In [19], a conceptual approach has been given to the classification of Lie algebras L with $L' := [L, L] \leq \mathfrak{z}(L)$ for small values of $\dim(L/\mathfrak{z}(L))$ and over an arbitrary field $\mathbb{K}$ (including the case $\operatorname{char}\mathbb{K} = 2$). In the present paper, we use this to find all automorphisms of Lie algebras L with $L' \leq \mathfrak{z}(L)$ and $\dim(L/L') = 4$, clarify the structure of the group $\operatorname{Aut}(L)$, and determine its orbits on L.

The restriction to the case $\dim(L/L') = 4$ is justified by the fact that the classification problem becomes wild for $\dim(L/L') > 4$, even if we assume that the ground field is algebraically closed, cf. [2], [16].

One motivation for the study of Lie algebras over arbitrary fields comes from group theory. If one wants to understand nilpotent groups, Lie methods are useful in many cases. For instance, we have (cf. [8, VIII 9.16]): If G is a nilpotent group of class at most 2 in which every element has a unique square root then $x + y := xy\sqrt{[y, x]}$ defines the addition of a Lie ring (i.e., a Lie algebra over the ring $\mathbb{Z}$ of integers) $\mathfrak{g} = (G, +, [\cdot, \cdot])$, and $\operatorname{Aut}(G) = \operatorname{Aut}(\mathfrak{g})$. Moreover, this addition coincides with the multiplication on any cyclic subgroup of G. If the group G has exponent $p \in \mathbb{P}$ (where $p \neq 2$ by our assumption on square roots) then the Lie ring may also be considered as a Lie algebra over the field with p elements. Our present work thus comprises a generalization of the results in [21]. See [22] for extensions in a more group-theoretic manner, and [6] for an investigation of a similar class of nilpotent groups.

Organization of the paper.

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1. Automorphisms of Heisenberg algebras

1.1 Definitions. A *generalized Heisenberg algebra* $\mathfrak{gh}(V,Z,\beta) := (V\times Z,[\cdot,\cdot]_\beta)$ is given by vector spaces V and Z together with an alternating bilinear map $\beta\colon V\times V\to Z$: the underlying vector space is $V\times Z$, and the Lie bracket is $[(v,x),(w,y)]_\beta := (0,\beta(v,w))$.

If the image $\beta(V\times V)$ generates Z, and $\{v\in V \mid \forall\, w\in V\colon \beta(v,w)=0\} = \{0\}$, we call $\mathfrak{gh}(V,Z,\beta)$ a *reduced Heisenberg algebra*. These conditions mean that $\{0\}\times Z$ equals both the center and the commutator algebra of $\mathfrak{gh}(V,Z,\beta)$.

Using the universal property of the tensor product, we obtain a unique linear surjection $\hat\beta\colon V\wedge V\to Z$ such that $\hat\beta(v\wedge w)=\beta(v,w)$ holds for all $v,w\in V$. If $\mathfrak{gh}(V,Z,\beta)$ is reduced then the kernel of $\hat\beta$ satisfies

$$\forall\, v\in V\smallsetminus\{0\}\colon \eta(\{v\}\times V)\not\subseteq \ker\hat\beta \tag{$*$}$$

where $\eta(v,w)=v\wedge w$. Conversely, every linear surjection $\gamma\colon V\wedge V\to Z$ satisfying condition $(*)$ yields a reduced Heisenberg algebra $\mathfrak{gh}(V,Z,\gamma\circ\eta))$.

Every nilpotent Lie algebra of class 2 is isomorphic to the direct sum of a reduced Heisenberg algebra and an abelian Lie algebra, cf. [19, 6.2].

1.2 Proposition. *Let $\mathfrak{gh}(V,Z,\beta)$ be a reduced Heisenberg algebra, and let A be an abelian Lie algebra. Let Σ_β denote the group of all linear bijections of V onto itself such that $\sigma(\ker\hat\beta)=\ker\hat\beta$. For $\sigma\in\Sigma_\beta$, let σ' denote the unique element of $\mathrm{GL}(Z)$ such that*

$$\forall\, u,v\in V\colon \sigma'(\beta(u,v))=\beta(\sigma(u),\sigma(v))\,. \tag{$**$}$$

Then the automorphisms of the Lie algebra $\mathfrak{gh}(V,Z,\beta)\times A$ are precisely the maps

$$(v,z,a)\mapsto \big(\sigma(v),\sigma'(z)+\tau(v)+\zeta(a),\alpha(a)+\xi(v)\big)\ ,$$

where $\sigma\in\Sigma_\beta$, $\tau\in\mathrm{Hom}(V,Z)$, $\zeta\in\mathrm{Hom}(A,Z)$, $\alpha\in\mathrm{GL}(A)$, and $\xi\in\mathrm{Hom}(V,A)$.

Proof. An easy computation shows that the given maps are automorphisms. The center of $\mathfrak{gh}(V,Z,\beta)\times A$ is $\{0\}\times Z\times A$, its commutator algebra is $\{0\}\times Z\times\{0\}$. Since these are invariant subalgebras, there are no other automorphisms than the ones we have described. ☐

For the case of reduced algebras this result has already been stated in [13, 4.4].

2. Tools from the classification of forms

The classification of Heisenberg algebras $\mathfrak{gh}(V,Z,\beta)$ with $\dim V=4$ and our determination of the corresponding groups of automorphisms and their orbits is based on the study of restrictions of the Pfaffian form, see 3.3 below.

Aiming at applications later on, we present well known results here, mainly on forms in at most four variables. For the reader's convenience we indicate proofs and include a discussion of quaternions over arbitrary fields.

2.1 The Arf Invariant. When dealing with quadratic forms over a field $\mathbb{K}$ with $\operatorname{char}\mathbb{K}=2$ one has to distinguish between diagonalizable quadratic forms and non-diagonalizable ones. Equivalence of the latter depends on Arf's invariant δ (cf. [4, 8.11]): for a non-diagonalizable form q in two variables given by $q(v) := v'Mv$ this is just $\delta(q) := \frac{\det M}{(\mathrm{tr}(iM))^2}+\wp$, where

$\wp := \{x + x^2 \mid x \in \mathbb{K}\}$. If we describe the same form by different matrices M and $\tilde{M}$ then $M - \tilde{M} = ti$ for some $t \in \mathbb{K}$ and $i = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Thus $\frac{\det M}{(\mathrm{tr}(iM))^2} - \frac{\det \tilde{M}}{(\mathrm{tr}(i\tilde{M}))^2} = \frac{t}{\mathrm{tr}(iM)} + \left(\frac{t}{\mathrm{tr}(iM)}\right)^2 \in \wp$ shows that δ is indeed an invariant of the form q.

It is easy to see that $\delta(q)$ does not change if we replace q by tq with $t \in \mathbb{K}^\times$, or if we pass to an equivalent form (replacing M by $A'MA$).

In order to understand the set of values of a non-diagonalizable form we multiply the form with a scalar such that it assumes the value 1 at some $v \in \mathbb{K}^2$. Then the form is equivalent to the form q given by $q(v) = v' \left(\begin{smallmatrix} 1 & t \\ 0 & d \end{smallmatrix}\right) v$, and may be interpreted as the (multiplicative) norm form of a suitable algebra, as follows.

We write u for the class of X modulo $(X^2 + tX + d)$ and consider the algebra $\mathbb{K}(u) := \mathbb{K}[X]/(X^2 + tX + d)$. Putting $\overline{a + bu} := a + tb - bu$ we obtain an involutory automorphism interchanging the two roots u and $t - u$ of $X^2 + tX + d$ in $\mathbb{K}(u)$; the corresponding norm is $(\overline{a + bu})(a+bu) = q(a, b)$. Clearly this norm is multiplicative. Moreover, a product in $\mathbb{K}(u)$ has norm 0 only if one of the factors has norm 0, and $\mathbb{K}(u)^\times = \{a + bu \in \mathbb{K}(u) \mid q(a, b) \neq 0\}$ is the group of units in $\mathbb{K}(u)$. Thus $V_{1,d} := \{a^2 + abt + b^2 d \mid a, b \in \mathbb{K}\} \smallsetminus \{0\} = \{\overline{v}v \mid v \in \mathbb{K}(u)^\times\}$ is a subgroup of the multiplicative group $\mathbb{K}^\times$.

2.2 Equivalence of Non-diagonalizable Forms in Characteristic Two. *Let q and $\tilde{q}$ be non-diagonalizable forms in two variables over a field $\mathbb{K}$ of characteristic 2. Then q and $\tilde{q}$ are equivalent precisely if their Arf invariants are equal and the forms share a non-zero value, i.e., there exist non-zero vectors v, w such that $q(v) = \tilde{q}(w)$.*

A set of representatives for the equivalence classes of non-diagonalizable forms is obtained by taking the forms described by the matrices in $\{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\} \cup \{a \left(\begin{smallmatrix} 1 & 1 \\ 0 & d \end{smallmatrix}\right) \mid d \in R_\wp, a \in R_d\}$ where $R_\wp$ and R_d are sets of representatives of the cosets in $\mathbb{K}/\wp$ and $\mathbb{K}^\times/V_{1,d}$, respectively.

Proof. We already know from 2.1 that $\delta(q)$ and $\delta(\tilde{q})$ coincide if q and $\tilde{q}$ are equivalent. Clearly, equivalent forms share a value (indeed, they have the same range).

Now assume, conversely, that q and $\tilde{q}$ have the same Arf invariant and that they share a non-zero value. Upon basis transformation we may assume that $0 \neq a := q(1, 0) = \tilde{q}(1, 0)$. Then the forms are represented by matrices $a \left(\begin{smallmatrix} 1 & x \\ 0 & c \end{smallmatrix}\right)$ and $a \left(\begin{smallmatrix} 1 & w \\ 0 & d \end{smallmatrix}\right)$, respectively, with $x \neq 0 \neq w$ because the forms are not diagonalizable. Our assumption $\delta(q) = \delta(\tilde{q})$ yields the existence of $k \in \mathbb{K}$ such that $\frac{c}{x^2} + \frac{d}{w^2} = k^2 + k$. Computing

$$\begin{pmatrix} 1 & 0 \\ kw & w/x \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & kw \\ 0 & w/x \end{pmatrix} = \begin{pmatrix} 1 & kw + w \\ kw & w^2(k^2 + k + c/x^2) \end{pmatrix} = \begin{pmatrix} 1 & kw + w \\ kw & d \end{pmatrix}$$

we see that the forms are equivalent.

In order to find the representatives we first choose a representative $d \in R_\wp$ for the Arf invariant of a given form. Then the form is equivalent to the form described by $k \left(\begin{smallmatrix} 1 & 1 \\ 0 & d \end{smallmatrix}\right)$ for any value $k \in \mathbb{K}^\times$ assumed by the form. It thus remains to choose a representative a for the coset $kV_{1,d}$. □

2.3 Diagonal Forms in Characteristic Two. We continue to discuss forms in two variables over a field $\mathbb{K}$ with $\mathrm{char}\,\mathbb{K} = 2$. If such a form is described by a diagonal matrix $\left(\begin{smallmatrix} x & 0 \\ 0 & z \end{smallmatrix}\right)$ a change of basis will have the effect of changing this matrix to

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2x + b^2z & acx + bdz \\ acx + bdz & c^2x + d^2z \end{pmatrix} .$$

This matrix describes the *same* quadratic form as the diagonal matrix $\left(\begin{smallmatrix} a^2x+b^2z & 0 \\ 0 & c^2x+d^2z \end{smallmatrix}\right)$. In other words: every diagonalizable form is in fact diagonal, and the action of $\mathrm{GL}_2\mathbb{K}$ on the space of diagonal forms is equivalent to the action on $\mathbb{K}^2$ given by

$$\omega_{\mathbb{K}}^{(2)}\colon \mathrm{GL}_2\mathbb{K} \times \mathbb{K}^2 \to \mathbb{K}\colon \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x \\ z \end{pmatrix} \right) \mapsto \begin{pmatrix} a^2x + b^2z \\ c^2x + d^2z \end{pmatrix} = \begin{pmatrix} a^2 & b^2 \\ c^2 & d^2 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} .$$

The orbits under $\omega_{\mathbb{K}}^{(2)}$ are the same as those under the natural action of $\mathrm{GL}_2\mathbb{K}^{\square}$ on $\mathbb{K}^2$, where $\mathbb{K}^{\square}$ denotes the subfield consisting of all squares in $\mathbb{K}$.

More generally, we consider a field extension $\mathbb{L}/\mathbb{K}$ such that $\mathbb{L}^{\square} \subseteq \mathbb{K}$ (for instance, an inseparable quadratic extension $\mathbb{L}/\mathbb{K}$, as in 5.6.b below). In that case let $R_{\mathbb{K}/\mathbb{L}^{\square}}$ be a set of representatives for the cosets in $\mathbb{K}/\mathbb{L}^{\square}$, and let $R^{(2)}_{\mathbb{K}/\mathbb{L}^{\square}}$ be a set that contains precisely one $\mathbb{L}^{\square}$-basis for each $\mathbb{L}^{\square}$-subspace of dimension 2 in $\mathbb{K}^2$. Then the orbits under the restriction of $\omega_{\mathbb{K}}^{(2)}$ to $\mathrm{GL}_2\mathbb{L} \times \mathbb{K}^2$ are represented by the elements of $\{(r,0)' \mid r \in R_{\mathbb{K}/\mathbb{L}^{\square}}\} \cup R^{(2)}_{\mathbb{K}/\mathbb{L}^{\square}}$.

2.4 Hermitian Forms and Quaternions. Let $\mathbb{L}/\mathbb{K}$ be a separable quadratic extension, let $\sigma\colon \mathbb{L} \to \mathbb{L}\colon x \mapsto \overline{x}$ denote the generator of the Galois group $\mathrm{Gal}(\mathbb{L}/\mathbb{K})$. We will need the norm $N_{\mathbb{L}/\mathbb{K}}\colon \mathbb{L} \to \mathbb{K}\colon x \mapsto \overline{x}x$ and the subgroup $N_{\mathbb{L}/\mathbb{K}}(\mathbb{L}^{\times})$ of $\mathbb{K}^{\times}$.

A (σ-)hermitian form $h\colon \mathbb{L}^2 \to \mathbb{L}$ will be described by its hermitian Gram matrix M_h via $h\left(\left(\begin{smallmatrix} a \\ x \end{smallmatrix}\right), \left(\begin{smallmatrix} b \\ y \end{smallmatrix}\right)\right) = (\overline{a}, \overline{x})\, M_h \left(\begin{smallmatrix} b \\ y \end{smallmatrix}\right)$. Forms h and g are equivalent precisely if there exists $A \in \mathrm{GL}_2\mathbb{L}$ such that $\overline{A}' M_h A = M_g$ where $\overline{A}$ is obtained from A by applying σ to each entry. We will call the hermitian matrices equivalent in this case. Without loss, we may concentrate on the case $M_h \neq 0$. Every hermitian matrix is equivalent to a diagonal one (necessarily, with entries from the ground field $\mathbb{K}$) but it is not easy to decide about equivalence of two given diagonal matrices, in general.

If h is non-degenerate and isotropic then M_h is equivalent to $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. In particular, all non-degenerate isotropic forms are equivalent to the one described by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Any degenerate hermitian matrix is equivalent to $\left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$ for some $a \in \mathbb{K}$, and $\left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} b & 0 \\ 0 & 0 \end{smallmatrix}\right)$ are equivalent precisely if $aN(\mathbb{L}^{\times}) = bN(\mathbb{L}^{\times})$.

It remains to understand the hermitian matrices describing anisotropic forms. Every such matrix is equivalent to one of the shape $\left(\begin{smallmatrix} a & 0 \\ 0 & c \end{smallmatrix}\right)$ with determinant $ac \neq 0$; and two such matrices can only be equivalent if their determinants are in the same coset modulo $N(\mathbb{L}^{\times})$. However, this condition is not sufficient for equivalence, in general.

For any anisotropic hermitian form h on $\mathbb{L}^2$ we define $V_h := \{h(X,X) \mid X \in \mathbb{L}^2 \smallsetminus \{(0,0)'\}\}$. The set

$$\mathbb{H}^h_{\mathbb{L}/\mathbb{K}} := \{A \in \mathbb{L}^{2\times 2} \mid \overline{A}' M_h A = M_h \det A\}$$

forms a quaternion field, see [12] (and Section 6 below). Proportional forms lead to the same quaternion field, of course. Conversely, equality $\mathbb{H}^g_{\mathbb{L}/\mathbb{K}} = \mathbb{H}^h_{\mathbb{L}/\mathbb{K}}$ implies that there exists $t \in \mathbb{K}^{\times}$ with $g = th$. Explicitly, we have for the form h_c given by $M_{h_c} := \left(\begin{smallmatrix} 1 & 0 \\ 0 & c \end{smallmatrix}\right)$:

$$\mathbb{H}^c_{\mathbb{L}/\mathbb{K}} := \mathbb{H}^h_{\mathbb{L}/\mathbb{K}} = \left\{ \begin{pmatrix} a & -c\overline{x} \\ x & \overline{a} \end{pmatrix} \,\middle|\, a, x \in \mathbb{L} \right\} ,$$

and mapping $A = \left(\begin{smallmatrix} a & -c\overline{x} \\ x & \overline{a} \end{smallmatrix}\right) \in \mathbb{H}^c_{\mathbb{L}/\mathbb{K}}$ to $\widetilde{A} := \left(\begin{smallmatrix} \overline{a} & c\overline{x} \\ -x & a \end{smallmatrix}\right)$ gives an anti-automorphism κ of $\mathbb{H}^c_{\mathbb{L}/\mathbb{K}}$. The map $N_c\colon (\mathbb{H}^c_{\mathbb{L}/\mathbb{K}}) \to \mathbb{K}\colon A \mapsto \det_{\mathbb{L}} A = \widetilde{A}A$ is an anisotropic quadratic form (called the *norm* of $\mathbb{H}^c_{\mathbb{L}/\mathbb{K}}$). The restriction of the norm to $(\mathbb{H}^c_{\mathbb{L}/\mathbb{K}})^{\times}$ is a group homomorphism onto the subgroup V_{h_c} of $\mathbb{K}^{\times}$.

We extend the definition to cover isotropic but non-degenerate forms, as well, considering $\mathbb{H}^c_{\mathbb{L}/\mathbb{K}} = \left\{ \left(\begin{smallmatrix} a & -c\overline{x} \\ x & \overline{a} \end{smallmatrix}\right) \,\middle|\, a, x \in \mathbb{L} \right\}$ also in the cases where there exists $(x,y)' \in \mathbb{L}^2 \setminus \{(0,0)'\}$ with $\overline{x}x + c\overline{y}y = 0$. Note, however, that in this case the algebra $\mathbb{H}^c_{\mathbb{L}/\mathbb{K}}$ is smaller than the full set of matrices such that $\overline{A}' M_{h_c} A = M_{h_c} \det A$; for instance, the latter property is satisfied for $\left(\begin{smallmatrix} 0 & x \\ 0 & y \end{smallmatrix}\right)$ whenever $\overline{x}x + c\overline{y}y = 0$. The norm N_{h_c} remains a multiplicative quadratic form but $0 \in N_{h_c}\big((\mathbb{H}^c_{\mathbb{L}/\mathbb{K}})^\times\big)$ if the form h_c is isotropic. Such a quaternion algebra will be called a *split quaternion algebra*.

An element $X \in \mathbb{H}^c_{\mathbb{L}/\mathbb{K}}$ is invertible precisely if $N_{h_c}(X) \neq 0$; the inverse is $\frac{1}{N_{h_c}(X)}\widetilde{X}$. Thus the algebra $\mathbb{H}^c_{\mathbb{L}/\mathbb{K}}$ is a skewfield if, and only if, the norm form is anisotropic.

In any case, we call $\operatorname{tr}(A) := \widetilde{A} + A$ the *trace* of $A \in \mathbb{H}^c_{\mathbb{L}/\mathbb{K}}$; this is just the usual trace of $A \in \mathbb{L}^{2\times 2}$. Note that the trace describes the polar form $f_{N_{h_c}}$ via $f_{N_{h_c}}(X,Y) = N_{h_c}(X+Y) - N_{h_c}(X) - N_{h_c}(Y) = \widetilde{X}Y + \widetilde{Y}X = \operatorname{tr}\big(\widetilde{Y}X\big)$. Thus the elements perpendicular to the neutral element 1 are just those with trace 0.

As usual, we identify the field $\mathbb{K}$ with the subring $\mathbb{K}1$ of $\mathbb{H}^c_{\mathbb{L}/\mathbb{K}}$; this is just the center of the quaternion algebra. The embedding of $\mathbb{L}$ in $\mathbb{H}^c_{\mathbb{L}/\mathbb{K}}$ is slightly more delicate: we identify $a \in \mathbb{L}$ with $\left(\begin{smallmatrix} a & 0 \\ 0 & \overline{a} \end{smallmatrix}\right)$.

2.5 Equivalence of Anisotropic Hermitian Forms. *Let $\mathbb{L}/\mathbb{K}$ be a separable quadratic extension, let σ be the generator of the Galois group $\operatorname{Gal}(\mathbb{L}/\mathbb{K})$, and let g and h be anisotropic σ-hermitian forms on $\mathbb{L}^2$, described by their hermitian Gram matrices $M_g, M_h \in \mathbb{L}^{2\times 2}$, respectively. Then the following are equivalent:*

- **a.** *The hermitian matrices (and thus the forms) are equivalent up to a scalar, i.e., there exists $A \in \mathrm{GL}_2\mathbb{L}$ with $\mathbb{K}\overline{A}' M_g A = \mathbb{K} M_h$.*
- **b.** *There exists $w \in \mathbb{L}^\times$ such that $\det M_g = N_{\mathbb{L}/\mathbb{K}}(w) \det M_h$.*
- **c.** *The quaternion fields are conjugates, i.e., there is $A \in \mathrm{GL}_2\mathbb{L}$ with $A\mathbb{H}^g_{\mathbb{L}/\mathbb{K}}A^{-1} = \mathbb{H}^h_{\mathbb{L}/\mathbb{K}}$.*

The forms are equivalent if, and only if, we have $V_g = V_h$ and any one of the conditions a, b or c is satisfied. A form h is, in particular, equivalent to $-h$ precisely if $-1 \in N_h(\mathbb{H}^h_{\mathbb{L}/\mathbb{K}})$.

A set of representatives of equivalence classes of hermitian matrices is obtained as

$$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} \,\middle|\, r \in R_N \right\} \cup \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} t & 0 \\ 0 & tr \end{pmatrix} \,\middle|\, r \in R_N, t \in R_{N_r} \right\}$$

where $R_N \subseteq \mathbb{K}^\times$ is a set of representatives for the cosets modulo $N_{\mathbb{L}/\mathbb{K}}(\mathbb{L}^\times)$ and $R_{N_r} \subseteq \mathbb{K}^\times$ is a set of representatives for the cosets modulo $N_{h_r}\big((\mathbb{H}^r_{\mathbb{L}/\mathbb{K}})^\times\big)$ for the anisotropic form h_r with hermitian matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & r \end{smallmatrix}\right)$.

Proof. Picking orthogonal bases with respect to g or h we find $a, b, c, d \in \mathbb{K}^\times$ such that M_g is equivalent to $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ and M_h is equivalent to $\left(\begin{smallmatrix} c & 0 \\ 0 & d \end{smallmatrix}\right)$.

Condition a clearly implies b. If b is satisfied we have $(ab)^{-1}cd \in N_{\mathbb{L}/\mathbb{K}}(\mathbb{L}^\times)$. Then $\left(\begin{smallmatrix} 1 & 0 \\ 0 & b/a \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & d/c \end{smallmatrix}\right)$ are equivalent, and a holds.

Finally, we note that conditions a and c are equivalent because the only σ-hermitian forms invariant under $\mathbb{H}^g_{\mathbb{L}/\mathbb{K}}$ are the scalar multiples of h.

Every non-degenerate isotropic hermitian form on $\mathbb{L}^2$ is equivalent to the form h_{-1} described by $M_{h_{-1}} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. The rest of the assertions is clear because $h \in \mathbb{K}g$ is equivalent to g precisely if $V_h = V_g$. ∎

See 6.14 below for examples of quaternion fields with different behavior with respect to the quotient $\mathbb{K}^\times/N_h\big((\mathbb{H}^h_{\mathbb{L}/\mathbb{K}})^\times\big)$ and also regarding the question whether -1 belongs to the group $N_h\big((\mathbb{H}^h_{\mathbb{L}/\mathbb{K}})^\times\big)$ of norms.

Inner automorphisms of quaternion fields and similitudes of the norm form

2.6 Conjugacy in Quaternion Fields. *Let $\mathbb{H} = \mathbb{H}^c_{\mathbb{L}/\mathbb{K}}$ be a quaternion field over a field of arbitrary characteristic, with norm form $N = N_{h_c}$ and involution $\kappa\colon x \mapsto \tilde{x}$.*

- **a.** *For $v, x \in \mathbb{H}$ there exists $a \in \mathbb{H}^\times$ with $ava^{-1} = x$ precisely if v and x have the same norm and the same trace. In particular, pure quaternions (i.e., quaternions with vanishing trace) are conjugates if, and only if, they have the same norm.*
- **b.** *Now consider $v, w, x, y \in \mathbb{H}$ such that $w \notin \mathbb{K}v$ and $y \notin \mathbb{K}x$. There exists $a \in \mathbb{H}^\times$ such that $ava^{-1} = w$ and $axa^{-1} = y$ if, and only if, we have $N(v) = N(x)$, $N(w) = N(y)$, $\mathrm{tr}(v) = \mathrm{tr}(w)$, $\mathrm{tr}(x) = \mathrm{tr}(y)$, and $f_N(v, w) = f_N(x, y)$.*
- **c.** *There exist $a, b \in \mathbb{H}^\times$ with $av\tilde{a}N(b) = x$ if, and only if, there exists $z \in \mathbb{H}$ such that $N(x) = N(v)N(z)^2$ and $\mathrm{tr}(x) = \mathrm{tr}(v)N(z)$.*

Proof. Multiplicativity of the norm form yields that conjugation with a induces an orthogonal map on $\mathbb{H}$. It remains to prove the non-trivial implications, i.e., those asserting conjugacy.

Assume $N(v) = N(x)$ and $\mathrm{tr}(v) = \mathrm{tr}(x)$. If $x \neq \tilde{v}$ then $a := x - \tilde{v}$ satisfies $\tilde{a} = \tilde{x} - v = \mathrm{tr}(x) - x - v = -x + \tilde{v} = -a$ and we compute $ava^{-1} = (x - \tilde{v})va^{-1} = (xv - N(v))a^{-1} = (xv - N(x))a^{-1} = x(v - \tilde{x})a^{-1} = v$. If $x = \tilde{v}$ we pick $a \in \{1, v\}^\perp \smallsetminus \{0\}$. Then $0 = f_N(a, v) = \tilde{a}v + \tilde{v}a = -av + \tilde{v}a$ gives $ava^{-1} = \tilde{v}$. This proves assertion a.

Under the assumptions made in b, we may also assume $x = v$ because of assertion a. It suffices to consider the case $w \neq y$. Then $c := y - w$ satisfies $\tilde{c} = (\mathrm{tr}(y) - y) - (\mathrm{tr}(w) - w) = w - y = -c$, and we compute $cwc^{-1} = (\tilde{w} - \tilde{y})wc^{-1} = (N(w) - \tilde{y}w)c^{-1} = (N(y) - \tilde{y}w)c^{-1} = \tilde{y}$. On the other hand, we use $\tilde{v}w + \tilde{w}v = f_N(v, w) = f_N(x, y) = f_N(v, y) = \tilde{v}y + \tilde{y}v$ to compute $cvc^{-1} = (\tilde{w} - \tilde{y})vc^{-1} = (\tilde{w}v - \tilde{y}v)c^{-1} = (\tilde{v}y - \tilde{v}w)c^{-1} = \tilde{v}$. It remains to pick $b \in \{1, v, y\}^\perp \smallsetminus \{0\}$; then $b\tilde{v}b^{-1} = v$ and $b\tilde{y}b^{-1} = y$ complete the proof of assertion b.

If $x = av\tilde{a}N(b)$ then $N(x) = N(v)N(ab)^2$ and $\mathrm{tr}(x) = \mathrm{tr}(ava^{-1}N(a)N(b)) = \mathrm{tr}(v)N(ab)$. Conversely, assume that there is $z \in \mathbb{H}^\times$ such that $N(x) = N(v)N(z)^2$ and $\mathrm{tr}(x) = \mathrm{tr}(v)N(z)$. Then x and $vN(z)$ have the same norm and the same trace, and assertion a yields $a \in \mathbb{H}^\times$ such that $x = avN(z)a^{-1} = av\tilde{a}N(z)N(a)^{-1} = av\tilde{a}N(za^{-1})$. □

2.7 Remark. The assertions of 2.6 do not remain valid if we consider a split quaternion algebra. In fact, such an algebra will contain divisors of zero, and the element $a := x - \tilde{u}$ used in the proof of 2.6.a might be non-invertible although $x \neq \tilde{u}$. It is known that every split quaternion algebra is isomorphic to the algebra of 2×2 matrices over the ground field, with the usual determinant playing the role of the norm form. The example of $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ shows that indeed 2.6.a does not extend to the split case without modification.

If $\mathrm{char}\,\mathbb{K} \neq 2$ it remains true that pure elements (i.e., matrices with vanishing trace) are conjugates if, and only if, they have the same norm because the trace of a non-trivial central element is non-zero. If $\mathrm{char}\,\mathbb{K} = 2$, however, our example consist of pure elements.

The following result contains a special case of the Skolem–Noether Theorem for central simple algebras (e.g., cf. [15, 8.4.2]).

2.8 Theorem. *Let $\mathbb{H}$ be a quaternion field over a field $\mathbb{K}$ of arbitrary characteristic. Then the group $\mathrm{Aut}(\mathbb{H})$ of $\mathbb{K}$-linear automorphisms acts faithfully on the orthogonal space $\mathrm{Pu}(\mathbb{H}) :=$*

$\{x \in \mathbb{H} \mid \operatorname{tr}(x) = 0\} = 1^\perp$*, inducing the group* $\mathrm{SO}(N|_{\mathrm{Pu}(\mathbb{H})})$*. Every element of* $\mathrm{Aut}(\mathbb{H})$ *is an inner automorphism.*

Proof. For any $x \in \mathbb{H}$ the assumption $x^2 \in \mathbb{K}$ implies $N(x) + x^2 = \operatorname{tr}(x)x \in \mathbb{K}$ and then $x \in \mathbb{K} \cup \mathrm{Pu}(\mathbb{H})$. Together with the fact that the center of $\mathbb{H}$ is just $\mathbb{K}$, the equality $\mathbb{K} \cup \mathrm{Pu}(\mathbb{H}) = \{x \in \mathbb{H} \mid x^2 \in \mathbb{K}\}$ now yields that $\mathrm{Pu}(\mathbb{H})$ is invariant under $\mathrm{Aut}(\mathbb{H})$.

Pick $x \in \mathrm{Pu}(\mathbb{H}) \setminus \mathbb{K}$ and $y \in \mathrm{Pu}(\mathbb{H}) \setminus \langle 1, x\rangle_{\mathbb{K}}$. Then $\langle 1, x\rangle_{\mathbb{K}}$ is a subfield of $\mathbb{H}$ and $xy \notin \langle 1, x, y\rangle_{\mathbb{K}}$ because $xy = r + sx + ty$ with $r, s, t \in \mathbb{K}$ implies that $(x - t)y \in \langle 1, x\rangle_{\mathbb{K}}$ and then $y \in (x - t)^{-1} \langle 1, x\rangle_{\mathbb{K}} = \langle 1, x\rangle_{\mathbb{K}}$, a contradiction. Therefore, the set $\mathrm{Pu}(\mathbb{H})$ generates the quaternion field as a $\mathbb{K}$-algebra, and $\mathrm{Aut}(\mathbb{H})$ acts faithfully on $\mathrm{Pu}(\mathbb{H})$; in fact, the stabilizer of x and y is trivial.

Since $N(x) = -x^2$ holds for $x \in \mathrm{Pu}(\mathbb{H})$ the group $\mathrm{Aut}(\mathbb{H})$ acts by orthogonal maps on $\mathrm{Pu}(\mathbb{H})$. Choosing a suitable $\mathbb{K}$-basis for $\mathbb{H}$ one easily sees that the $\mathbb{K}$-determinant of any left multiplication $x \mapsto ax$ is the square of $N(a)$. The right multiplication $x \mapsto xa = \widetilde{\tilde{a}\tilde{x}}$ then has determinant $N(\tilde{a}) = N(a)$, and every inner automorphism has determinant 1. Therefore, $\mathrm{Aut}(\mathbb{H})$ is a subgroup of $\mathrm{SO}(N)$ and induces a subgroup of $\mathrm{SO}(N|_{\mathrm{Pu}(\mathbb{H})})$.

According to 2.6.b the group of inner automorphisms acts transitively on any set of two-dimensional subspaces of given isometry type in $\mathrm{Pu}(\mathbb{H})$. The elements x and y above were chosen outside the radical[1] of the restriction of the polar form f_N to $\mathrm{Pu}(\mathbb{H})$. Therefore the stabilizer of x and y (as above) is trivial even in the group $\mathrm{SO}(N|_{\mathrm{Pu}(\mathbb{H})})$, and the transitivity assertion from 2.6.b yields that the group of inner automorphisms coincides with $\mathrm{SO}(N|_{\mathrm{Pu}(\mathbb{H})})$.

If $\operatorname{char} \mathbb{K} \neq 2$ it remains to show that no element of the coset $\mathrm{O}(N_{\mathrm{Pu}(\mathbb{H})}) \setminus \mathrm{SO}(N_{\mathrm{Pu}(\mathbb{H})})$ is induced by an element of $\mathrm{Aut}(\mathbb{H})$. As the involution $x \mapsto \tilde{x}$ is not an automorphism but represents the coset this is another consequence of the fact that each element of $\mathrm{SO}(N|_{\mathrm{Pu}(\mathbb{H})})$ is induced by an inner automorphism. □

Similitudes of quadratic forms

2.9 Definition. Let $q\colon V \to \mathbb{K}$ be a quadratic form. A *similitude* of q with *multiplier* r is a linear bijection $\lambda\colon V \to V$ such that $q(\lambda(v)) = rq(v)$ holds for each $v \in V$. The set $\mathrm{GO}(q)$ of all similitudes forms a subgroup of $\mathrm{GL}(V)$.

If $q \neq 0$ then every multiplier is non-zero. In this case, mapping $\lambda \in \mathrm{GO}(q)$ to its multiplier μ_λ gives a group homomorphism $\mu\colon \mathrm{GO}(q) \to \mathbb{K}^\times$ called the *multiplier map*.

The range of the multiplier map contains $\mathbb{K}^{\boxtimes}$ because $s\,\mathrm{id}$ is a similitude with multiplier s^2.

Every diagonalizable quadratic form $q\colon V \to \mathbb{K}$ over a field $\mathbb{K}$ of characteristic 2 is additive. Regarding $\mathbb{K}$ as a vector space over $\mathbb{K}^{\square}$ we may then view q as a semilinear map with respect to the field isomorphism $\varphi\colon \mathbb{K} \to \mathbb{K}^{\square}\colon x \mapsto x^2$. The form q is non-degenerate if, and only if, the kernel of this semilinear map is trivial. Thus the form is non-degenerate precisely if it is anisotropic.

2.10 Theorem. *Let* $q\colon V \to \mathbb{K}$ *be a diagonalizable non-degenerate quadratic form over a field* $\mathbb{K}$ *with* $\operatorname{char} \mathbb{K} = 2$.

- **a.** *The form* q *is anisotropic and* $\mathrm{O}(q)$ *is trivial.*
- **b.** *The subset* $\{0\} \cup \mathrm{GO}(q)$ *of* $\mathrm{End}_{\mathbb{K}}(V)$ *is a field, and the multiplier map* μ *extends to an isomorphism onto the subfield* $\mathbb{M} := \{0\} \cup \{\mu(\lambda) \mid \lambda \in \mathrm{GO}(q)\}$ *of* $\mathbb{K}$.

[1] This precaution is only relevant if $\operatorname{char} \mathbb{K} = 2$.

c. *We have* $\mathbb{K}^{\square} \leq \mathbb{M}$ *and* $\mathbb{M}/\mathbb{K}^{\square}$ *is a purely inseparable extension.*
d. *If* $\dim_{\mathbb{K}} V$ *is finite then* $\dim_{\mathbb{K}} V = (\dim_{\mathbb{M}} V)(\dim_{\mathbb{K}^{\square}} \mathbb{M})$ *and* $\dim_{\mathbb{K}^{\square}} \mathbb{M}$ *is a power of* 2*.*
e. *In particular, if* $\dim_{\mathbb{K}} V$ *is odd then* $\mathrm{GO}(q) = \mathbb{K}^{\times}\,\mathrm{id} = \mu^{-1}(\mathbb{K}^{\boxtimes})$*.*
f. *In any case, the group* $\mathrm{GO}(q)$ *acts regularly on* $V \setminus \{0\}$*. In particular, if* $G \leq \mathrm{GO}(q)$ *is transitive on* $V \setminus \{0\}$ *then* $G = \mathrm{GO}(q)$*.*

Proof. Every isotropic vector belongs to the kernel of the semi-linear map q. This kernel coincides with the radical of q because the form is diagonal. Thus q is an injective map. This yields that $\mathrm{O}(q)$ is trivial, as claimed.

For similitudes $\lambda, \sigma \in \mathrm{GO}(q)$ and $v \in V$ we compute $q\left((\lambda + \sigma)(v)\right) = q\left(\lambda(v)\right) + q\left(\sigma(v)\right) = \mu(\lambda)\,q(v) + \mu(\sigma)\,q(v) = \left(\mu(\lambda) + \mu(\sigma)\right)\,q(v)$. This shows that $\lambda + \sigma$ is a similitude with multiplier $\mu(\lambda) + \mu(\sigma)$ unless $\mu(\lambda) = \mu(\sigma)$. In the latter case we have $\lambda\sigma^{-1} \in \mathrm{O}(q)$ and $\lambda = \sigma$ by assertion a. Thus $\{0\} \cup \mathrm{GO}(q) \subseteq \mathrm{End}_{\mathbb{K}}(V)$ is additively closed and μ extends to an additive map onto $\mathbb{M}$. Since μ is multiplicative anyway, this extension is a field isomorphism.

The similitudes in $\mathbb{K}^{\times}\,\mathrm{id}$ are mapped to the elements of $\mathbb{K}^{\boxtimes}$ under μ. For every $m \in \mathbb{M}$ the minimal polynomial over $\mathbb{K}^{\square}$ divides $X^2 - m^2 \in \mathbb{K}^{\square}[X]$, and assertion c is proved.

From assertion c we infer that every intermediate field between $\mathbb{K}^{\square}$ and $\mathbb{M}$ either has infinite degree over $\mathbb{K}^{\square}$ or that degree is a power of 2. This gives the last two assertions d and e.

The last assertion follows from the fact that $\mathrm{O}(q)$ is trivial. □

2.11 Example. Let $\mathbb{L}/\mathbb{K}$ be a purely inseparable extension with $\mathrm{char}\,\mathbb{K} = 2$. Then $\mathbb{L}^{\square} \leq \mathbb{K}$ and the norm $N_{\mathbb{L}/\mathbb{K}}(x) \colon \mathbb{L} \to \mathbb{K} \colon x \mapsto x^2$ is a quadratic form. This form is diagonal, and $\mathrm{GO}(N_{\mathbb{L}/\mathbb{K}}) = \mathbb{L}^{\times}$ by 2.10.f. Forms of this type will occur in Section 6 below.

The following observation will be useful because it constrains the orbits under groups of similitudes.

2.12 Lemma. *Let* $q \colon V \to \mathbb{K}$ *be a non-degenerate quadratic form, and assume that* $\dim V$ *is odd. Then every multiplier is a square, and* $\mathrm{GO}(q) = \mathbb{K}^{\times}\,\mathrm{SO}(q)$*.*

Proof. Assume first that $\mathrm{char}\,\mathbb{K} \neq 2$. Then the set of all determinants of Gram matrices for q is a square class which we denote by $\mathrm{disc}\,q$, and a similitude λ with multiplier s will multiply $\mathrm{disc}\,q$ by $(\det\lambda)^2$. On the other hand, we have $\mathrm{disc}(sq) = s^{\dim V}\,\mathrm{disc}\,q$. Since $\dim V$ is odd, we immediately obtain that s is a square.

If $\mathrm{char}\,\mathbb{K} = 2$ and the form is not diagonalizable then the Gram matrix cannot be chosen in such a way that this choice is invariant under linear transformations. However, the polar form f_q is alternating in that case, and has even rank. The radical $V^{\perp}$ of the polar form is invariant under any similitude, and has odd dimension because $\dim V$ is odd. The restriction of q to $V^{\perp}$ is diagonalizable, and we know from 2.10 that s is a square in this case, too.

For any $\mu \in \mathrm{GO}(q)$ with multiplier s^2 we note that $s^{-1}\mu$ belongs to $\mathrm{O}(q)$, and $\mathrm{O}(q) = \mathrm{SO}(q) \cup (-\mathrm{id})\,\mathrm{SO}(q)$. Thus the last assertion follows. □

If $\dim V$ is even, the situation is more involved: depending on the form and the field in question, non-squares may occur as multipliers.

2.13 Corollary. *For any quaternion field* $\mathbb{H}$ *we have*

$$\begin{aligned} \mathrm{GO}(N|_{\mathrm{Pu}(\mathbb{H})}) = \mathbb{K}^{\times}\mathrm{SO}(N|_{\mathrm{Pu}(\mathbb{H})}) &= \left\{ (x \mapsto saxa^{-1}) \,\middle|\, a \in \mathbb{H}^{\times}, s \in \mathbb{K}^{\times} \right\} \\ &= \left\{ (x \mapsto sax\tilde{a}) \,\middle|\, a \in \mathbb{H}^{\times}, s \in \mathbb{K}^{\times} \right\} . \end{aligned}$$

3. Classification of reduced Heisenberg algebras

We need some explicit notation for the action of $\mathrm{GL}_n\mathbb{K}$ on $\mathbb{K}^n \wedge \mathbb{K}^n$.

3.1 Notation. Let $\mathbb{K}$ be a field, and let $b_0, \ldots, b_{n-1}$ denote the standard basis for $\mathbb{K}^n$. We will think of elements $v = \sum_{j<n} v_j b_j \in \mathbb{K}^n$ as columns, the transpose is written $v' := (v_0, \ldots, v_{n-1})$. For $v, w \in \mathbb{K}^n$ we obtain the *decomposable tensor* $v \otimes w := vw' = (v_j w_k)_{j,k<n}$. The elements $b_j \otimes b_k$, with $j, k < n$, form the standard basis for the space $\mathbb{K}^{n\times n}$ of $(n \times n)$ – matrices with entries from $\mathbb{K}$.

The set of *alternating tensors* is the linear span $\mathbb{K}^n \wedge \mathbb{K}^n$ of the elements of the form $v \wedge w := v \otimes w - w \otimes v$. These are the skew-symmetric matrices with zero diagonal (the latter condition follows from the former unless $\operatorname{char}\mathbb{K} = 2$). The elements $S^j_k := b_j \wedge b_k$ with $0 \le j < k < n$ form a basis for $\mathbb{K}^n \wedge \mathbb{K}^n$. With the bilinear map $\eta\colon \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}^n \wedge \mathbb{K}^n \colon (v, w) \mapsto v \wedge w$, we have $(\mathbb{K}^n \wedge \mathbb{K}^n, \eta)$ as an explicit model for the exterior product, satisfying the universal property: for every alternating bilinear map $\beta\colon V \times V \to Z$, there is a unique linear map $\hat\beta\colon \mathbb{K}^n \wedge \mathbb{K}^n \to Z$ such that $\beta = \hat\beta \circ \eta$.

The linear action of the group $\mathrm{GL}_n\mathbb{K}$ on $\mathbb{K}^n$ induces a linear action on the tensor product:

$$\omega\colon \mathrm{GL}_n\mathbb{K} \times (\mathbb{K}^n \otimes \mathbb{K}^n) \to \mathbb{K}^n \otimes \mathbb{K}^n \colon (A, M) \mapsto AMA' .$$

Obviously, the set $\mathbb{K}^n \wedge \mathbb{K}^n$ is invariant under this action. We write $A.X := AXA'$.

3.2 More Notation. The space $\mathbb{K}^2 \wedge \mathbb{K}^2$ is spanned by $i := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) = (1,0)' \wedge (0,1)'$. We will use this notation for blocks in elements of $\mathbb{K}^4 \wedge \mathbb{K}^4$ later on. By E_j we denote the identity matrix in $\mathbb{K}^{j\times j}$.

3.3 The Pfaffian Form and the Klein Quadric. We will be interested in the Grassmann space $\mathrm{Gr}^\wedge_{d,4}$ of d-dimensional subspaces of $\mathbb{K}^4 \wedge \mathbb{K}^4$, where $d \in \{1, 2, 3, 4, 5\}$. Note that ω induces an action on each one of the Grassmann spaces. The following belongs to classical line geometry; details and proofs may also be found in [19].

The group $\mathrm{GL}_4\mathbb{K}$ acts with exactly three orbits on $\mathbb{K}^4 \wedge \mathbb{K}^4$, represented by 0, S^0_1 and $S^0_1 + S^2_3$. Therefore, there are two orbits on $\mathrm{Gr}^\wedge_{1,4}$. The orbit of $\langle S^0_1\rangle_\mathbb{K}$ consists of subspaces $\langle X\rangle_\mathbb{K}$ with $X \in \mathbb{K}^4 \wedge \mathbb{K}^4 \smallsetminus \{0\}$ and $\det X = 0$. We use the basis $S^0_1, S^0_2, S^0_3, S^1_2, S^1_3, S^2_3$ to introduce homogeneous coordinates $[x_0, \ldots, x_5]$ for $\langle X\rangle_\mathbb{K}$, where

$$X = x_0 S^0_1 + x_1 S^0_2 + x_2 S^0_3 + x_3 S^1_2 + x_4 S^1_3 + x_5 S^2_3 = \begin{pmatrix} 0 & x_0 & x_1 & x_2 \\ -x_0 & 0 & x_3 & x_4 \\ -x_1 & -x_3 & 0 & x_5 \\ -x_2 & -x_4 & -x_5 & 0 \end{pmatrix} .$$

The orbit of $\langle S^0_1\rangle_\mathbb{K}$ may then be described as the quadric Q (known as the Klein quadric) defined by the *Pfaffian form* $\mathrm{pf}(x_0, x_1, x_2, x_3, x_4, x_5) := x_0x_5 - x_1x_4 + x_2x_3$, cf. [3, § 5, no. 2]. This is a quadratic form of Witt index 3 on $\mathbb{K}^4 \wedge \mathbb{K}^4$. The complement of Q is the second orbit in $\mathrm{Gr}^\wedge_{1,4}$.

We re-arrange the basis, using $S^0_1, S^0_2, S^0_3, S^2_3, -S^1_3, S^1_2$. With respect to the new basis, the Pfaffian form itself may be described as $\mathrm{pf}(v) = v' M_{\mathrm{pf}} v$, and the polar form f_{pf} becomes $f_{\mathrm{pf}}(v, w) = v' J w$ with the Gram matrix J, where $M_{\mathrm{pf}}, J \in \mathbb{K}^{6\times 6}$ are defined as $M_{\mathrm{pf}} := \left(\begin{smallmatrix} 0 & E_3 \\ 0 & 0 \end{smallmatrix}\right)$ and $J := M_{\mathrm{pf}} + M'_{\mathrm{pf}} := \begin{pmatrix} 0 & E_3 \\ -E_3 & 0 \end{pmatrix}$.

The group $\mathrm{GL}_4\mathbb{K}$ acts by similitudes with respect to pf. This yields a homomorphism δ from $\mathrm{GL}_4\mathbb{K}$ to $\mathrm{GO}(\mathrm{pf})$ with kernel $\{\mathrm{id}, -\mathrm{id}\}$. We will use the induced groups $\mathrm{PGL}_4\mathbb{K}$ and $\mathrm{PGO}_6\mathbb{K}$

on the projective spaces (or on the quadric): the homomorphism δ induces an isomorphism from $\mathrm{PGL}_4\mathbb{K}$ onto a subgroup $\mathrm{PGO}^+(\mathrm{pf})$ of index 2 in $\mathrm{PGO}(\mathrm{pf})$, see [19, 3.11].

The Klein quadric provides a model for the space $\mathcal{L}$ of lines in the 3-dimensional projective space over $\mathbb{K}$ via the bijection $\lambda\colon \mathcal{L} \to Q\colon \langle u, v\rangle_{\mathbb{K}} \mapsto \langle u\wedge v\rangle_{\mathbb{K}}$. This can be used to understand the action of $\mathrm{GL}_4\mathbb{K}$ as $\mathrm{PGO}^+(\mathrm{pf})$, cf. [19, 3.4 ff]:

3.4 Lemma. **a.** *The group* $\mathrm{GL}_4\mathbb{K}$ *acts with precisely three orbits on the set of pairs of lines, represented by the pairs* (L_0, L_0), (L_0, K_0), *and* (L_0, K_1), *where* $L_0 := \langle b_0, b_1\rangle_{\mathbb{K}}$ *and* $K_j := \langle b_0 + jb_2, b_3\rangle_{\mathbb{K}}$.

b. *Two lines* $K, L \in \mathcal{L}$ *share a point if, and only if, their images* $\lambda(K)$ *and* $\lambda(L)$ *are orthogonal with respect to* q.

c. *The maximal totally singular subspaces with respect to* q *are just the images of maximal sets of pairwise confluent lines. There are two types of such sets: pencils* $\mathcal{L}_p := \{L \in \mathcal{L} \mid p < L\}$ *and, dually, line sets of planes* $\mathcal{L}_P := \{L \in \mathcal{L} \mid L < P\}$.

d. *The action of* $\mathrm{GL}_4\mathbb{K}$ *on the set* $\mathcal{M}_3$ *of maximal totally singular subspaces has two orbits, represented by* $\lambda(\mathcal{L}_p) = \langle S_1^0, S_2^0, S_3^0\rangle_{\mathbb{K}}$ *and* $J(\lambda(\mathcal{L}_p)) = \lambda(\mathcal{L}_P) = \langle S_2^1, S_3^1, S_3^2\rangle_{\mathbb{K}}$, *where* $p = \langle b_0\rangle_{\mathbb{K}}$, *and* $P = \langle b_1, b_2, b_3\rangle_{\mathbb{K}}$.

e. *The group* $\mathrm{GL}_4\mathbb{K}$ *acts transitively on the set* $\mathcal{M}_2$ *of* 2*-dimensional maximal totally singular subspaces. We may use* $\lambda(\mathcal{L}_p) \cap \lambda(\mathcal{L}_{P'}) = \langle S_1^0, S_2^0\rangle_{\mathbb{K}}$ *as a representative.* □

The orbits on $\mathrm{Gr}^\wedge_{2,4}$, $\mathrm{Gr}^\wedge_{3,4}$, $\mathrm{Gr}^\wedge_{4,4}$ and $\mathrm{Gr}^\wedge_{5,4}$ (i.e., on the sets of lines, planes, three-spaces, and hyperplanes, respectively, in the projective space $\mathcal{P}$ coordinatized by $\mathbb{K}^4 \wedge \mathbb{K}^4$) may be described using the Klein quadric Q. We introduce some more notation.

3.5 Definitions. We consider the following lines in $\mathcal{P}$:

$$E := \langle S_1^0, S_2^0\rangle_{\mathbb{K}}, \qquad T := \langle S_1^0, S_3^0 + S_2^1\rangle_{\mathbb{K}}, \quad \text{and} \quad S := \langle S_1^0, S_3^2\rangle_{\mathbb{K}}.$$

The orthogonal spaces (with respect to q) are

$$\begin{aligned} E^\perp &= \langle S_1^0, S_2^0, S_3^0, S_2^1\rangle_{\mathbb{K}}, \\ T^\perp &= \langle S_1^0, S_2^0, S_3^0 - S_2^1, S_3^1\rangle_{\mathbb{K}} \\ \text{and} \quad S^\perp &= \langle S_2^0, S_3^0, S_2^1, S_3^1\rangle_{\mathbb{K}}, \qquad \text{respectively.} \end{aligned}$$

We will also use the planes

$$\begin{aligned} F &:= \langle S_1^0, S_2^0, S_3^0\rangle_{\mathbb{K}}, & E+T &:= \langle S_1^0, S_2^0, S_3^0 + S_2^1\rangle_{\mathbb{K}}, \\ E+S &:= \langle S_1^0, S_2^0, S_3^2\rangle_{\mathbb{K}}, & T+S &:= \langle S_1^0, S_3^0 + S_2^1, S_3^2\rangle_{\mathbb{K}}. \end{aligned}$$

With respect to the given bases, the restriction of pf to the subspace X may be described by an upper triangular matrix m_X, where

$$m_E = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad m_T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad m_S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad m_F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$m_{E+T} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad m_{E+S} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad m_{T+S} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$m_{E^\perp} = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix}, \quad m_{T^\perp} = \begin{pmatrix} 0&0&0&0\\ 0&0&0&-1\\ 0&0&1&0\\ 0&0&0&0 \end{pmatrix}, \quad m_{S^\perp} = \begin{pmatrix} 0&0&0&-1\\ 0&0&1&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}.$$

Note that the Gram matrix for the polar form of the restriction is $m_X + (m_X)'$.

In order to describe certain subspaces that have only a small intersection with the Klein quadric, we use

$$\begin{aligned} P^t_{a,c} &:= \left\langle S^0_1 + aS^2_3,\ S^0_3 + cS^1_2 + tS^2_3 \right\rangle_{\mathbb{K}} \\ P^t_{a,b,c} &:= \left\langle S^0_1 + aS^2_3,\ S^0_2 - bS^1_3,\ S^0_3 + cS^1_2 + tS^2_3 \right\rangle_{\mathbb{K}} . \end{aligned}$$

The following has been proved in [19, 5.6].

3.6 Theorem. **a.** *The* $\mathrm{GL}_4\mathbb{K}$*-orbits in* $\mathrm{Gr}^\wedge_{1,4}$ *are represented by* $\langle S^0_1\rangle_{\mathbb{K}}$ *and* $\langle S^0_1 + S^2_3\rangle_{\mathbb{K}}$.

b. *The* $\mathrm{GL}_4\mathbb{K}$*-orbits in* $\mathrm{Gr}^\wedge_{2,4}$ *are represented by a set* $\{E, T, S\} \cup \mathcal{P}_1$*, where* $\mathcal{P}_1$ *denotes a (possibly empty) set of nonsingular lines.*

c. *The* $\mathrm{GL}_4\mathbb{K}$*-orbits in* $\mathrm{Gr}^\wedge_{3,4}$ *are represented by a set* $\{F, J(F), E+T, E+S, T+S\} \cup \mathcal{P}_2 \cup \mathcal{P}_3$*, where* $\mathcal{P}_2$ *denotes a (possibly empty) set of nonsingular planes, and* $\mathcal{P}_3$ *is a (possibly empty) set of planes of the form* $\langle S^0_2\rangle_{\mathbb{K}} + \ell$*, where* ℓ *is a nonsingular line contained in* $(S^0_2)^\perp$.

d. *The* $\mathrm{GL}_4\mathbb{K}$*-orbits in* $\mathrm{Gr}^\wedge_{4,4}$ *are represented by* $\mathcal{U}^\perp := \{U^\perp \mid U \in \mathcal{U}\}$*, where* $\mathcal{U}$ *is an arbitrary set of representatives in* $\mathrm{Gr}^\wedge_{2,4}$.

e. *The* $\mathrm{GL}_4\mathbb{K}$*-orbits in* $\mathrm{Gr}^\wedge_{5,4}$ *are represented by* $\langle S^0_1\rangle^\perp_{\mathbb{K}}$ *and* $\langle S^0_1 + S^2_3\rangle^\perp_{\mathbb{K}}$.

For $\ker\hat\beta \in \{F, E^\perp, \langle S^0_1\rangle^\perp_{\mathbb{K}}\}$ *the Heisenberg algebra* $\mathfrak{gh}(\mathbb{K}^4, (\mathbb{K}^4 \wedge \mathbb{K}^4)/\ker\hat\beta, \beta)$ *is not reduced. If* $\ker\hat\beta$ *runs over the remaining representatives we obtain a complete system of pairwise non-isomorphic reduced Heisenberg algebras of dimension* 4 *modulo their center.* □

3.7 Remarks. One may always choose $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$ using members from the collection of spaces $P^t_{a,c}$ or $P^t_{a,b,c}$ (in fact, we could choose *all* our representatives to be of this form, but prefer E, S and T). We indicate some special cases, cf. [19, 5.3]:

a. If $\mathbb{K}$ is a euclidean field (e.g. $\mathbb{K} = \mathbb{R}$) we may choose $\mathcal{P}_1 := \{P^0_{1,1}\}$, $\mathcal{P}_2 := \{P^0_{1,1,1}\}$, and $\mathcal{P}_3 := \{P^0_{1,0,1}\}$.

b. Let d, t be chosen such that the restriction of pf to $P^t_{1,d}$ is anisotropic. Then X^2+tX+d is an irreducible polynomial over $\mathbb{K}$. The Heisenberg algebras corresponding to the spaces $\left(P^0_{1,d}\right)^\perp = \langle S^0_2, S^1_3, S^0_1 - S^2_3, S^0_3 - dS^1_2\rangle_{\mathbb{K}}$ and $\left(P^1_{1,d}\right)^\perp = \langle S^0_2, S^1_3, S^0_1 - S^2_3 - S^1_2, S^0_3 - dS^1_2\rangle_{\mathbb{K}}$ may be interpreted as Heisenberg algebras of dimension 3 over the field extension $\mathbb{L}$ obtained by adjoining a root of $X^2 + tX + d$ to $\mathbb{K}$, cf. [19, 8.3]. In 5.3 below we give an explicit construction, starting from an embedding of $\mathbb{L}$ into the ring $\mathbb{K}^{2\times 2}$ and such that the corresponding embedding of $\mathrm{GL}_2\mathbb{L}$ by block matrices in $\mathrm{GL}_4\mathbb{K}$ leaves a suitable element of the orbit of $P^t_{1,d}$ invariant. This explicit description also helps to understand the elements of $\mathcal{P}_3$, see 5.4 below.

c. There is a connection between nonsingular subspaces of dimension 3 and Heisenberg algebras defined using a quaternion field over $\mathbb{K}$, see [19, 8.4] and Section 6 below. If $\mathrm{char}\,\mathbb{K} \neq 2$ then every anisotropic subspace of dimension 3 in $\mathbb{K}^4 \wedge \mathbb{K}^4$ belongs to the orbit of $P^0_{1,u,v}$ for suitable $u, v \in \mathbb{K}$ (see [19, 8.6]), and thus to the orbit of $\ker\widehat{\beta_{\mathbb{H}}}$ for the quaternion field $\mathbb{H} := \mathbb{H}^{u,v}_{\mathbb{K}}$. This connection allows to describe the relevant part of the automorphism groups of these Heisenberg algebras, see 6.6 and 6.10 below.

d. If $\mathbb{K}$ is quadratically closed (e.g., if $\mathbb{K} = \mathbb{C}$) then there are no nonsingular lines, and each of the sets $\mathcal{P}_j$ is empty.

e. If $\mathbb{K}$ is finite we may chose $\mathcal{P}_1 = \{P^t_{1,d}\}$, $\mathcal{P}_2 := \emptyset$, and $\mathcal{P}_3 := \{P^t_{1,0,d}\}$, for any irreducible polynomial $X^2 + tX + d$ over $\mathbb{K}$.

4. The cases where Σ_β can be computed directly

In order to determine $\mathrm{Aut}(\mathfrak{gh}(V, Z, \beta))$ for a reduced Heisenberg algebra $\mathfrak{gh}(V, Z, \beta)$ it suffices to determine Σ_β. This is comparatively easy if $V = \mathbb{K}^4$ and the kernel of $\hat\beta$ is one of the "generic" subspaces[2] of $\mathbb{K}^4 \wedge \mathbb{K}^4$, i.e., belongs to the orbit of $\langle S_1^0\rangle_\mathbb{K}$, $\langle S_1^0 + S_3^2\rangle_\mathbb{K}$, E, T, S, $J(F)$, $E+T$, $E+S$, $T+S$, or the orthogonal space of one of $\langle S_1^0\rangle_\mathbb{K}$, $\langle S_1^0 + S_3^2\rangle_\mathbb{K}$, E, T, or S. In fact, each of these spaces (except $\langle S_1^0 + S_3^2\rangle_\mathbb{K}$) has considerable intersection with the Klein quadric, and this intersection is invariant under the stabilizer Σ_β. The resulting conditions mean that $\Sigma_\beta \le \mathrm{GL}_4\mathbb{K}$ fixes a certain system of lines of the 3-dimensional projective space in each of these cases; and this allows to determine Σ_β. Details are given in this section.

The situation is more complicated if $\hat\beta$ meets the Klein quadric in only a few points, or none at all. This situation requires the existence of anisotropic forms in 2 or 3 variables; thus it only occurs for specially chosen fields, and is not "generic". We postpone the discussion of these cases to Sections 5 and 6 where we will use field extensions and quaternion algebras associated to these anisotropic forms to determine Σ_β.

For any subspace $X \le \mathbb{K}^4 \wedge \mathbb{K}^4$ the orthogonal space $X^\perp$ has the same stabilizer because $\mathrm{GL}_4\mathbb{K}$ acts by similitudes. In most of the following cases, the transitivity assertions that are implicit in the statements about the orbit representatives are easy to check by a direct computation: one has to apply the respective stabilizer to the given representatives.

4.1 Proposition. *The stabilizer of $\langle S_1^0\rangle_\mathbb{K}$ is the stabilizer of the line $\langle b_0, b_1\rangle_\mathbb{K} = \lambda^{-1}(S_1^0)$:*

$$(\mathrm{GL}_4\mathbb{K})_{\langle S_1^0\rangle_\mathbb{K}} = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \;\middle|\; A, C \in \mathrm{GL}_2\mathbb{K},\ B \in \mathbb{K}^{2\times 2} \right\} = (\mathrm{GL}_4\mathbb{K})_{\langle S_1^0\rangle_\mathbb{K}^\perp} .$$

The orbits of this stabilizer on $\mathbb{K}^4$ are represented by 0, b_0, b_2, those on $(\mathbb{K}^4 \wedge \mathbb{K}^4)/\langle S_1^0\rangle_\mathbb{K}$ by $\langle S_1^0\rangle_\mathbb{K}$, $S_2^0 + \langle S_1^0\rangle_\mathbb{K}$, $S_2^0 + S_3^1 + \langle S_1^0\rangle_\mathbb{K}$, $S_3^2 + \langle S_1^0\rangle_\mathbb{K}$ and those on $(\mathbb{K}^4 \wedge \mathbb{K}^4)/\langle S_1^0\rangle_\mathbb{K}^\perp$ by $\langle S_1^0\rangle_\mathbb{K}^\perp$, $S_3^2 + \langle S_1^0\rangle_\mathbb{K}^\perp$. □

The space $\langle S_1^0 + S_3^2\rangle_\mathbb{K}$ will be considered next. This describes a point outside the Klein quadric Q but the orthogonal space $\langle S_1^0 + S_3^2\rangle_\mathbb{K}^\perp$ has large intersection with Q. The treatment will be complicated; this is due to the fact that we discuss the representation of $\mathrm{Sp}_4\mathbb{K}$ as a group of orthogonal transformations on a space of dimension 5 (which is, indeed, the space $\langle S_1^0 + S_3^2\rangle_\mathbb{K}^\perp$). This representation gives rise to one of the interesting exceptional isomorphisms between classical groups, corresponding to the isomorphism of simple Lie algebras of types C_2 and B_2.

4.2 Proposition. *The stabilizer of $\langle S_1^0 + S_3^2\rangle_\mathbb{K}$ is a conjugate of that of $N := \langle S_2^0 + S_3^1\rangle_\mathbb{K}$, and*

$$(\mathrm{GL}_4\mathbb{K})_N = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \;\middle|\; \begin{array}{c} A, B, C, D \in \mathbb{K}^{2\times 2}, \\ AB' - BA' = 0 = CD' - DC' \\ \exists s \in \mathbb{K}^\times : AD' - BC' = sE_2 \end{array} \right\} = (\mathrm{GL}_4\mathbb{K})_{N^\perp} .$$

[2] We only consider subspaces that belong to *reduced* Heisenberg algebras, and ignore F, $E^\perp$, $\langle S_1^0\rangle_\mathbb{K}^\perp$, cf. 3.6.

This is the group $\mathrm{GSp}_4\mathbb{K}$ *of all similitudes of the non-degenerate alternating form mapping* $(x,y) \in \mathbb{K}^4 \times \mathbb{K}^4$ *to* $x'(S_2^0 + S_3^1)y$. *The group* $\mathrm{GSp}_4\mathbb{K}$ *acts transitively both on* $\mathbb{K}^4 \smallsetminus \{0\}$ *and on* $\big((\mathbb{K}^4 \wedge \mathbb{K}^4)/N^\perp\big) \smallsetminus \{0\}$. *In order to describe the orbits on* N, *on the orthogonal space* $N^\perp = \langle S_1^0, S_2^0 - S_3^1, S_3^0, S_3^2, S_2^1\rangle_\mathbb{K}$, *and on* $(\mathbb{K}^4 \wedge \mathbb{K}^4)/N$ *we choose a set* R_* *of representatives for the cosets forming the multiplicative group* $\mathbb{K}^\times/\mathbb{K}^{\boxtimes}$ *of square classes; here* $\mathbb{K}^{\boxtimes} := \{s^2 \mid s \in \mathbb{K}^\times\}$ *is the multiplicative group of squares. If* $\operatorname{char}\mathbb{K} = 2$ *we need to pick a set* R_+ *of representatives for the orbits of* $\mathbb{K}^{\boxtimes}$ *on the additive group* $\mathbb{K}/\mathbb{K}^{\square}$ *where* $\mathbb{K}^{\square} := \{x^2 \mid x \in \mathbb{K}\}$ *is the subfield of squares, and a set* $R_\wp$ *of representatives for the cosets in the additive group* $\mathbb{K}/\wp$, *cf. 2.1.*

We have to distinguish cases:

- **a.** *If an orbit of* $\mathrm{GSp}_4\mathbb{K}$ *on* $\mathbb{K}^4 \wedge \mathbb{K}^4$ *contains an element* X *with* $q(X) \neq 0$ *then* q *assumes on that orbit only values from a single square class.*
- **b.** *If* $\operatorname{char}\mathbb{K} \neq 2$ *then the orbits on* $(\mathbb{K}^4 \wedge \mathbb{K}^4)/N$ *are represented by the elements of the set* $\mathcal{R}_* := \{N\} \cup \{S_1^0 + rS_3^2 + N \mid r \in R_* \cup \{0\}\}$.
- **c.** *If* $\operatorname{char}\mathbb{K} = 2$ *then the orbits in* $N^\perp/N$ *are represented by the elements of* $\mathcal{R}_+ := \{N\} \cup \big\{(S_1^0 + rS_3^2) + N \,\big|\, r \in R_+\big\}$, *and those in* $\big((\mathbb{K}^4 \wedge \mathbb{K}^4)/N\big) \smallsetminus \big(N^\perp/N\big)$ *are represented by* $\mathcal{R}_\wp := \{S_3^1 + N\} \cup \big\{S_3^1 + (S_1^0 + rS_3^2) + N \,\big|\, r \in R_\wp\big\}$.
- **d.** *In any case, the orbits on* N *are represented by* $\big\{r(S_2^0 + S_3^1) \,\big|\, r \in R_* \cup \{0\}\big\}$.

Note that $\mathcal{R}_+$ *is an infinite set whenever* $\mathbb{K}$ *is not a perfect field: the additive group of* $\mathbb{K}$ *forms a vector space over the subfield* $\mathbb{K}^\square$ *of squares; in the corresponding affine space the set* $\mathcal{R}_+$ *represents the line* $\mathbb{K}^\square 1$ *and the* $\mathbb{K}^\square$*-planes through that line.*

The set $R_\wp$ *can be chosen inside* $\mathbb{K}^\square$ *because* r^2 *belongs to* $r + \wp$.

Proof. Only the assertions about the orbits need a proof. For the sake of easy reference later on, we note that the stabilizer of N contains the subgroups

$$\Delta := \left\{ \begin{pmatrix} aA & 0 \\ 0 & (A^{-1})' \end{pmatrix} \,\middle|\, \begin{matrix} A \in \mathrm{GL}_2\mathbb{K} \\ a \in \mathbb{K}^\times \end{matrix} \right\},\ \Lambda := \left\{ \begin{pmatrix} E_2 & 0 \\ X & E_2 \end{pmatrix} \,\middle|\, \begin{matrix} X \in \mathbb{K}^{2\times 2} \\ X' = X \end{matrix} \right\},\ \Upsilon := \Lambda' .$$

It is well known[3] that $\mathrm{Sp}_4\mathbb{K}$ acts transitively on $\mathbb{K}^4 \smallsetminus \{0\}$. We offer a direct argument: let $x, y \in \mathbb{K}^2$ and assume that not both of these vectors are 0. Our claim then is that there exists $\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in \mathrm{Sp}_4\mathbb{K} \subseteq \mathrm{GSp}_4\mathbb{K}$ mapping $\left(\begin{smallmatrix} x \\ y\end{smallmatrix}\right)$ to $(1,0,0,0)'$. If $y = 0$ we choose $A \in \mathrm{GL}_2\mathbb{K}$ with $Ax = \left(\begin{smallmatrix} 1 \\ 0\end{smallmatrix}\right)$, put $D := (A')^{-1}$ and $B = 0 = C$. If $y \neq 0$ we find $B \in \mathrm{GL}_2\mathbb{K}$ such that $By = \left(\begin{smallmatrix} 1 \\ 0\end{smallmatrix}\right)$, and a symmetric matrix $S \in \mathbb{K}^{2\times 2}$ with $S\left(\begin{smallmatrix} 1 \\ 0\end{smallmatrix}\right) = -(B')^{-1}x$. Now $A := 0$, $C := -(B')^{-1}$ and $D := -SB$ yields an element of $\mathrm{Sp}_4\mathbb{K}$ with the required properties.

The cosets in $(\mathbb{K}^4 \wedge \mathbb{K}^4)/N^\perp$ are represented by elements of $\langle S_2^0\rangle_\mathbb{K}$; the nontrivial representatives are in a single orbit under the subgroup $\left\{ \left(\begin{smallmatrix} sE_2 & 0 \\ 0 & E_2\end{smallmatrix}\right) \,\middle|\, s \in \mathbb{K}^\times \right\}$ of $\mathrm{GSp}_4\mathbb{K}$. The same subgroup also shows that $\big\{r(S_2^0 + S_3^1) \,\big|\, r \in R_* \cup \{0\}\big\}$ contains a set of representatives for the orbits on N. Different elements of this set can not be fused into the same orbit because the values $q(r(S_2^0 + S_3^1)) = -r^2$ all belong to the same square class; and assertion d is proved. We note that this observation also implies assertion a: On each $\mathrm{GSp}_4\mathbb{K}$-orbit on $\mathbb{K}^4 \wedge \mathbb{K}^4$ the form q assumes only values from a single square class, or only the value 0.

It remains to determine the orbits on $(\mathbb{K}^4 \wedge \mathbb{K}^4)/N$. We consider an arbitrary element $M = \left(\begin{smallmatrix} ai & B \\ -B' & ci\end{smallmatrix}\right)$ of $\mathbb{K}^4 \wedge \mathbb{K}^4$. Note that $q(M) = ac - \det B$. Assume first $M \in N^\perp$; then $B \in \mathbb{K}^{2\times 2}$ has trace 0 and $-B'i$ is symmetric. We search for a representative with $B = 0$ in the orbit of M, so assume $B \neq 0$.

[3] From Witt's Theorem, see [15] § 9 or [5] § 11, p. 21 and § 16, p. 35, cf. [20] 7.4 or [1] Thm. 3.9 for $\operatorname{char}\mathbb{K} \neq 2$.

If $a \neq 0$ we use $\left(\begin{smallmatrix} E_2 & 0 \\ -a^{-1}B'i & E_2 \end{smallmatrix}\right) \in \Lambda$ to transform M into $\left(\begin{smallmatrix} ai & 0 \\ 0 & di \end{smallmatrix}\right)$, where $d = c - a^{-1}\det B$. A suitable element of Δ maps this into an element of $\left\{S_1^0 + rS_3^2 \,\middle|\, r \in R_* \cup \{0\}\right\}$. The case $a = 0 \neq c$ is reduced to the previous one by an application of $\left(\begin{smallmatrix} 0 & E_2 \\ -E_2 & 0 \end{smallmatrix}\right) \in \mathrm{GSp}_4\mathbb{K}$.

If $a = 0 = c$ we may assume that B is not a scalar multiple of E_2 because otherwise M belongs to $0 + N \in \mathcal{R}_*$. We pick a symmetric matrix X such that BX is not symmetric. Then $\left(\begin{smallmatrix} E_2 & X \\ 0 & E_2 \end{smallmatrix}\right) \in \Upsilon$ can be used to transform M into a matrix with $a \neq 0$.

Thus we have shown that $\mathcal{R}_*$ contains a full set of representatives for the orbits of cosets of elements in $(N^\perp + N)/N$.

If $\operatorname{char}\mathbb{K} \neq 2$ then the orthogonal space is a complement to N and of course invariant under the stabilizer of N. Different elements of $\mathcal{R}_*$ belong to different orbits because their images under q belong to different square classes (cf. assertion a), and assertion b of the theorem is established.

The situation changes in two respects if $\operatorname{char}\mathbb{K} = 2$: since $N \leq N^\perp$ it is possible that different elements of $\mathcal{R}_*$ belong to the same orbit, and we have to consider representatives outside $N^\perp$ as well.

We apply $\left(\begin{smallmatrix} E_2 & 0 \\ a^2 i & E_2 \end{smallmatrix}\right) \in \Lambda$ to $(S_1^0 + rS_3^2) + N$ and obtain $(S_1^0 + (r + a^2)S_3^2) + a(S_2^0 + S_3^1) + N = (S_1^0 + (r + a^2)S_3^2) + N$. This shows that the orbits in $N^\perp/N$ are represented by the elements of $\{N\} \cup \left\{(S_1^0 + rS_3^2) + N \,\middle|\, r \in R_+\right\}$, as claimed.

In order to treat orbits without representatives in $N^\perp$ we consider $M = \left(\begin{smallmatrix} ai & T \\ -T' & ci \end{smallmatrix}\right)$ where the trace t of T is different from 0. We write $p := \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and find that $B := T - tp$ has trace 0; thus Bi is symmetric.

If $a \neq 0$ we use $\left(\begin{smallmatrix} E_2 & 0 \\ -a^{-1}B'i & E_2 \end{smallmatrix}\right) \in \Lambda$ to find $\left(\begin{smallmatrix} ai & tp \\ -tp & di \end{smallmatrix}\right)$ in the orbit of M. Now we use $A := \left(\begin{smallmatrix} t^{-1} & 0 \\ 0 & t^{-1}a \end{smallmatrix}\right) \in \mathrm{GL}_2\mathbb{K}$ and $\left(\begin{smallmatrix} tA & 0 \\ 0 & A \end{smallmatrix}\right) \in \Delta$ to transform $\left(\begin{smallmatrix} ai & tp \\ -tp & di \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} i & p \\ -p & si \end{smallmatrix}\right)$ with $s = t^{-2}ad$. As above, the cases with $a = 0$ are reduced to this case; note that $T \notin \mathbb{K}E_2$.

It remains to identify sets of representatives from $\mathcal{T}_+ := \left\{(S_1^0 + rS_3^2) + N \,\middle|\, r \in \mathbb{K}\right\}$ and $\mathcal{T}_\wp := \{S_3^1 + N\} \cup \left\{S_3^1 + (S_1^0 + rS_3^2) + N \,\middle|\, r \in \mathbb{K}\right\}$. Since $N^\perp$ is invariant under the action, we know that an element of $\mathcal{T}_+$ is never in the orbit of an element of $\mathcal{T}_\wp$.

We compute $q\left(aS_3^1 + (S_1^0 + rS_3^2) + N\right) = \left\{r + u^2 + au \,\middle|\, u \in \mathbb{K}\right\}$. Since $\mathrm{GSp}_4\mathbb{K}$ can change the values of the Pfaffian form only by square factors we know that $aS_3^1 + (S_1^0 + rS_3^2) + N$ and $aS_3^1 + (S_1^0 + tS_3^2) + N$ can only belong to the same orbit if there exists $v \in \mathbb{K}^\times$ such that $v^2 t = r + u^2 + au$.

If $a = 0$ we apply the element $\left(\begin{smallmatrix} E_2 & 0 \\ 0 & vE_2 \end{smallmatrix}\right) \in \Delta$ to see that indeed the elements $(S_1^0 + rS_3^2) + N$ and $(S_1^0 + tS_3^2) + N$ belong to the same orbit if $s \in \mathbb{K}^{\boxtimes}(r + \mathbb{K}^{\square})$. Thus we have proved that $\mathcal{R}_+$ represents the orbits inside $\mathcal{T}_+$.

If $a = 1$ we use $F := \left(\begin{smallmatrix} E_2 & 0 \\ ui & E_2 \end{smallmatrix}\right)$ which belongs to Λ because $\operatorname{char}\mathbb{K} = 2$, and observe $F\left(S_3^1 + (S_1^0 + rS_3^2)\right)F' = S_3^1 + S_1^0 + (r - u^2)S_3^2 + u(S_2^0 + S_3^1) = S_3^1 + S_1^0 + tS_3^2 + u(S_2^0 + S_3^1)$ as required.

Finally, assume that $S_3^1 + (S_1^0 + rS_3^2) + N$ and $S_3^1 + (S_1^0 + tS_3^2) + N$ belong to the same orbit under the stabilizer of N. Then there exists a multiplier $u^2 \in \mathbb{K}^{\boxtimes}$ such that $r + \wp = q\left(S_3^1 + (S_1^0 + rS_3^2) + N\right) = u^2\, q\left(S_3^1 + (S_1^0 + tS_3^2) + N\right) = u^2 t + u^2\wp$. This yields $\wp = u^2\wp$. We claim that $u = 1$ if $r + \wp \neq t + \wp$. In fact, for each $x \in \mathbb{K}$ we have $u^2(x^2 + x) \in u^2\wp = \wp$ and then $(u+1)ux = u^2(x^2 + x) + \left((ux)^2 + (ux)\right) \in \wp + \wp = \wp$. Thus $(u+1)u\,\mathbb{K} \subseteq \wp$, and $(u+1)u = 0$ follows because $r + \wp \neq t + \wp$ implies $\wp \neq \mathbb{K}$. Now $u = 1$ is the only solution for $(u+1)u = 0$ in $\mathbb{K}^\times$. $\square$

4.3 Proposition. *The stabilizer* $(\mathrm{GL}_4\mathbb{K})_E$ *of* E *coincides with the stabilizer of the point-plane flag* $(\langle b_0\rangle_\mathbb{K}, \langle b_0, b_1, b_2\rangle_\mathbb{K})$. *Thus*

$$(\mathrm{GL}_4\mathbb{K})_E = \left\{ \left(\begin{array}{ccc} a & b' & c \\ 0 & D & e \\ 0 & 0 & f \end{array} \right) \middle| \begin{array}{cc} a, c, f \in \mathbb{K}, & af \neq 0, \\ b, e \in \mathbb{K}^2, & D \in \mathrm{GL}_2\mathbb{K} \end{array} \right\} = (\mathrm{GL}_4\mathbb{K})_{E^\perp} .$$

The orbits on $\mathbb{K}^4$ *are represented by* 0, b_0, b_1, b_3, *those on* $(\mathbb{K}^4 \wedge \mathbb{K}^4)/E$ *are represented by* E, $S_3^0 + E$, $S_2^1 + E$, $S_3^1 + E$, *and those on* $(\mathbb{K}^4 \wedge \mathbb{K}^4)/E^\perp$ *by* $E^\perp$, $S_3^1 + E^\perp$.

Proof. The statement about the stabilizer follows from the fact that E corresponds (via λ, see 3.4.e) to the set of lines passing through $\langle b_0\rangle_\mathbb{K}$ and lying in the plane $\langle b_0, b_1, b_2\rangle_\mathbb{K}$. ☐

4.4 Proposition. *The stabilizer* $(\mathrm{GL}_4\mathbb{K})_T$ *of* T *is*

$$(\mathrm{GL}_4\mathbb{K})_T = \left\{ \left(\begin{array}{cc} A & B \\ 0 & c\sigma A\sigma \end{array} \right) \middle| \begin{array}{cc} A \in \mathrm{GL}_2\mathbb{K}, & B \in \mathbb{K}^{2\times 2}, \\ c \in \mathbb{K} \setminus \{0\} & \end{array} \right\} = (\mathrm{GL}_4\mathbb{K})_{T^\perp} ,$$

where $\sigma = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. *The orbits on* $\mathbb{K}^4$ *are represented by* 0, b_0, b_3, *those on* $(\mathbb{K}^4 \wedge \mathbb{K}^4)/T$ *are represented by* T, $S_2^0 + T$, $S_3^0 + T$, $S_3^1 + T$, *and those on* $(\mathbb{K}^4 \wedge \mathbb{K}^4)/T^\perp$ *by* $T^\perp$, $S_3^0 + T^\perp$, $S_3^2 + T^\perp$.

Proof. The intersection of T with the Klein quadric is the single point $\langle S_1^0\rangle_\mathbb{K}$. Thus the stabilizer of T leaves the line $\langle b_0, b_1\rangle_\mathbb{K}$ invariant, and

$$(\mathrm{GL}_4\mathbb{K})_T \leq \left\{ \left(\begin{array}{cc} A & B \\ 0 & C \end{array} \right) \middle| A, C \in \mathrm{GL}_2\mathbb{K}, B \in \mathbb{K}^{2\times 2} \right\} .$$

Evaluating the requirement

$$\left(\begin{array}{cc} A & B \\ 0 & C \end{array} \right) . (S_3^0 + S_2^1) = \left(\begin{array}{cc} A & B \\ C & D \end{array} \right) \in \langle S_1^0, S_3^0 + S_2^1\rangle_\mathbb{K}$$

gives the condition $C \in \mathbb{K}^\times \sigma A\sigma$, as claimed. In order to see that the orbits of $S_3^0 + T^\perp$ and $S_3^2 + T^\perp$ are large enough, it is helpful to observe $\left(\begin{smallmatrix} A & 0 \\ 0 & \sigma A\sigma \end{smallmatrix}\right) . S_3^0 \in (\det A) S_3^0 + T^\perp$, and to use $B = \left(\begin{smallmatrix} f & 0 \\ 0 & 0 \end{smallmatrix}\right) \sigma A\sigma$. ☐

4.5 Proposition. *The stabilizer* $(\mathrm{GL}_4\mathbb{K})_S$ *of* S *is*

$$(\mathrm{GL}_4\mathbb{K})_S = \left\{ \left(\begin{array}{cc} A & 0 \\ 0 & B \end{array} \right) \middle| A, B \in \mathrm{GL}_2\mathbb{K} \right\} \cup \left\{ \left(\begin{array}{cc} 0 & A \\ B & 0 \end{array} \right) \middle| A, B \in \mathrm{GL}_2\mathbb{K} \right\} = (\mathrm{GL}_4\mathbb{K})_{S^\perp} .$$

The orbits on $\mathbb{K}^4$ *are represented by* 0, b_0, $b_0 + b_2$, *those on* $(\mathbb{K}^4 \wedge \mathbb{K}^4)/S$ *are represented by* S, $S_2^0 + S$, $S_2^0 + S_3^1 + S$, *and those on* $(\mathbb{K}^4 \wedge \mathbb{K}^4)/S^\perp$ *by* $S^\perp$, $S_1^0 + S^\perp$, $S_1^0 + S_3^2 + S^\perp$.

Proof. The description of the stabilizer follows immediately from the observation that S meets the Klein quadric in exactly two points, corresponding to the two lines $\langle b_0, b_1\rangle_\mathbb{K}$ and $\langle b_2, b_3\rangle_\mathbb{K}$, respectively. ☐

4.6 Proposition. *The stabilizer* $(\mathrm{GL}_4\mathbb{K})_{J(F)}$ *of* $J(F) = \langle S_2^1, S_3^1, S_3^2\rangle_\mathbb{K}$ *coincides with the stabilizer of the plane* $(\langle b_1, b_2, b_3\rangle_\mathbb{K})$. *Thus*

$$(\mathrm{GL}_4\mathbb{K})_{J(F)} = \left\{ \left(\begin{array}{cc} a & 0 \\ b & C \end{array} \right) \middle| a \in \mathbb{K} \setminus \{0\}, b \in \mathbb{K}^3, C \in \mathrm{GL}_3\mathbb{K} \right\} .$$

The orbits on $\mathbb{K}^4$ *are represented by* 0, b_1, b_0, *and those on* $(\mathbb{K}^4 \wedge \mathbb{K}^4)/J(F)$ *are represented by* $J(F)$, $S_1^0 + J(F)$.

Proof. This follows from the fact (see 3.4.d) that $J(F)$ corresponds to the set of all lines in the plane $P = \langle b_1, b_2, b_3\rangle_{\mathbb{K}}$. □

4.7 Proposition. *The stabilizer of $E+T$ is*

$$(\mathrm{GL}_4\mathbb{K})_{E+T} = \left\{ \begin{pmatrix} a & b' & c \\ 0 & D & e \\ 0 & 0 & \frac{1}{a}\det D \end{pmatrix} \,\middle|\, \begin{matrix} a, c \in \mathbb{K}, & a \neq 0, \\ b, e \in \mathbb{K}^2, & D \in \mathrm{GL}_2\mathbb{K} \end{matrix} \right\}.$$

The orbits on $\mathbb{K}^4$ are represented by 0, b_0, b_1, b_3, and the orbits on $(\mathbb{K}^4 \wedge \mathbb{K}^4)/(E+T)$ are represented by $E+T$, $S_3^0 + (E+T)$, $S_3^1 + (E+T)$.

Proof. The intersection of $E+T$ with the Klein quadric is just E. Thus $(\mathrm{GL}_4\mathbb{K})_{E+T}$ is contained in $(\mathrm{GL}_4\mathbb{K})_E$, see 4.3. Evaluating the condition

$$\begin{pmatrix} a & b' & c \\ 0 & D & e \\ 0 & 0 & f \end{pmatrix}.(S_3^0 + S_2^1) = \begin{pmatrix} a & b' & c \\ 0 & D & e \\ 0 & 0 & f \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & i & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ b & D' & 0 \\ c & e' & f \end{pmatrix} \in E+T$$

we obtain the description of the stabilizer. □

4.8 Proposition. *The stabilizer $(\mathrm{GL}_4\mathbb{K})_{E+S}$ of $E+S$ is*

$$(\mathrm{GL}_4\mathbb{K})_{E+S} = \left\{ \begin{pmatrix} a & b & 0 & c \\ 0 & d & 0 & 0 \\ 0 & e & f & g \\ 0 & 0 & 0 & h \end{pmatrix} \,\middle|\, \begin{matrix} a, d, f, h \in \mathbb{K}^\times \\ b, c, e, g \in \mathbb{K} \end{matrix} \right\} \cup \left\{ \begin{pmatrix} 0 & c & a & b \\ 0 & 0 & 0 & d \\ f & g & 0 & e \\ 0 & h & 0 & 0 \end{pmatrix} \,\middle|\, \begin{matrix} b, d, e, h \in \mathbb{K}^\times \\ a, c, f, g \in \mathbb{K} \end{matrix} \right\}.$$

This group has 5 orbits on $\mathbb{K}^4$, represented by 0, b_0, $b_0 + b_2$, $b_0 + b_2 + b_3$, $b_1 + b_3$, and 4 orbits on $(\mathbb{K}^4 \wedge \mathbb{K}^4)/(E+S)$, represented by $E+S$, $S_3^0 + (E+S)$, $S_3^1 + (E+S)$, $S_3^0 + S_2^1 + (E+S)$.

Proof. The plane $E+S$ meets the Klein quadric in two lines, namely E and $\langle S_2^0, S_3^2\rangle_{\mathbb{K}}$. Thus the stabilizer fixes their intersection point $\langle S_2^0\rangle_{\mathbb{K}}$ which corresponds to the line $\langle b_0, b_2\rangle_{\mathbb{K}}$ while the planes $\langle b_0, b_1, b_2\rangle_{\mathbb{K}}$ and $\langle b_0, b_2, b_3\rangle_{\mathbb{K}}$ are either swapped or left invariant. Now easy calculations yield the stabilizer. □

Instead of $T+S$, we consider a different representative of the orbit $\mathrm{GL}_4\mathbb{K}.(T+S)$, namely $K := \langle S_2^0, S_3^0 + S_2^1, S_3^1\rangle_{\mathbb{K}} = \left\{ \left(\begin{smallmatrix} 0 & X \\ -X & 0 \end{smallmatrix} \right) \,\middle|\, X \in \mathbb{K}^{2\times 2}, X' = X \right\}$. Note that the orthogonal spaces $(T+S)^\perp$ and $K^\perp = \langle S_1^0, S_3^0 - S_2^1, S_3^2\rangle_{\mathbb{K}}$ belong to the orbit of $T+S$, as well.

4.9 Proposition. *The stabilizer $(\mathrm{GL}_4\mathbb{K})_{T+S}$ of $T+S$ is a conjugate of $(\mathrm{GL}_4\mathbb{K})_K = \Phi\Delta$, where*

$$\Phi := \left\{ \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix} \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2\mathbb{K} \right\}, \quad \Delta := \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \,\middle|\, A \in \mathrm{GL}_2\mathbb{K} \right\}.$$

Note that Φ and Δ centralize each other, their intersection is the center of $\mathrm{GL}_4\mathbb{K}$, and each of these factors is isomorphic to $\mathrm{GL}_2\mathbb{K}$.

The orbits of $\Phi\Delta$ on $\mathbb{K}^4$ are represented by 0, b_0, $b_0 + b_3$. In order to understand the orbits on $(\mathbb{K}^4 \wedge \mathbb{K}^4)/K$, we have to consider the multiplicative group $\mathbb{K}^{\boxtimes}$ of squares, pick a set R_* of representatives for the group $\mathbb{K}^\times/\mathbb{K}^{\boxtimes}$ of square classes, and distinguish the cases:

a. *If* $\operatorname{char}\mathbb{K} \neq 2$ *then the orbits on* $(\mathbb{K}^4 \wedge \mathbb{K}^4)/K$ *are represented by the elements of the set*

$$\{0\} \cup \left\{ S_2^0 + rS_3^1 + K \;\middle|\; r \in R_* \cup \{0\} \right\} .$$

b. *If* $\operatorname{char}\mathbb{K} = 2$ *we also need a set* $R_\wp$ *of coset representatives for the additive group* $\mathbb{K}/\wp$*, cf. 2.1. The orbits on* $(\mathbb{K}^4 \wedge \mathbb{K}^4)/K$ *are represented by the elements of the set*

$$\{0\} \cup \left\{ S_2^0 + rS_3^1 + K \;\middle|\; r \in R_* \cup \{0\} \right\} \cup \left\{ S_1^0 + S_3^0 + cS_3^2 \;\middle|\; c \in R_\wp \right\} .$$

Proof. First of all, we remark that

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

map $T + S$ to K and $K^\perp$, respectively. It is easy to check that Φ and Δ are subgroups of $(\mathrm{GL}_4\mathbb{K})_K$. The plane K intersects the Klein quadric in the non-degenerate conic

$$\mathcal{C} := \left\{ \left\langle \left(\begin{smallmatrix} 0 & X \\ -X & 0 \end{smallmatrix} \right) \right\rangle_{\mathbb{K}} \;\middle|\; X' = X \neq 0, \det X = 0 \right\} .$$

The group $\Phi\Delta$ acts on $\mathcal{C}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \cdot \begin{pmatrix} 0 & X \\ -X & 0 \end{pmatrix} = (ad - bc) \begin{pmatrix} 0 & AXA' \\ -AXA' & 0 \end{pmatrix} .$$

It is easy to see that this action of Δ on $\mathcal{C}$ is sharply 3-transitive; in fact, we have given here an explicit representation of $\mathrm{PGL}_2\mathbb{K}$ as the group of similitudes of the quadratic form $\det$ on the space of symmetric 2×2 matrices yielding a sharply 3-transitive action on the conic.

The stabilizer of the three points $\langle S_2^0 \rangle_{\mathbb{K}}$, $\langle S_3^1 \rangle_{\mathbb{K}}$, and $\langle S_2^0 + S_3^0 + S_2^1 + S_3^1 \rangle_{\mathbb{K}}$ in $(\mathrm{GL}_4\mathbb{K})_K$ is the group Φ. Thus we have $(\mathrm{GL}_4\mathbb{K})_K = \Phi\Delta$, as claimed.

In order to understand the action of $\Phi\Delta$ on $(\mathbb{K}^4 \wedge \mathbb{K}^4)/K$, we first describe quadratic forms on $\mathbb{K}^2$ by upper matrices: For $X := \left(\begin{smallmatrix} r & s \\ t & u \end{smallmatrix} \right)$ we put $q_X \colon v \mapsto v'Xv$ and $\widehat{X} := \left(\begin{smallmatrix} r & s+t \\ 0 & u \end{smallmatrix} \right)$. Clearly we have $q_X = q_{\widehat{X}}$, and a basis transformation $v \mapsto Bv$ transforms q_X to $q_{B'XB}$.

For the cosets in $(\mathbb{K}^4 \wedge \mathbb{K}^4)/K$ we use representatives of the form

$$\rho \left(\begin{smallmatrix} x & y_1 \\ y_2 & z \end{smallmatrix} \right) := xS_1^0 + (y_1 + y_2)S_3^0 + zS_3^2 = \begin{pmatrix} 0 & x & 0 & y_1 + y_2 \\ -x & 0 & 0 & 0 \\ 0 & 0 & 0 & z \\ -y_1 - y_2 & 0 & -z & 0 \end{pmatrix} .$$

Note that $\rho(X) = \rho(\widehat{X})$ holds for each $X \in \mathbb{K}^{2\times 2}$. A straightforward computation gives $(B, A).\widehat{X} - \det(A)\, \widehat{BXB'} \in K$ for each pair $(B, A) \in \Phi \times \Delta$. Thus the action of $\Phi\Delta$ on $(\mathbb{K}^4 \wedge \mathbb{K}^4)/K$ is equivalent to the action on the space of quadratic forms on $\mathbb{K}^2$.

If $\operatorname{char}\mathbb{K} \neq 2$ every quadratic form q_X is diagonalizable: i.e., there exists $B \in \mathrm{GL}_2\mathbb{K}$ with $\widehat{BXB'} = \left(\begin{smallmatrix} r & 0 \\ 0 & u \end{smallmatrix} \right)$ with $r, u \in \mathbb{K}$. If the form is non-zero, we may also assume $r \neq 0$. Replacing the second basis vector by a scalar multiple just multiplies u by the square of that factor. Choosing A such that $\det A = r^{-1}$ gives the assertion for $\operatorname{char}\mathbb{K} \neq 2$.

If $\operatorname{char}\mathbb{K} = 2$ we have to distinguish between diagonalizable forms (where the equivalence classes are again described by the square classes) and non-diagonalizable forms. Since the action of $\Phi\Delta$ allows to pick representatives that assume the value 1, the latter orbits are characterized by the Arf invariant, see 2.2. This yields the assertion for $\operatorname{char}\mathbb{K} = 2$. □

For the case where $\operatorname{char}\mathbb{K} \neq 2$, we obtain an alternative description of $(\mathrm{GL}_4\mathbb{K})_{T+S}$ in 6.15 below, in terms of the split quaternion algebra.

4.10 Remark. For a finite field $\mathbb{F}$ of characteristic 2, we have $|\mathbb{K}/\wp(\mathbb{K})| = 2$. Anything may happen in the infinite case, even if $\mathbb{K}$ is a perfect field.

5. Examples involving field extensions

Subspaces that meet the Klein quadric in only few points (or even no points at all) present particular problems when classifying them or when determining the automorphism groups of the corresponding Heisenberg algebras. Therefore, we will now discuss connections between anisotropic subspaces, quadratic extension fields, and (in Section 6) quaternion fields, together with constructions of Heisenberg algebras related to these structures.

5.1 Example. An explicit model for the (unique) isomorphism type of reduced Heisenberg algebra $\mathfrak{gh}(V,Z,\beta)$ with $\dim V = 2$ and $\dim Z = 1$ is $\mathfrak{gh}(\mathbb{K}^2,\mathbb{K},\det)$, where $\det(v,w)$ is the usual determinant of the 2×2 matrix with columns v,w. This algebra is *the* Heisenberg algebra used to explain the uncertainty principle.

5.2 Example. Among the reduced Heisenberg algebras $\mathfrak{gh}(V,Z,\beta)$ with $\dim V = 4$ and $\dim Z = 2$, we find the direct product $\mathfrak{gh}(\mathbb{K}^2,\mathbb{K},\det)\times\mathfrak{gh}(\mathbb{K}^2,\mathbb{K},\det)$. This algebra is isomorphic to $\mathfrak{gh}(\mathbb{K}^4,\mathbb{K}^2,\beta)$, where the kernel of $\hat\beta$ is $S^\perp = \langle S_2^0, S_3^0, S_2^1, S_3^1\rangle_{\mathbb{K}}$, see 3.5.

5.3 Examples from Quadratic Extensions. Let $\mathbb{L} = \mathbb{K}(u)$ be a quadratic extension field of $\mathbb{K}$, where the minimal polynomial of u over $\mathbb{K}$ is $X^2 + tX + d$. As a $\mathbb{K}$-algebra, the Heisenberg algebra $\mathfrak{gh}(\mathbb{L}^2,\mathbb{L},\det)$ is then isomorphic to $\mathfrak{gh}(\mathbb{K}^4,\mathbb{K}^2,\beta_u)$, where the kernel of $\widehat{\beta_u}$ belongs to the orbit of $(P^t_{1,d})^\perp$, cf. 3.7.

Note that we may choose $t\in\{0,1\}$, and that $t=1$ is only needed if $\operatorname{char}\mathbb{K}=2$ and the extension is a separable one (cf. [15, 8.11, p. 313]).

Here is a more explicit way to describe the action of $\mathrm{GL}_2\mathbb{L}$ on $\ker\widehat{\beta_u}$: We identify $\mathbb{L}$ with $\left\{\left(\begin{smallmatrix} x & -yd\\ y & x+yt\end{smallmatrix}\right) \,\middle|\, x,y\in\mathbb{K}\right\}$ and $\mathbb{L}^{2\times 2}$ with the set $\left\{\left(\begin{smallmatrix} A & B\\ C & D\end{smallmatrix}\right) \,\middle|\, A,B,C,D\in\mathbb{L}\right\}$ of block matrices. Thus u is identified with $\left(\begin{smallmatrix} 0 & -d\\ 1 & t\end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & -d\\ 1 & -t\end{smallmatrix}\right)$.

For $\delta := \left(\begin{smallmatrix} d & 0\\ 0 & -1\end{smallmatrix}\right)$ we observe $\delta A' = A\delta$ for all $A\in\mathbb{L}$. Now the set

$$P_{\mathbb{L}} := \left\{\left(\begin{smallmatrix} 0 & X\delta\\ -X\delta & 0\end{smallmatrix}\right) \,\middle|\, X\in\mathbb{L}\right\} = \left\langle dS_2^0 - S_3^1, dS_3^0 + dS_2^1 - tS_3^1\right\rangle_{\mathbb{K}}$$

is a subspace of $\mathbb{K}^4\wedge\mathbb{K}^4$, belongs to the orbit of $P^t_{1,d}$ and is invariant under the action of $\mathrm{GL}_2\mathbb{L}\le\mathrm{GL}_4\mathbb{K}$. Using $i = \left(\begin{smallmatrix} 0 & 1\\ -1 & 0\end{smallmatrix}\right)$ we obtain the orthogonal space as

$$P_{\mathbb{L}}^\perp = \left\{\left(\begin{smallmatrix} xi & Yi\\ -(Yi)' & zi\end{smallmatrix}\right) \,\middle|\, x,z\in\mathbb{K}, Y\in\mathbb{L}\right\} = \langle S_1^0, dS_2^0 + S_3^1 + tS_2^1, S_3^0 - S_2^1, S_3^2\rangle_{\mathbb{K}}\,.$$

Of course the orthogonal space $P_{\mathbb{L}}^\perp$ is also invariant under $\mathrm{GL}_2\mathbb{L}$; this space serves as a model for $\ker\widehat{\beta_u}$. For $X\in\mathbb{L}$ we observe $iX'i^{-1} = \overline{X}$, where $\overline{r+su} := r - su + st$. This helps to verify invariance of $P_{\mathbb{L}}^\perp$ explicitly; here we use the fact that $t\neq 0$ only occurs if $\operatorname{char}\mathbb{K}=2$.

The intersection of the projective 3-space coordinatized by $P_{\mathbb{L}}^\perp$ with the Klein quadric is the ellipsoid

$$\mathcal{O} := \left\{\left\langle\begin{pmatrix} xi & Yi\\ -(Yi)' & zi\end{pmatrix}\right\rangle_{\mathbb{K}} \,\middle|\, x,z\in\mathbb{K}, Y\in\mathbb{L}, Y\overline{Y} = xzE_2\right\}.$$

It is easy to see that $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\in \mathrm{GL}_2\mathbb{L}$ interchanges $S_1^0=\left(\begin{smallmatrix}i&0\\0&0\end{smallmatrix}\right)$ with $S_3^2=\left(\begin{smallmatrix}0&0\\0&i\end{smallmatrix}\right)$, and that the orbit $\left\{\left\langle\left(\begin{smallmatrix}B\overline{B}i & Bi\\ -(Bi)' & i\end{smallmatrix}\right)\right\rangle_{\mathbb{K}} \,\middle|\, B\in\mathbb{L}\right\}$ of $\langle S_3^2\rangle_{\mathbb{K}}$ under the stabilizer $\left\{\left(\begin{smallmatrix}A&B\\0&D\end{smallmatrix}\right) \,\middle|\, A,D\in\mathbb{L}^\times, B\in\mathbb{L}\right\}$ of $\langle S_1^0\rangle_{\mathbb{K}}$ in $\mathrm{GL}_2\mathbb{L}$ equals $\mathcal{O}\smallsetminus\{\langle S_1^0\rangle_{\mathbb{K}}\}$.

5.4 Corollary. *The representatives in $\mathcal{P}_1$ may be chosen in the form $P_{\mathbb{L}}^{\perp}$, and those in $\mathcal{P}_3$ may be chosen from the collection of spaces of the form $P_{\mathbb{L}}^0 := P_{\mathbb{L}}\oplus\langle S_1^0\rangle_{\mathbb{K}}$. Here $\mathbb{L}$ and $P_{\mathbb{L}}$ are constructed as in 5.3 using an irreducible polynomial X^2+tX+d over $\mathbb{K}$ with $2t=0$.* □

5.5 Lemma. *Let X^2+tX+d be irreducible over $\mathbb{K}$, with $2t=0$. As in 5.3 we choose $\mathbb{L}$ and $P_{\mathbb{L}}$ and write $i:=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ and $\delta:=\left(\begin{smallmatrix}d&0\\0&-1\end{smallmatrix}\right)$.*

- **a.** *If the extension $\mathbb{L}/\mathbb{K}$ is separable then $\mathbb{K}^{2\times2}=\mathbb{L}i\oplus\mathbb{L}\delta$. In this case the orthogonal space $P_{\mathbb{L}}^{\perp}$ is a vector space complement to $P_{\mathbb{L}}$ in $\mathbb{K}^4\wedge\mathbb{K}^4$.*
- **b.** *If the extension $\mathbb{L}/\mathbb{K}$ is inseparable then $\mathbb{L}i=\mathbb{L}\delta$. In this case the polar form vanishes on $P_{\mathbb{L}}$, and $P_{\mathbb{L}}\subset P_{\mathbb{L}}^{\perp}$.*
- **c.** *If $t=0$ then $\mathbb{K}^{2\times2}=\mathbb{L}\oplus\mathbb{L}\delta$.*
- **d.** *If $t=1$ then $\mathbb{K}^{2\times2}=\mathbb{L}\oplus\mathbb{L}i$.*
- **e.** *If $d=1=t$ then $\mathbb{L}=\mathbb{L}\delta$.*
- **f.** *If $d=1$ and $t=0$ then $\mathbb{L}=\mathbb{L}i$.*

Proof. For $X:=\left(\begin{smallmatrix}a&-bd\\b&a+bt\end{smallmatrix}\right)$ and $Y:=\left(\begin{smallmatrix}x&-yd\\y&x+yt\end{smallmatrix}\right)$ in $\mathbb{L}$ we compute $Xi=\left(\begin{smallmatrix}bd&a\\-a-bt&b\end{smallmatrix}\right)$ and $Y\delta=\left(\begin{smallmatrix}xd&yd\\yd&-x-yt\end{smallmatrix}\right)$. Equality $Xi=Y\delta$ thus implies $b=x$, $a=yd$ and then $-bt=2a$, $2x=-yt$. If the extension is separable, we have $\operatorname{char}\mathbb{K}\neq2$ or $t\neq0$. In either case, we infer $A=0=X$. This means $\mathbb{L}i\cap\mathbb{L}\delta=\{0\}$. Using $-(Xi)'=iX'=\overline{X}i$ and $-(Y\delta)'=-\delta Y'=-Y\delta$ we infer $\mathbb{K}^4\wedge\mathbb{K}^4=\left\{\left(\begin{smallmatrix}ai&Xi\\\overline{X}i&ci\end{smallmatrix}\right)\,\middle|\, a,c\in\mathbb{K}, X\in\mathbb{L}\right\}\oplus\left\{\left(\begin{smallmatrix}0&Y\delta\\-Y\delta&0\end{smallmatrix}\right)\,\middle|\, Y\in\mathbb{L}\right\}=P_{\mathbb{L}}^{\perp}\oplus P_{\mathbb{L}}$ as claimed.

The extension $\mathbb{L}/\mathbb{K}$ is inseparable if $\operatorname{char}\mathbb{K}=2$ and $t=0$. In this case the sets in question coincide: $\mathbb{L}i=\left\{\left(\begin{smallmatrix}bd&a\\a&b\end{smallmatrix}\right)\,\middle|\, a,b\in\mathbb{K}\right\}=\mathbb{L}\delta$. □

5.6 Theorem. *Let X^2+tX+d be irreducible over $\mathbb{K}$, with $2t=0$. We choose $\mathbb{L}$ and $P_{\mathbb{L}}$ as in 5.3 and put $P_{\mathbb{L}}^0:=P_{\mathbb{L}}\oplus\langle S_1^0\rangle_{\mathbb{K}}$. Moreover, we write $\xi:=\left(\begin{smallmatrix}1&t\\0&-1\end{smallmatrix}\right)$ and $\Xi:=\left(\begin{smallmatrix}\xi&0\\0&\xi\end{smallmatrix}\right)\in\mathrm{GL}_4\mathbb{K}$. Then conjugation by ξ induces the (possibly trivial) generator of $\mathrm{Gal}(\mathbb{L}/\mathbb{K})$ on $\mathbb{L}$, and*

$$(\mathrm{GL}_4\mathbb{K})_{P_{\mathbb{L}}}=\mathrm{GL}_2\mathbb{L}\,\langle\Xi\rangle=(\mathrm{GL}_4\mathbb{K})_{P_{\mathbb{L}}^{\perp}} \text{ where } \mathrm{GL}_2\mathbb{L}=\left\{\begin{pmatrix}A&B\\C&D\end{pmatrix}\,\middle|\,\begin{matrix}A,B,C,D\in\mathbb{L},\\ AD-BC\neq0\end{matrix}\right\},$$

$$(\mathrm{GL}_4\mathbb{K})_{P_{\mathbb{L}}^0}=\left\{\begin{pmatrix}A&B\\0&D\end{pmatrix}\,\middle|\, A,D\in\mathbb{L}^\times, B\in\mathbb{K}^{2\times2}\right\}\langle\Xi\rangle\,.$$

The group $(\mathrm{GL}_4\mathbb{K})_{P_{\mathbb{L}}}$ acts with 2 orbits on $\mathbb{K}^4$, represented by 0 and b_0, and with 2 orbits on $(\mathbb{K}^4\wedge\mathbb{K}^4)/P_{\mathbb{L}}^{\perp}$, represented by $P_{\mathbb{L}}$ and any other coset of $P_{\mathbb{L}}$. The orbits on $(\mathbb{K}^4\wedge\mathbb{K}^4)/P_{\mathbb{L}}$ are more complicated, we have to distinguish cases:

- **a.** *If the extension $\mathbb{L}/\mathbb{K}$ is separable (in particular, if $\operatorname{char}\mathbb{K}\neq2$) then the action of $\mathrm{GL}_2\mathbb{L}$ on $(\mathbb{K}^4\wedge\mathbb{K}^4)/P_{\mathbb{L}}$ is equivalent to the action on the space of hermitian 2×2 matrices, cf. 2.5. Here Ξ acts as $-\mathrm{id}$ on the set of diagonal matrices, and an orbit under $\mathrm{GL}_2\mathbb{L}\,\langle\Xi\rangle$ is the union of two different $\mathrm{GL}_2\mathbb{L}$-orbits if, and only if, the norm group of the corresponding quaternion field does not contain -1.*
- **b.** *Now assume that $\mathbb{L}/\mathbb{K}$ is an inseparable extension. Then the action of $\mathrm{GL}_2\mathbb{L}$ on $P_{\mathbb{L}}^{\perp}/P_{\mathbb{L}}$ is equivalent to the action of $\mathrm{GL}_2\mathbb{L}^{\square}\leq\mathrm{GL}_2\mathbb{K}$ on $\mathbb{K}^2$. The orbits under that action are represented by the elements of $\{(r,0)'\mid r\in R_{\mathbb{K}/\mathbb{L}^{\square}}\}\cup R^{(2)}_{\mathbb{K}/\mathbb{L}^{\square}}$ (see 2.3).*

The $\mathrm{GL}_2\mathbb{L}$*-orbits on* $\big((\mathbb{K}^4\wedge\mathbb{K}^4)/P_{\mathbb{L}}\big)\smallsetminus\big(P_{\mathbb{L}}^{\perp}/P_{\mathbb{L}}\big)$ *are represented by the elements of the set* $\left\{\rho_z+P_{\mathbb{L}}\;\middle|\; z\in R_{\#}\right\}$ *where* $\rho_z := \left(\begin{smallmatrix} i & E_2\\ E_2 & zi\end{smallmatrix}\right)$ *and* $R_{\#}$ *is a set of representatives for the orbits under the action*

$$\#\colon \mathrm{SL}_2\mathbb{L}\times\mathbb{K}^2\to\mathbb{K}^2\colon \left(\begin{pmatrix} A & B\\ C & D\end{pmatrix},\begin{pmatrix} x_1\\ x_2\end{pmatrix}\right)\mapsto\begin{pmatrix} A^2x_1+B^2x_2+(AB+(AB)')i\\ C^2x_1+D^2x_2+(CD+(CD)')i\end{pmatrix}.$$

Note that no point is fixed under this action of $\mathrm{SL}_2\mathbb{L}$ *by affine transformations of* $\mathbb{K}^2$ *(viewed as an affine space over* $\mathbb{L}^{\square}\le\mathbb{K}$*).*

c. *In any case (separable or not), the orbits of* $(\mathrm{GL}_4\mathbb{K})_{P^0_{\mathbb{L}}}$ *on* $\mathbb{K}^4$ *are represented by* 0*,* b_0*,* b_2*.*

d. *For the orbits on* $(\mathbb{K}^4\wedge\mathbb{K}^4)/P^0_{\mathbb{L}}$ *we pick a set* $R_N\subseteq\mathbb{K}^{\times}$ *of representatives for the cosets modulo* $N_{\mathbb{L}/\mathbb{K}}(\mathbb{L}^{\times})\,\langle -1\rangle$*. Then the orbits on* $(\mathbb{K}^4\wedge\mathbb{K}^4)/P^0_{\mathbb{L}}$ *are represented by the elements of* $\{P^0_{\mathbb{L}},(S^0_2+S^1_3)+P^0_{\mathbb{L}}\}\cup\left\{cS^2_3\;\middle|\;c\in R_N\right\}$ *— again, irrespective of separability.*

Proof. Clearly the group $\mathrm{GL}_2\mathbb{L}=\left\{\left(\begin{smallmatrix} A & B\\ C & D\end{smallmatrix}\right)\in\mathrm{GL}_4\mathbb{K}\;\middle|\; A,B,C,D\in\mathbb{L},\ AD-BC\ne 0\right\}$ is contained in $(\mathrm{GL}_4\mathbb{K})_{P_{\mathbb{L}}}$ and acts transitively on the intersection $\mathcal{O}$ of the Klein quadric with the projective 3-space coordinatized by $P_{\mathbb{L}}\smallsetminus\{0\}^{\perp}$, cf. 5.3. Thus it remains to determine the stabilizer of $\langle S^0_1\rangle_{\mathbb{K}}$ in $(\mathrm{GL}_4\mathbb{K})_{P_{\mathbb{L}}}$. Evaluating the condition

$$\begin{pmatrix} A & B\\ 0 & D\end{pmatrix}\begin{pmatrix} 0 & X\delta\\ -X\delta & 0\end{pmatrix}\begin{pmatrix} A' & 0\\ B' & D'\end{pmatrix}=\begin{pmatrix} AX\delta B'-BX\delta A' & AX\delta D'\\ -DX\delta A' & 0\end{pmatrix}\in P_{\mathbb{L}}$$

we find that for each $X\in\mathbb{L}^{\times}$ there exists $L_X\in\mathbb{L}^{\times}$ such that $AX\delta D'=L_X\delta$. Specializing $X=1\in\mathbb{L}$ we obtain $\delta D'=A^{-1}L_1\delta$ and then $AXA^{-1}=L_XL_1^{-1}\in\mathbb{L}$. Thus A belongs to the normalizer of $\mathbb{L}$ in $\mathrm{GL}_2\mathbb{K}$. According to Schur's Lemma (e.g., see [9, 3.5, p. 118] or [11, Ch. XVII, Prop. 1.1]), this normalizer is $\mathbb{L}^{\times}\operatorname{Gal}(\mathbb{L}/\mathbb{K})=\mathbb{L}^{\times}\langle\xi\rangle$. As Ξ belongs to $(\mathrm{GL}_4\mathbb{K})_{P_{\mathbb{L}}}$ we may assume $A\in\mathbb{L}^{\times}$ from now on. Then $D=\delta(A^{-1}L_1)'\delta^{-1}=A^{-1}L_1$ also lies in $\mathbb{L}^{\times}$. There remains the condition that $BX\delta A'$ is symmetric for each $X\in\mathbb{L}$. Specializing $X=1$ and $X=u=\left(\begin{smallmatrix} 0 & -d\\ 1 & t\end{smallmatrix}\right)$ we find $BA\in\mathbb{L}$ and therefore $B\in\mathbb{L}$. We have thus proved that $(\mathrm{GL}_4\mathbb{K})_{P_{\mathbb{L}}}=\mathrm{GL}_2\mathbb{L}\,\langle\Xi\rangle$.

The stabilizer of $P^0_{\mathbb{L}}$ in $(\mathrm{GL}_4\mathbb{K})$ fixes $\langle S^0_1\rangle_{\mathbb{K}}$ because this is the only intersection point of the Klein quadric with the plane coordinatized by $P^0_{\mathbb{L}}$. Thus $(\mathrm{GL}_4\mathbb{K})_{P^0_{\mathbb{L}}}$ is contained in $\left\{\left(\begin{smallmatrix} A & B\\ 0 & D\end{smallmatrix}\right)\;\middle|\; A,D\in\mathrm{GL}_2\mathbb{K}, B\in\mathbb{K}^{2\times 2}\right\}$. The elements of the stabilizer are characterized by the condition

$$\begin{pmatrix} AX\delta B'-BX\delta A' & AX\delta D'\\ -DX\delta A' & 0\end{pmatrix}\in P^0_{\mathbb{L}}.$$

As in the case discussed before, the upper right entry yields $A\in\mathbb{L}^{\times}\langle\xi\rangle$; we may assume $A\in\mathbb{L}$, and then $D\in\mathbb{L}$ follows. However, the entry on the upper left does not mean any restriction now, and we obtain $(\mathrm{GL}_4\mathbb{K})_{P^0_{\mathbb{L}}}=\left\{\left(\begin{smallmatrix} A & B\\ 0 & D\end{smallmatrix}\right)\;\middle|\; A,D\in\mathbb{L}^{\times},B\in\mathbb{K}^{2\times 2}\right\}\langle\Xi\rangle$, as claimed.

The assertions about orbits on $\mathbb{K}^4$ are easily verified for both stabilizers. In order to understand the orbits on $(\mathbb{K}^4\wedge\mathbb{K}^4)/P_{\mathbb{L}}^{\perp}$ we pick a vector space complement W for $P_{\mathbb{L}}^{\perp}$ in $\mathbb{K}^4\wedge\mathbb{K}^4$. In the separable case we may use $W=P_{\mathbb{L}}=\left\{\left(\begin{smallmatrix} 0 & X\delta\\ -X\delta & 0\end{smallmatrix}\right)\;\middle|\; X\in\mathbb{L}\right\}$ while $W=\left\{\left(\begin{smallmatrix} 0 & X\\ X' & 0\end{smallmatrix}\right)\;\middle|\; X\in\mathbb{L}\right\}$ is a suitable choice in the inseparable case. In both cases it is easy to see that the action of $M\in\mathrm{GL}_2\mathbb{K}$ on W is given by multiplication of X with $\det_{\mathbb{L}}M$. Therefore, the action on the set of non-trivial cosets in $(\mathbb{K}^4\wedge\mathbb{K}^4)/P_{\mathbb{L}}^{\perp}$ is transitive.

If the extension is separable then the action of $\mathrm{GL}_2\mathbb{L}\,\langle\Xi\rangle$ on $(\mathbb{K}^4\wedge\mathbb{K}^4)/P_{\mathbb{L}}$ is equivalent to the action on the invariant orthogonal complement $P_{\mathbb{L}}^{\perp}$. Using 5.5 we see that this action is equivalent to the action on the space $\left\{\left(\begin{smallmatrix} a & X \\ \overline{X} & c \end{smallmatrix}\right) \;\middle|\; a, c \in \mathbb{K}, X \in \mathbb{L}\right\}$ of hermitian 2×2 matrices:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} a & X \\ \overline{X} & c \end{pmatrix} := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a & X \\ \overline{X} & c \end{pmatrix} \begin{pmatrix} \overline{A} & \overline{C} \\ \overline{B} & \overline{D} \end{pmatrix} .$$

The orbit structure on $(\mathbb{K}^4 \wedge \mathbb{K}^4)/P_{\mathbb{L}}$ thus corresponds to the classification of hermitian forms over $\mathbb{L}$, cf. 2.5. An easy computation shows that Ξ acts as $-\mathrm{id}$ on the diagonal forms. The assertion about fusion of $\mathrm{GL}_2\mathbb{L}$-orbits under $\langle\Xi\rangle$ now also follows from 2.5.

If $\mathbb{L}/\mathbb{K}$ is inseparable (i.e., if $\operatorname{char}\mathbb{K} = 2$ and $t = 0$) we note that $\left\{\left(\begin{smallmatrix} xi & Y \\ Y' & zi \end{smallmatrix}\right) \;\middle|\; x, z \in \mathbb{K}, Y \in \mathbb{L}\right\}$ is a vector space complement for $P_{\mathbb{L}} = \left\{\left(\begin{smallmatrix} 0 & Y\delta \\ Y\delta & 0 \end{smallmatrix}\right) \;\middle|\; Y \in \mathbb{L}\right\}$. The action of $\mathrm{GL}_2\mathbb{L}$ on $P_{\mathbb{L}}^{\perp}/P_{\mathbb{L}}$ is described by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} xi & 0 \\ 0 & zi \end{pmatrix} \begin{pmatrix} A' & C' \\ B' & D' \end{pmatrix} = \begin{pmatrix} (A^2x + B^2z)i & 0 \\ 0 & (C^2x + D^2z)i \end{pmatrix} .$$

This looks like the usual action on diagonalizable quadratic forms (cf. 2.3) but with the group $\mathrm{GL}_2\mathbb{L}$ replacing $\mathrm{GL}_2\mathbb{K}$: if x and z are linearly independent over $\mathbb{L}^{\square}$ then the orbit of $(x, z)'$ consists of all bases for $\langle x, y\rangle_{\mathbb{L}^{\square}}$. If x and z are linearly dependent over $\mathbb{L}^{\square}$ then the orbit of $(x, z)'$ contains $(y, 0)'$ where $\langle y\rangle_{\mathbb{L}^{\square}} = \langle x, z\rangle_{\mathbb{L}^{\square}}$. This gives the assertion about the orbits in $P_{\mathbb{L}}^{\perp}/P_{\mathbb{L}}$.

For $A, B \in \mathbb{L}$ there exists $b \in \mathbb{K}$ with $B' - B = bi$, and both $AB' - AB = A(B - B') = bAi$ and $AB' + BA' - (AB + (AB)') = b(A - \overline{A})i$ belong to $\mathbb{L}i$. Thus $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{GL}_2\mathbb{L}$ maps $\left(\begin{smallmatrix} xi & Y \\ Y' & zi \end{smallmatrix}\right) + P_{\mathbb{L}}$ to

$$\begin{pmatrix} (A^2x + B^2z)i + AYB + (AYB)' & (AD - BC)Y \\ Y'(AD - BC)' & (C^2x + D^2z)i + CYD + (CYD)' \end{pmatrix} + P_{\mathbb{L}} .$$

From 5.5.c and 5.5.b we know $\mathbb{K}^{2\times 2} = \mathbb{L} \oplus \mathbb{L}\delta$ and $\mathbb{L}i = \mathbb{L}\delta$. Thus each $\mathrm{GL}_2\mathbb{L}$-orbit on $\left((\mathbb{K}^4 \wedge \mathbb{K}^4)/P_{\mathbb{L}}\right) \smallsetminus \left(P_{\mathbb{L}}^{\perp}/P_{\mathbb{L}}\right)$ contains a representative of the form $\rho_v + P_{\mathbb{L}}$ where $v = (v_1, v_2) \in \mathbb{K}^2$ and $\rho_v := \left(\begin{smallmatrix} v_1 i & E_2 \\ E_2 & v_2 i \end{smallmatrix}\right)$. Since $\rho_w + P_{\mathbb{L}} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \rho_v + P_{\mathbb{L}}$ implies $AD - BC = 1$, we are left with the action $\#$ of $\mathrm{SL}_2\mathbb{L}$, as claimed.

It remains to determine the orbits of $(\mathrm{GL}_4\mathbb{K})_{P_{\mathbb{L}}^0}$ on $(\mathbb{K}^4 \wedge \mathbb{K}^4)/P_{\mathbb{L}}^0$. If $\mathbb{L}/\mathbb{K}$ is separable then $\mathbb{K}^{2\times 2} = \mathbb{L}\delta \oplus \mathbb{L}i$ and we may choose representatives for cosets modulo $P_{\mathbb{L}}^0$ from the vector space complement $\left\{\left(\begin{smallmatrix} 0 & Xi \\ iX' & ci \end{smallmatrix}\right) \;\middle|\; c \in \mathbb{K}, X \in \mathbb{L}\right\}$ to $P_{\mathbb{L}}^0 = \left\{\left(\begin{smallmatrix} ai & X\delta \\ -X\delta & 0 \end{smallmatrix}\right) \;\middle|\; a \in \mathbb{K}, X \in \mathbb{L}\right\}$. If $X \neq 0$ then the orbit of $\left(\begin{smallmatrix} 0 & Xi \\ iX' & 0 \end{smallmatrix}\right)$ contains $S_2^0 + S_3^1 = \left(\begin{smallmatrix} 0 & E_2 \\ E_2 & 0 \end{smallmatrix}\right)$. If $c \neq 0$ then $\left(\begin{smallmatrix} E_2 & -c^{-1}X \\ 0 & D \end{smallmatrix}\right) \in (\mathrm{GL}_4\mathbb{K})_{P_{\mathbb{L}}^0}$ maps $\left(\begin{smallmatrix} 0 & Xi \\ iX' & ci \end{smallmatrix}\right) + P_{\mathbb{L}}^0$ to $\left(\begin{smallmatrix} 0 & 0 \\ 0 & D\overline{D}ci \end{smallmatrix}\right) + P_{\mathbb{L}}^0$. We may achieve $D\overline{D}c \in R_N$ and assertion d follows from the fact that $\Xi \left(\begin{smallmatrix} 0 & 0 \\ 0 & ci \end{smallmatrix}\right) \Xi' = \left(\begin{smallmatrix} 0 & 0 \\ 0 & -ci \end{smallmatrix}\right)$.

If, finally, the extension is inseparable then $\mathbb{L}\delta = \mathbb{L}i$ has trivial intersection with $\mathbb{L}$. Thus we may use the complement $\left\{\left(\begin{smallmatrix} 0 & X \\ -X' & ci \end{smallmatrix}\right) \;\middle|\; c \in \mathbb{K}, X \in \mathbb{L}\right\}$. It is easy to see that $S_2^0 + S_3^1$ belongs to the orbit of $\left(\begin{smallmatrix} 0 & X \\ -X' & 0 \end{smallmatrix}\right)$ if $X \in \mathbb{L}^{\times}$. For $c \neq 0$ we use $\left(\begin{smallmatrix} E_2 & c^{-1}Xi \\ 0 & D \end{smallmatrix}\right) \in (\mathrm{GL}_4\mathbb{K})_{P_{\mathbb{L}}^0}$ in order to map $\left(\begin{smallmatrix} 0 & X \\ X' & ci \end{smallmatrix}\right) + P_{\mathbb{L}}^0$ to $\left(\begin{smallmatrix} 0 & 0 \\ 0 & D\overline{D}ci \end{smallmatrix}\right) + P_{\mathbb{L}}^0$. Now we may achieve $D\overline{D}c \in R_N$. This gives assertion d also in the inseparable case. □

5.7 Remark. For the case $\operatorname{char}\mathbb{K} \neq 2$ the assertion about $\Sigma_{\beta_u} = (\mathrm{GL}_4\mathbb{K})_{P_{\mathbb{L}}}$ in 5.6 is just a special case of [6, Th. 1.1.1]. In fact, since automorphisms of the *group* $\mathrm{GH}(\mathbb{L}^2, \mathbb{L}, \det)$ are considered in [6] the cited result yields all automorphisms of the *Lie ring* $\mathfrak{gh}_{\mathbb{Z}}(\mathbb{L}^2, \mathbb{L}, \det)$:

apart from the automorphisms of $\mathfrak{gh}_{\mathbb{K}}(\mathbb{L}^2, \mathbb{L}, \det)$ we also have the automorphisms induced by arbitrary field automorphisms of $\mathbb{L}$, and not only those from $\mathrm{Gal}(\mathbb{L}/\mathbb{K})$. Moreover, we have to add *arbitrary* additive maps τ from $\mathbb{L}^2$ to $\mathbb{L}$, and not only the $\mathbb{K}$-linear ones. See also 8.1.

6. Examples involving quaternion algebras

We will use quaternion algebras to describe the alternating maps $\beta\colon \mathbb{K}^4 \times \mathbb{K}^4 \to \mathbb{K}^3$ where $\ker \widehat{\beta} \in \mathcal{P}_2$ describes an anisotropic plane, having no point in common with the Klein quadric. Our construction and our results will not depend critically on $\operatorname{char} \mathbb{K}$ up to the point where we investigate the action on $(\mathbb{K}^4 \wedge \mathbb{K}^4)/\ker \widehat{\beta}$ in 6.7 and 6.8.

6.1 Anisotropic planes and quaternions. We recall from 3.7 that the orbit of any member of $\mathcal{P}_2$ contains a representative of the form $P^t_{1,c,d}$ with $2t = 0$. A cyclic permutation of the basis vectors b_0, b_1, b_2 shows that this orbit also contains

$$W := W^t_{c,d} := \langle S^0_3 - S^1_2, cS^0_1 - S^2_3, S^1_3 + dS^0_2 + tS^0_3\rangle_{\mathbb{K}} = \left\{ \begin{pmatrix} xci & Yi \\ \overline{Y}i & -xi \end{pmatrix} \,\middle|\, x \in \mathbb{K}, Y \in \mathbb{L} \right\}$$

for $\mathbb{L} := \left\{ \left(\begin{smallmatrix} x & -yd \\ y & x+yt \end{smallmatrix}\right) \,\middle|\, x, y \in \mathbb{K} \right\}$. Since the restriction of the Pfaffian form to W is assumed to be anisotropic the polynomial $X^2 + tX + d$ is irreducible in $\mathbb{K}[X]$. The subalgebra $\mathbb{L}$ of $\mathbb{K}^{2\times 2}$ is isomorphic to the extension field $\mathbb{K}[X]/(X^2 + tX + d)$, cf. 5.3. We also recall that conjugation by $\xi := \left(\begin{smallmatrix} 1 & t \\ 0 & -1 \end{smallmatrix}\right)$ induces the (possibly trivial) generator of the Galois group $\mathrm{Gal}(\mathbb{L}/\mathbb{K})$ on $\mathbb{L}$, mapping $A := \left(\begin{smallmatrix} x & -yd \\ y & x+yt \end{smallmatrix}\right)$ to $\overline{A} := \left(\begin{smallmatrix} x+yt & yd \\ -y & x \end{smallmatrix}\right)$. We use the root $u := \left(\begin{smallmatrix} 0 & -d \\ 1 & t \end{smallmatrix}\right)$ of $X^2 + tX + d$ in $\mathbb{L}$; note that $\overline{u} = t - u$.

If the extension $\mathbb{L}/\mathbb{K}$ is separable then the hermitian matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & c \end{smallmatrix}\right)$ describes an anisotropic hermitian form h on $\mathbb{L}^2$, and $\mathbb{H} := \mathbb{H}^c_{\mathbb{L}/\mathbb{K}} = \left\{ \left(\begin{smallmatrix} A & -c\overline{B} \\ B & \overline{A} \end{smallmatrix}\right) \,\middle|\, A, B \in \mathbb{L} \right\}$ is a quaternion field, cf. 2.4. We identify $A \in \mathbb{L}$ with $\left(\begin{smallmatrix} A & 0 \\ 0 & \overline{A} \end{smallmatrix}\right)$ and put $I := \left(\begin{smallmatrix} 0 & -cE_2 \\ E_2 & 0 \end{smallmatrix}\right)$; then $\mathbb{H} = \mathbb{L} \oplus I\mathbb{L}$.

The matrices in $\mathbb{H}$ may be considered as matrices for left multiplications $\lambda_a\colon x \mapsto ax$. In fact, with respect to the basis

$$\begin{pmatrix} E_2 & 0 \\ 0 & E_2 \end{pmatrix}, \quad \begin{pmatrix} u & 0 \\ 0 & \overline{u} \end{pmatrix}, \quad \begin{pmatrix} 0 & -cE_2 \\ E_2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -c\overline{u} \\ u & 0 \end{pmatrix}$$

we find that λ_{A+IB} is described by $\left(\begin{smallmatrix} A & -c\overline{B} \\ B & \overline{A} \end{smallmatrix}\right)$. Applying Schur's Lemma (cf. [9, 3.5, p. 118] or [11, Ch. XVII, Prop. 1.1]) we infer that the centralizer $\mathbb{H}^\rho$ of $\mathbb{H}^\times$ in $\mathrm{GL}_4\mathbb{K}$ consists of the matrices for right multiplications $\rho_a\colon x \mapsto xa$, with respect to the same basis. A straightforward computation yields that ρ_{A+IB} is described by the matrix $\left(\begin{smallmatrix} A & -cB\xi \\ B\xi & A \end{smallmatrix}\right)$. Thus $\mathbb{H}^\rho = \left\{ \left(\begin{smallmatrix} A & -cB\xi \\ B\xi & A \end{smallmatrix}\right) \,\middle|\, (A, B) \in \mathbb{L}^2 \smallsetminus \{(0,0)\} \right\}$.

If $\mathbb{L}/\mathbb{K}$ is inseparable then ξ is the identity matrix, and $\mathbb{H} := \left\{ \left(\begin{smallmatrix} A & -cB \\ B & A \end{smallmatrix}\right) \,\middle|\, A, B \in \mathbb{L} \right\}$ becomes a commutative field which (again by Schur's Lemma) coincides with its centralizer, indeed $\mathbb{H}^\times = \mathbb{H}^\rho$ in that case.

In any case, we define $\tilde{a} := \left(\begin{smallmatrix} \overline{A} & c\overline{B} \\ -B & A \end{smallmatrix}\right)$ for $a = \left(\begin{smallmatrix} A & -c\overline{B} \\ B & \overline{A} \end{smallmatrix}\right)$ and obtain an anti-automorphism of $\mathbb{H}$. Note that this anti-automorphism is the identity if $\mathbb{L}/\mathbb{K}$ is inseparable. We use the *norm* $N\colon \mathbb{H} \to \mathbb{K}\colon a \mapsto \tilde{a}a$.

6.2 Definition. If $\mathbb{L}/\mathbb{K}$ is separable then $\mathrm{Pu}(\mathbb{H}) := \{X \in \mathbb{H} \mid \tilde{X} = -X\} = 1^\perp$. For $\operatorname{char}\mathbb{K} = 2$ we have $1 \in \mathrm{Pu}(\mathbb{H})$ and $\mathrm{Pu}(\mathbb{H}) = \mathbb{K} \oplus I\mathbb{L}$. In the inseparable case the norm form on $\mathbb{H}$ has trivial polar form, and $1^\perp = \mathbb{H}$. We extend the definition of $\mathrm{Pu}(\mathbb{H})$ to this case quite arbitrarily, putting $\mathrm{Pu}(\mathbb{H}) := \mathbb{K} \oplus I\mathbb{L}$ as in the remaining cases where $\operatorname{char}\mathbb{K} = 2$.

In any case, we find that the restriction $N|_{\mathrm{Pu}(\mathbb{H})}$ of the norm is equivalent to $-\mathrm{pf}|_W$.

6.3 Lemma. **a.** *If the extension $\mathbb{L}/\mathbb{K}$ is separable then $\mathrm{GO}(N|_{\mathrm{Pu}(\mathbb{H})}) = \mathbb{K}^\times \mathrm{SO}(N|_{\mathrm{Pu}(\mathbb{H})}) = \{(x \mapsto sax\tilde{a}) \mid a \in \mathbb{H}^\times, s \in \mathbb{K}^\times\}$ and $\mathrm{SO}(N|_{\mathrm{Pu}(\mathbb{H})}) = \{(x \mapsto axa^{-1}) \mid a \in \mathbb{H}^\times\}$.*

b. *In the inseparable case each one of the groups $\mathrm{O}(\mathrm{pf}|_W)$, $\mathrm{O}(N)$, $\mathrm{O}(N|_{\mathrm{Pu}(\mathbb{H})})$ is trivial; note that $\mathrm{Pu}(\mathbb{H}) = \mathbb{K} \oplus I\mathbb{L}$ by definition 6.2. The groups $\mathrm{GO}(\mathrm{pf}|_W)$ and $\mathrm{GO}(N|_{\mathrm{Pu}(\mathbb{H})})$ of similitudes coincide with $\mathbb{K}^\times \mathrm{id}$ but $\mathrm{GO}(N) = \mathbb{H}^\times$.*

Proof. For a separable extension $\mathbb{L}/\mathbb{K}$ the assertion has been proved in 2.13, cf. 2.8.

Now assume that the extension $\mathbb{L}/\mathbb{K}$ is inseparable. For $x = x_0 + x_1u + I(x_2 + x_3u)$ with $x_0, x_1, x_2, x_3 \in \mathbb{K}$ the norm is given by $N(x) = x^2 = x_0^2 + dx_1^2 + cx_2^2 + cdx_3^2$. Thus it is diagonalizable and anisotropic, and the assertion follows from 2.10 and 2.12. □

6.4 Lemma. *The space W is invariant under $\mathbb{H}^\times$ and under $\mathbb{H}^\rho$. Both the action of $\mathbb{H}^\times$ on W and the action via $(a, X) \mapsto aX\tilde{a}$ on $\mathrm{Pu}(\mathbb{H})$ (see 6.2) are equivalent to that on $\mathbb{K} \times \mathbb{L}$ given by*

$$\left(\begin{pmatrix} A & -c\overline{B} \\ B & \overline{A} \end{pmatrix}, (x, Y)\right) \mapsto \left((A\overline{A} - cB\overline{B})x - ABY - \overline{ABY}\,,\, 2cA\overline{B}x + A^2Y - c\overline{B^2Y}\right).$$

Any element $\rho_a \in \mathbb{H}^\rho$ induces the multiplication by its norm $\tilde{a}a$ on W. Thus the action of $\mathbb{H}^\times\mathbb{H}^\rho$ on W is equivalent to an action by similitudes of the quadratic form $N|_{\mathrm{Pu}(\mathbb{H})}$.

Proof. Choose $j \in \mathbb{L} \setminus \{0\}$ with $\overline{j} = -j$ in the separable case, and put $j := 1$ if $\mathbb{L}/\mathbb{K}$ is inseparable. A straightforward calculation shows that mapping (x, Y) to $\left(\begin{smallmatrix} cxi & Yi \\ \overline{Y}i & -xi \end{smallmatrix}\right) \in W$ or to $\left(\begin{smallmatrix} xj & Yj \\ c^{-1}\overline{Y}j & -xj \end{smallmatrix}\right) \in \mathrm{Pu}(\mathbb{H})$, respectively, gives equivalences as claimed. For the rest of the assertion, it remains to compute

$$\begin{pmatrix} A & -cB\xi \\ B\xi & A \end{pmatrix}\begin{pmatrix} cxi & Yi \\ \overline{Y}i & -xi \end{pmatrix}\begin{pmatrix} A' & (B\xi)' \\ -(cB\xi)' & A' \end{pmatrix} = (A\overline{A} + cB\overline{B})\begin{pmatrix} cxi & Yi \\ \overline{Y}i & -xi \end{pmatrix}. \qquad \square$$

6.5 Lemma. *If $M \in \mathrm{GL}_4\mathbb{K}$ fixes b_0 and induces a scalar multiple of id on W then $M = \mathrm{id}$.*

Proof. We consider an element $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ of the stabilizer of b_0 in $\mathrm{GL}_4\mathbb{K}$; then $A = \left(\begin{smallmatrix} 1 & b \\ 0 & a \end{smallmatrix}\right)$, $C = \left(\begin{smallmatrix} 0 & x \\ 0 & y \end{smallmatrix}\right)$ and $B, D \in \mathbb{K}^{2\times 2}$. Moreover, we assume that M induces $\alpha\,\mathrm{id}$ on W.

Evaluating the condition (C1): $M\left(\begin{smallmatrix} ci & 0 \\ 0 & i \end{smallmatrix}\right)M' = \left(\begin{smallmatrix} \alpha ci & 0 \\ 0 & \alpha i \end{smallmatrix}\right)$ we find $\alpha = \det D$ and $B = \left(\begin{smallmatrix} z & w \\ 0 & 0 \end{smallmatrix}\right)$; then $a = \det A = \alpha$ follows.

The condition (C2): $M\left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)M' = \left(\begin{smallmatrix} 0 & \alpha i \\ \alpha i & 0 \end{smallmatrix}\right)$ yields $z = 0$. Putting this into (C1) we find that the second column of C equals $c^{-1}w$ times the first column of D.

Finally, we evaluate (C3): $M\left(\begin{smallmatrix} 0 & ui \\ \overline{u}i & 0 \end{smallmatrix}\right)M' = \left(\begin{smallmatrix} 0 & \alpha ui \\ \alpha\overline{u}i & 0 \end{smallmatrix}\right)$. From $0 = B\overline{u}iA' + AuiC' = -w\alpha i$ we infer $w = 0$. Thus both B and C are zero, and there remain the conditions $A(ui)D' = \alpha ui$ from (C3) and $AiD' = \alpha i$ from (C2). Now $iD'i^{-1} = D^{-1}\det D = D^{-1}\alpha$ and the latter equality give $A = D$. The first equality then yields that A centralizes u. But this means $A = \left(\begin{smallmatrix} 1 & b \\ 0 & a \end{smallmatrix}\right) \in \mathbb{L}$, and $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ follows. □

6.6 Theorem. *For $\mathbb{H}$, $\mathbb{H}^\rho$ and W as in 6.1 we have $\Sigma_W = \mathbb{H}^\times\mathbb{H}^\rho$.*

Proof. We know from 6.4 that the multiplicative group $\mathbb{H}^\times\mathbb{H}^\rho$ is contained in the stabilizer $(\mathrm{GL}_4\mathbb{K})_W$. The subgroup $\{\lambda_a\rho_a^{-1} \mid a \in \mathbb{H}^\times\}$ of $\mathbb{H}^\times\mathbb{H}^\rho$ induces the full group $\mathrm{SO}(N|_{\mathrm{Pu}(\mathbb{H})})$ on W and $\mathrm{GO}(N|_{\mathrm{Pu}(\mathbb{H})}) = \mathbb{K}^\times \mathrm{SO}(N|_{\mathrm{Pu}(\mathbb{H})})$, cf. 6.3. Thus it suffices to consider elements $M \in \mathrm{GL}_4\mathbb{K}$ that induce scalar multiples of the identity on W. Adapting M by a further element of $\mathbb{H}^\rho$ (which induces a scalar multiple of the identity on W by 6.4) we may assume that M fixes b_0. Now the result follows from 6.5. □

It remains to understand the action of $\Sigma_W = \mathbb{H}^\times\mathbb{H}^\rho$ on $(\mathbb{K}^4 \wedge \mathbb{K}^4)/W$. Straightforward computations yield:

6.7 Lemma. *If $\mathbb{L}/\mathbb{K}$ is a separable extension then the action of the group $\mathbb{H}^\times\mathbb{H}^\rho$ on $W^\perp = \left\{\left(\begin{smallmatrix} cxi & Y\delta \\ Y\delta & xi\end{smallmatrix}\right) \mid x \in \mathbb{K}, Y \in \mathbb{L}\right\}$ is quasi-equivalent to that on W; indeed $G := \left(\begin{smallmatrix} E_2 & 0 \\ 0 & \xi\end{smallmatrix}\right) \in \mathrm{GL}_4\mathbb{K}$ satisfies $G\,\mathbb{H}^\times G^{-1} = \mathbb{H}^\rho$ and $GWG' = W^\perp$.* □

If $\operatorname{char}\mathbb{K} \neq 2$ then 6.7 describes the action on $(\mathbb{K}^4 \wedge \mathbb{K}^4)/W$ because $W^\perp$ is a vector space complement to W in $\mathbb{K}^4 \wedge \mathbb{K}^4$. The remaining case $\operatorname{char}\mathbb{K} = 2$ is more involved. We treat the inseparable case, as well.

6.8 Lemma. *Assume $\operatorname{char}\mathbb{K} = 2$ and $P^t_{1,c,d} \in \mathcal{P}_2$. We write $W := W^t_{c,d}$ and $\mathbb{H} := \mathbb{H}_{\mathbb{K}}^{-d,-c}$ if $t = 1$ and $\mathbb{H} := \mathbb{K}(\sqrt{d}, \sqrt{c})$ otherwise. In any case, put $\mathbb{L} := \mathbb{K}[X]/(X^2 + tX + d)$. The action $\omega\colon (\mathbb{H}^\times\mathbb{H}^\rho) \times (\mathbb{K}^4 \wedge \mathbb{K}^4)/W \to (\mathbb{K}^4 \wedge \mathbb{K}^4)/W$ can be described as follows.*

- **a.** *If $t = 1$ then the action ω is equivalent to the action $\omega_1\colon (\mathbb{H}^\times\mathbb{H}^\rho) \times (\mathbb{K} \times \mathbb{L}) \to \mathbb{K} \times \mathbb{L}$ given by*

$$\omega_1\big(\lambda_{A+IB}, (x, Y)\big) = N(A + IB)\,(x, Y)\ ,$$
$$\omega_1\big(\rho_{C+ID}, (x, Y)\big) = \big(N(C + ID)x, CDu^{-1}x + C^2Y + (1 + u^{-1})D^2\overline{Y}\big)\ .$$

- **b.** *If $t = 0$ then $\mathbb{H}$ is commutative, $\mathbb{H}^\times\mathbb{H}^\rho = \mathbb{H}^\times$ and the action ω is equivalent to the action $\omega_0\colon \mathbb{H}^\times \times (\mathbb{K} \times \mathbb{L}) \to \mathbb{K} \times \mathbb{L}$ given by*

$$\omega_0\big(\lambda_{A+IB}, (x, Y)\big) = (A + IB)^2(x, Y)\,.$$

Proof. Assume $t = 1$. Then $\left\{\left(\begin{smallmatrix} xi & Y\delta \\ Y\delta & 0\end{smallmatrix}\right) \mid x \in \mathbb{K}, Y \in \mathbb{L}\right\}$ is a vector space complement to W because the extension is separable. Using $u = \left(\begin{smallmatrix} 0 & -d \\ 1 & 1\end{smallmatrix}\right)$ and the relations $\xi i = u^{-1}\delta$, $\xi\delta\xi' = (1 + u^{-1})\delta$ we compute

$$\begin{pmatrix} A & c\overline{B} \\ B & \overline{A}\end{pmatrix}\left(\begin{pmatrix} xi & Y\delta \\ Y\delta & 0\end{pmatrix} + W\right)\begin{pmatrix} A' & B' \\ c\overline{B}' & \overline{A}'\end{pmatrix} = (A\overline{A} + cB\overline{B})\begin{pmatrix} xi & Y\delta \\ Y\delta & 0\end{pmatrix} + W$$

and

$$\begin{pmatrix} C & cD\xi \\ D\xi & C\end{pmatrix}\left(\begin{pmatrix} xi & Y\delta \\ Y\delta & 0\end{pmatrix} + W\right)\begin{pmatrix} C' & (D\xi)' \\ (D\xi)' & C'\end{pmatrix}$$
$$= \begin{pmatrix} (C\overline{C} + cD\overline{D})xi & \big(CDu^{-1}x + C^2Y + c(1 + u^{-1})D^2\overline{Y}\big)\,\delta \\ \big(CDu^{-1}x + C^2Y + c(1 + u^{-1})D^2\overline{Y}\big)\,\delta & 0\end{pmatrix} + W\,.$$

Mapping $\left(\begin{smallmatrix} xi & Y\delta \\ Y\delta & 0\end{smallmatrix}\right) + W$ to (x, Y) is a bijection from $(\mathbb{K}^4 \wedge \mathbb{K}^4)/W$ onto $\mathbb{K} \times \mathbb{L}$ that gives the equivalence to the action ω_1.

Now assume $t = 0$. Then $i \notin \mathbb{L}$ and $\mathbb{K}^{2\times 2} = \mathbb{L} \oplus \mathbb{L}i$ yields that $\{\left(\begin{smallmatrix} xi & Y \\ Y' & 0 \end{smallmatrix}\right) \mid x \in \mathbb{K}, Y \in \mathbb{L}\}$ is a vector space complement to W. Using $Z' \in Z + \mathbb{K}i$ and $iZ' = Zi$ we compute

$$\begin{pmatrix} A & cB \\ B & A \end{pmatrix}\begin{pmatrix} xi & Y \\ Y' & 0 \end{pmatrix}\begin{pmatrix} A' & B' \\ cB' & A' \end{pmatrix} = \begin{pmatrix} A^2xi + c(BY'A' + AYB') & ABxi + AYA' + cBY'B' \\ ABxi + AY'A' + cBYB' & B^2xi + AY'B' + BYA' \end{pmatrix}$$
$$\in \begin{pmatrix} (A^2 + cB^2)xi + c\psi(A,B,Y) & (A^2+cB^2)Y \\ \big((A^2+cB^2)Y\big)' & 0 \end{pmatrix} + W$$

for $\psi(A,B,Y) := BY'A' + AYB' + AY'B' + BYA' = B(Y+Y')A' + A(Y+Y')B' = ByiA' + AyiB' = 0$. It remains to note $(A + IB)^2 = A^2 + I^2B^2 = A^2 + cB^2$. □

In marked contrast to the case where $\operatorname{char}\mathbb{K} \neq 2$ (cf. 6.7 and 6.4) the action ω_1 in 6.8 is *not* (quasi-) equivalent[4] to the action $\omega_2 \colon (\mathbb{H}^\times\mathbb{H}^\rho) \times \mathrm{Pu}(\mathbb{H}) \to \mathrm{Pu}(\mathbb{H})$ given by $\omega_2\left((\lambda_a, \rho_b), x\right) = N(b)ax\tilde{a}$: the subspace $\{0\}\times\mathbb{L}$ is invariant under ω_1 but there is no two-dimensional invariant subspace under ω_2. However, we have the following.

6.9 Lemma. *The restriction of ω_1 to $\{0\} \times \mathbb{L}$ is equivalent to the action induced by ω_2 on the quotient $\mathrm{Pu}(\mathbb{H})/\mathbb{K}$ modulo the subspace $\mathbb{K}$ which is invariant under ω_2.*

The orbits in $\{0\} \times \mathbb{L}$ are represented by a set $R_{W^\perp} \subseteq \mathrm{Pu}(\mathbb{H})$ such that $\{x^2 \mid x \in R_{W^\perp}\}$ represents the orbits $v^2N(\mathbb{H}^{\boxtimes}) + \mathbb{K}^{\square}$ of the group $N(\mathbb{H}^{\boxtimes})$ on $\{v^2 + \mathbb{K}^{\square} \mid v \in \mathrm{Pu}(\mathbb{K})\}$.

The orbits in $\mathbb{K}^\times \times \mathbb{L}$ are represented by $R_{\mathbb{H}^\times} \times \{0\}$ where $R_{\mathbb{H}^\times}$ is a set of representatives for the cosets in the multiplicative group $\mathbb{K}^\times/N(\mathbb{H}^\times)$.

Proof. Applying the element G from 6.7 to $(W + W^\perp)/(W \cap W^\perp)$ interchanges the two irreducible summands $W/(W \cap W^\perp)$ and $W^\perp/(W \cap W^\perp)$.

From 2.6 we infer that $x, v \in \mathrm{Pu}(\mathbb{H})$ (i.e., both with trace 0) represent cosets $x+\mathbb{K}$ and $v+\mathbb{K}$ in the same ω_2-orbit precisely if there exists $z \in \mathbb{H}^\times$ such that $\{N(x+k) \mid k \in \mathbb{K}\} = x^2 + \mathbb{K}^{\square}$ has nonempty intersection with $\{N(v+k)N(z^2) \mid k \in \mathbb{K}\} = v^2N(z^2) + \mathbb{K}^{\square}$.

For any $x \in \mathbb{K}^\times$ the orbit $\{(x, CDu^{-1}x) \mid (C,D) \in \mathbb{L}^2 \smallsetminus \{(0,0)\}\}$ of $(x,0)$ meets each one of the cosets $\mathbb{K} \times \{Y\}$. Therefore, it suffices to consider the action on the quotient modulo $\{0\} \times \mathbb{L}$ to prove the last claim. □

6.10 Theorem. *The orbit of $\ker\widehat{\beta} \in \mathcal{P}_2$ under $\mathrm{GL}_4\mathbb{K}$ contains an element of the form $P^t_{1,c,d}$ and then also $W := W^t_{c,d}$ as in 6.1. We identify $\mathbb{K}^4$ with $\mathbb{H} := \mathbb{H}^{-d,-c}_{\mathbb{K}}$ if $X^2 + tX + d$ is separable and $\mathbb{H} := \mathbb{K}(\sqrt{d}, \sqrt{c})$ otherwise.*

- **a.** *In any case the group $\Sigma_W = \mathbb{H}^\times\mathbb{H}^\rho$ acts with two orbits on $\mathbb{H}$, represented by 0 and 1.*
- **b.** *If $\operatorname{char}\mathbb{K} \neq 2$ then the orbits of Σ_W on $(\mathbb{K}^4 \wedge \mathbb{K}^4)/W$ are represented by a set $R_{\mathbb{H}} \subseteq \mathrm{Pu}(\mathbb{H})$ such that for each coset of $N(\mathbb{H}^\times)/N(\mathbb{H}^{\boxtimes})$ there is exactly one element in $R_{\mathbb{H}}$.*
- **c.** *If $\operatorname{char}\mathbb{K} = 2$ and $t = 1$ then the action of Σ_W is described in 6.8.a. The orbits on $(\mathbb{K}^4 \wedge \mathbb{K}^4)/W$ are represented by $R_{\mathbb{H}^\times} \cup R_{W^\perp}$ as in 6.9.*
- **d.** *If $\operatorname{char}\mathbb{K} = 2$ and $t = 0$ then the orbits of Σ_W on $(\mathbb{K}^4 \wedge \mathbb{K}^4)/W$ are represented by the elements of $\{rZ \mid r \in R_H \cup \{0\}, Z \in R_W\}$ where R_H and R_W are sets of representatives for the cosets in $\mathbb{H}^\times/\mathbb{H}^{\boxtimes}$ and for the one-dimensional subspaces of $(\mathbb{K}^4 \wedge \mathbb{K}^4)/W$, respectively. Thus the number of orbits equals the cardinality of the (infinite) field $\mathbb{K}$.*

Proof. According to [19, 8.6] (cf. 3.7.c) we find an element of the form $P^t_{1,c,d}$ the orbit of $\ker\widehat{\beta}$. The rest follows from 6.6, 6.7, 2.6, 6.8, and 6.9. □

[4] The situation is different in the inseparable case where $\omega_0 = \omega_2$ because the multiplication is commutative and $N(a) = a^2$.

6.11 Remark. If $\operatorname{char}\mathbb{K} \neq 2$ one can give a very nice description of an alternating map β with $\ker\widehat{\beta} = W^0_{c,d}$. Writing $\mathbb{H} := \mathbb{H}^c_{\mathbb{L}/\mathbb{K}}$ for $\mathbb{L} = \mathbb{K}(\sqrt{-d})$ and $x \mapsto \tilde{x}$ for the standard involution on $\mathbb{H}$ we obtain the alternating map $\beta_\mathbb{H} \colon \mathbb{H}\times\mathbb{H} \to \operatorname{Pu}(\mathbb{H}) \colon (x,y) \mapsto \tilde{x}y - \tilde{y}x$. Evaluating this map at pairs of the basis elements $b_0 = -h_3$, $b_1 = 1$, $b_2 = h_1$, $b_3 = h_2$ one finds $\ker\widehat{\beta_\mathbb{H}} = \langle S^0_1 + S^2_3, S^0_2 + dS^1_3, S^0_3 - cS^1_2\rangle_\mathbb{K} = P^0_{1,-d,-c}$, and $\ker\widehat{\beta_\mathbb{H}}$ lies in the orbit of $W^0_{c,d}$.

For the classical quaternion field $\mathbb{H} = \mathbb{H}^1_{\mathbb{C}/\mathbb{R}}$ over the field $\mathbb{R}$ the group $\operatorname{Aut}(\mathfrak{gh}(\mathbb{H},P,\beta_\mathbb{H}))$ acts with only three orbits on $\mathfrak{gh}(\mathbb{H},P,\beta_\mathbb{H})$. In that case the Heisenberg algebra $\mathfrak{gh}(\mathbb{H},P,\beta_\mathbb{H})$ is *almost homogeneous* (in the sense of [17], [18] and [7] where $\mathfrak{gh}(\mathbb{H},P,\beta_\mathbb{H})$ occurs as $\mathrm{H}^4_\mathbb{H}$). For a general quaternion field, the group $\Sigma_{\beta_\mathbb{H}}$ still acts transitively but there may be more than two orbits on P, cf. 6.14.

We will interpret our result 6.10 for several cases explicitly in 6.14 and 6.15 below. We introduce some more notation (which appears to be quite standard for quaternion algebras if $\operatorname{char} \neq 2$).

6.12 Notation. Let $\mathbb{L}/\mathbb{K}$ be a quadratic field extension where $\mathbb{L} \cong \mathbb{K}[X]/(X^2+tX+d)$ for some irreducible polynomial $X^2+tX+d \in \mathbb{K}[X]$, and pick $c \in \mathbb{K}^\times$. In order to indicate briefly the construction of $\mathbb{L}$ we will denote the quaternion algebra $\mathbb{H}^c_{\mathbb{L}/\mathbb{K}}$ also by $\mathbb{H}^{-d,-c}_\mathbb{K}$.

The $\mathbb{K}$-algebra $\mathbb{H}^{-d,-c}_\mathbb{K}$ can also be described using a basis $h_0 = 1$, h_1, h_2, $h_3 = h_2h_1$ where $h_1 \in \mathbb{L}$ is a root of X^2+tX+d and $h_2 \in \mathbb{L}^\perp$ is a root of X^2+c; then $h_1h_2 = th_2 - h_3$. This description does not depend on the fact that $\mathbb{L} = \mathbb{K}[X]/(X^2+tX+d)$ is a field; we will use the notation $\mathbb{H}^{-d,-c}_\mathbb{K}$ for any pair $(d,c) \in (\mathbb{K}\smallsetminus\{0\})^2$ to denote a 4-dimensional associative $\mathbb{K}$-algebra with a basis $h_0 = 1$, h_1, h_2, h_3 satisfying $h_1^2 = -th_1 - d$, $h_2^2 = -c$, $h_2h_1 = h_3$, $h_1h_2 = th_2 - h_3$ for $t = 0$ if $\operatorname{char}\mathbb{K} \neq 2$ and $t = 1$ if $\operatorname{char}\mathbb{K} = 2$.

Such a quaternion algebra is a quaternion field precisely if its norm form is anisotropic; in all other cases it is isomorphic to $\mathbb{H}^{1,1}_\mathbb{K}$.

6.13 Examples. For a quaternion algebra $\mathbb{H} = \mathbb{H}^{-d,-c}_\mathbb{K}$ the set $\mathbb{H}^\boxtimes$ need not be closed under multiplication (the interesting set for us is indeed $N(\mathbb{H}^\boxtimes) = \{N(x)^2 \mid x \in \mathbb{H}^\times\}$ which is closed under multiplication). For example, we have $-2 = h_1^2$ and $-3 = h_2^2$ in $(\mathbb{H}^{-1,-1}_\mathbb{Q})^\boxtimes$ but their product $6 = (-2)(-3)$ does not belong to $(\mathbb{H}^{-1,-1}_\mathbb{Q})^\boxtimes$ because $\{x^2 \mid x \in \operatorname{Pu}(\mathbb{H}^{-1,-1}_\mathbb{Q})\} \subseteq \{q \in \mathbb{Q} \mid q \leq 0\}$ and $\{x \in \mathbb{H}^{-1,-1}_\mathbb{Q} \mid x^2 \in \mathbb{Q}\} = \mathbb{Q} \cup \operatorname{Pu}(\mathbb{H}^{-1,-1}_\mathbb{Q})$ while $6 \notin \mathbb{Q}^\boxtimes$.

As a second interesting example we mention $\mathbb{R}^{2\times 2} \cong \mathbb{H}^{1,1}_\mathbb{R}$ (see 6.15 below); here $\left(\begin{smallmatrix}-1 & 2\\ 0 & -1\end{smallmatrix}\right)$ is not a square but $\left(\begin{smallmatrix}-1 & 2\\ 0 & -1\end{smallmatrix}\right) = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)^2 \left(\begin{smallmatrix}1 & -1\\ 0 & 1\end{smallmatrix}\right)^2$.

6.14 Examples. We consider $\mathbb{H} = \mathbb{H}^{-d,-c}_\mathbb{K}$ and $N(\mathbb{H}^\times)/N(\mathbb{H}^\boxtimes)$ for different explicit choices of $\mathbb{K}$ and $(-d,-c)$:

- **a.** For $\mathbb{K} = \mathbb{R}$ and $(-d,-c) = (-1,-1)$ we find $N(\mathbb{H}^\times) = \{r \in \mathbb{R} \mid r > 0\} = N(\mathbb{H}^\boxtimes)$ and $|N(\mathbb{H}^\times)/N(\mathbb{H}^\boxtimes)| = 1$.
- **b.** For $\mathbb{K} = \mathbb{Q}$ and $(-d,-c) = (-1,-1)$ we have $N(\mathbb{H}^\times) = \{r \in \mathbb{Q} \mid r > 0\}$ by Lagrange's four-square theorem (cf. [14, II.8.3]). The group $\mathbb{Q}^\times$ is isomorphic to a direct sum of $\mathbb{Z}/2\mathbb{Z}$ and countably many infinite cyclic groups. The norm group $N(\mathbb{H}^\times)$ is the unique subgroup of index 2 and the quotient $N(\mathbb{H}^\times)/N(\mathbb{H}^\boxtimes)$ is an elementary abelian group of countably infinite rank.
- **c.** According to Fermat's theorem on sums of two squares (cf. [14, II.8.1]) an odd prime number p is the sum of two squares of integers precisely if $p \equiv 1 \pmod 4$. For $\mathbb{K} = \mathbb{Q}$ and $(-d,-c) = (-1,-c)$ with any $c \in \mathbb{Q}^\times$ this implies that the group $N(\mathbb{H}^\times)$ contains infinitely many primes. Thus $N(\mathbb{H}^\times)/N(\mathbb{H}^\boxtimes)$ is countably infinite in these cases, as well.

d. Let $p > 2$ be a prime number, and let $\mathbb{K}$ be a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers. Then there exists, up to isomorphism, precisely one quaternion field $\mathbb{H}$ over $\mathbb{K}$, and $\mathbb{K}^\times = N(\mathbb{H}^\times)$, cf. [10, VI 2.10]. There are four square classes in $\mathbb{K}^\times$ (see [10, VI, 2.22]), and $N(\mathbb{H}^\times)/N(\mathbb{H}^\boxtimes) = \mathbb{K}^\times/\mathbb{K}^\boxtimes \cong (\mathbb{Z}/2\mathbb{Z})^2$.

e. Now let $\mathbb{K}$ be a finite extension of the field $\mathbb{Q}_2$ of 2-adic numbers, of degree e. Again, there exists precisely one quaternion field $\mathbb{H}$ over $\mathbb{K}$ (up to isomorphism), and $\mathbb{K}^\times = N(\mathbb{H}^\times)$. However, there are 2^{e+2} square classes in $\mathbb{K}^\times$, and $N(\mathbb{H}^\times)/N(\mathbb{H}^\boxtimes) = \mathbb{K}^\times/\mathbb{K}^\boxtimes \cong (\mathbb{Z}/2\mathbb{Z})^{e+2}$, cf. [10, VI 2.23].

6.15 Split Quaternions in Odd Characteristic. Let $\mathbb{K}$ be any field with $\operatorname{char}\mathbb{K} \neq 2$. It is well known that then the quaternion algebra $\mathbb{H}^{1,1}_{\mathbb{K}}$ splits, and is therefore isomorphic to the algebra $\mathbb{K}^{2\times 2}$; the multiplicative form is then $N(x) = \det(x)$. See [19, 8.7], where an explicit isomorphism is given to show that $\ker \widehat{\beta_{\mathbb{K}^{2\times 2}}}$ belongs to the orbit of $T+S$ under the group $\mathrm{GL}_4\mathbb{K}$. The automorphisms can be read off from 6.6 or from 4.9. Note that the multiplicative group of the split quaternion algebra is (isomorphic to) $\mathrm{GL}_2\mathbb{K}$; the subgroup $\mathrm{SL}_2\mathbb{K}$ induces a group isomorphic to $\mathrm{PSL}_2\mathbb{K}$ on P, acting as the group of proper hyperbolic motions with respect to the form $N|_P$.

We translate our result 4.9 into the description using split quaternions:

6.16 Theorem. *If $\operatorname{char}\mathbb{K} \neq 2$ then there are three orbits of $\Sigma_{\beta_{\mathbb{K}^{2\times 2}}}$ on $\mathbb{K}^{2\times 2}$; characterized by the rank of their members. Representatives for the orbits are thus $\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$, and $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$.*

Each orbit on P (i.e., on the set of matrices with vanishing trace) is obtained by fusion of a conjugacy class with all its images under multiplication with scalars. These orbits are represented by the elements of the set $\{0\} \cup \left\{ \left(\begin{smallmatrix}0&-d\\1&0\end{smallmatrix}\right) \,\middle|\, d \in R_* \cup \{0\} \right\}$ where R_* is, again, a set of representatives for the cosets in $\mathbb{K}^\times/\mathbb{K}^\boxtimes$.

Proof. The orbits on $\mathbb{K}^{2\times 2}$ are clear from 6.6 or from 4.9. Those on P can be read off from 6.6 and 2.6: we have $N(x) = \det x$ and the elements of P are those with trace 0. Thus elements of P are conjugates if, and only if, they have the same norm, and belong to the same orbit under $\Sigma_{\beta_{\mathbb{K}^{2\times 2}}}$ if their determinants differ by a square in $\mathbb{K}^\times$.

Now $\operatorname{char}\mathbb{K} \neq 2$ yields that 0 is the only scalar multiple of the identity matrix in P. For every other element $x \in P$ we may choose a representative for the orbit of x in Frobenius normal form $X_d := \left(\begin{smallmatrix}0&-d\\1&0\end{smallmatrix}\right)$, where $d \in R_*$ is the representative of $\det(x)\mathbb{K}^\square$. ☐

7. Results

Let $H := \mathfrak{gh}(V, Z, \beta)$ be a reduced Heisenberg algebra with $\dim V = 4$. From 1.2 we know that the orbits of $\mathrm{Aut}(\mathfrak{gh}(V, Z, \beta))$ are controlled by the orbits of Σ_β on V and on Z, respectively. In particular, we find $\omega(\mathfrak{gh}(V, Z, \beta)) = \omega_V + \omega_Z + 1$ where ω_V denotes the number of orbits in $H \smallsetminus Z$ and ω_Z the number of those in $Z \smallsetminus \{0\}$. The numbers ω_V and ω_Z can be read off from our discussion of the possible cases for $\ker\hat{\beta}$ in Sections 4, 5 and 6 above. Table 7.1 collects these results; the column "reference" indicates the place where the corresponding result is proved.

7.1 Numbers of Orbits under Automorphisms of Reduced Heisenberg Algebras.

$\ker \widehat{\beta}$	ω_V	ω_Z	ω	References
$\langle S_1^0 \rangle_{\mathbb{K}}$	2	3	6	4.1
$\langle S_1^0 + S_3^2 \rangle_{\mathbb{K}}^{\perp}$ $\langle S_1^0 + S_3^2 \rangle_{\mathbb{K}}, \operatorname{char}\mathbb{K} \neq 2$ $\langle S_1^0 + S_3^2 \rangle_{\mathbb{K}}, \operatorname{char}\mathbb{K} = 2$	1 1 1	1 $1 + \lvert R_* \rvert$ $2 + \lvert R_+ \rvert + \lvert R_\wp \rvert$	3 $3 + \lvert R_* \rvert$ $4 + \lvert R_+ \rvert + \lvert R_\wp \rvert$	4.2
E	3	3	7	4.3
T $T^{\perp}$	2 2	3 2	6 5	4.4
S $S^{\perp}$	2 2	2 2	5 5	4.5
$J(F)$	2	1	4	4.6
$E + T$	3	2	6	4.7
$E + S$	4	3	8	4.8
$T + S, \operatorname{char}\mathbb{K} \neq 2$ $T + S, \operatorname{char}\mathbb{K} = 2$	2 2	$1 + \lvert R_* \rvert$ $1 + \lvert R_* \rvert + \lvert R_\wp \rvert$	$4 + \lvert R_* \rvert$ $4 + \lvert R_* \rvert + \lvert R_\wp \rvert$	4.9, 6.6
$P_{\mathbb{L}}^{\perp} \in \mathcal{P}_1^{\perp}$ $P_{\mathbb{L}} \in \mathcal{P}_1, \mathbb{L}/\mathbb{K}$ sep. $P_{\mathbb{L}} \in \mathcal{P}_1, \mathbb{L}/\mathbb{K}$ insep.	1 1 1	1 $\mathrm{HF}(\mathbb{L}/\mathbb{K})$ $\lvert R_{\mathbb{K}/\mathbb{L}^\square} \rvert + \lvert R^{(2)}_{\mathbb{K}/\mathbb{L}^\square} \rvert$ $+ \lvert R_\# \rvert - 1$	3 $2 + \mathrm{HF}(\mathbb{L}/\mathbb{K})$ $\lvert R_{\mathbb{K}/\mathbb{L}^\square} \rvert + \lvert R^{(2)}_{\mathbb{K}/\mathbb{L}^\square} \rvert$ $+ \lvert R_\# \rvert + 1$	5.6
$W^0_{c,d} \in \mathcal{P}_2, \operatorname{char}\mathbb{K} \neq 2$ $W^1_{c,d} \in \mathcal{P}_2, \operatorname{char}\mathbb{K} = 2$ $W^0_{c,d} \in \mathcal{P}_2, \operatorname{char}\mathbb{K} = 2$	1 1 1	$\lvert N(\mathbb{H}^\times)/N(\mathbb{H}^\boxtimes) \rvert$ $\lvert R_{\mathbb{H}^\times} \rvert + \lvert R_{W^\perp} \rvert$ $\lvert R_H \rvert + \lvert R_W \rvert$	$2 + \lvert N(\mathbb{H}^\times)/N(\mathbb{H}^\boxtimes) \rvert$ $2 + \lvert R_{\mathbb{H}^\times} \rvert + \lvert R_{W^\perp} \rvert$ $2 + \lvert R_H \rvert + \lvert R_W \rvert = \lvert \mathbb{K} \rvert$	6.10
$P^0_{\mathbb{L}} \in \mathcal{P}_3$	2	$1 + \lvert R_N \rvert$	$4 + \lvert R_N \rvert$	5.6

7.2 Notation. In Table 7.1 we use the following notation.

R_* denotes a set of coset representatives for the classes in $\mathbb{K}^\times/\mathbb{K}^\boxtimes$.

R_+ is a set of representatives for the orbits of $\mathbb{K}^\boxtimes$ on the additive group $\mathbb{K}/\mathbb{K}^\square$.

$R_\wp$ is (if $\operatorname{char}\mathbb{K} = 2$) a set of coset representatives for the additive group $\mathbb{K}/\wp$ where $\wp := \{x + x^2 \mid x \in \mathbb{K}\}$, cf. 2.1.

$\mathrm{HF}(\mathbb{L}/\mathbb{K})$ denotes the number of equivalence classes of non-zero hermitian forms on $\mathbb{L}^2$, cf. 2.5.

$R_{\mathbb{K}/\mathbb{L}^\square}$ and $R^{(2)}_{\mathbb{K}/\mathbb{L}^\square}$ represent the two kinds of orbits under the action of $\mathrm{GL}_2\mathbb{L}$ on $\mathbb{K}^2$, see 2.3.

$R_\#$ represents the orbits under the action $\#$ of $\mathrm{SL}_2\mathbb{L}$ on $\mathbb{K}^2$, see 5.6.

$R_{\mathbb{H}^\times}$ is a set of representatives for $\mathbb{K}^\times/N(\mathbb{H}^\times)$ where $\mathbb{H} = \mathbb{H}_{\mathbb{K}}^{-d,-c}$.

$R_{W^\perp} \subseteq \mathrm{Pu}(\mathbb{H})$ is (for a quaternion field $\mathbb{H}$ over $\mathbb{K}$ with $\operatorname{char}\mathbb{K} = 2$) a set such that the set $\{x^2 \mid x \in R_{W^\perp}\}$ represents the orbits $v^2 N(\mathbb{H}^\boxtimes) + \mathbb{K}^\square$ of $N(\mathbb{H}^\boxtimes)$ on $\{v^2 + \mathbb{K}^\square \mid v \in \mathrm{Pu}(\mathbb{K})\}$.

R_H is (if $\mathbb{H} = \mathbb{K}(\sqrt{d}, \sqrt{c})$ is a purely inseparable extension of degree 4) a set of coset representatives for $\mathbb{H}^\times/\mathbb{H}^\boxtimes$.

R_W is a set of representatives for the one-dimensional subspaces of $(\mathbb{K}^4 \wedge \mathbb{K}^4)/W$, see 6.10.

R_N is a set of coset representatives for the multiplicative group $\mathbb{K}^\times/(N_{\mathbb{L}/\mathbb{K}}(\mathbb{L}^\times)\langle -1 \rangle)$, cf. 5.6.

8. Open problems

8.1 Problem. How is $\mathrm{Aut}(\mathfrak{gh}_{\mathbb{Z}}(V,Z,\beta))$ related to $\mathrm{Aut}(\mathfrak{gh}(V,Z,\beta))$, or, more generally, how is the automorphism group $\mathrm{Aut}(\mathfrak{n})$ of a nilpotent Lie algebra $\mathfrak{n}$ over $\mathbb{K}$ related to the automorphism group $\mathrm{Aut}_{\mathbb{Z}}(\mathfrak{n})$ of the Lie ring? Definitely, the subgroup

$$H := \left\{ (v,z,a) \mapsto (v, z+\tau(v), \alpha(a)+\zeta(v,z)) \;\middle|\; \begin{array}{c} \tau \in \mathrm{Hom}(V,Z), \alpha \in \mathrm{GL}(A), \\ \zeta \in \mathrm{Hom}(V \times Z, A) \end{array} \right\}$$

from 1.2 has to be enlarged to

$$H_{\mathbb{Z}} := \left\{ (v,z,a) \mapsto (v, z+\tau(v), \alpha(a)+\zeta(v,z)) \;\middle|\; \begin{array}{c} \tau \in \mathrm{Hom}_{\mathbb{Z}}(V,Z), \alpha \in \mathrm{Aut}_{\mathbb{Z}}(A), \\ \zeta \in \mathrm{Hom}_{\mathbb{Z}}(V \times Z, A) \end{array} \right\};$$

we must use arbitrary additive homomorphisms instead of $\mathbb{K}$-linear ones. Is it true that $\mathrm{Aut}(\mathfrak{gh}_{\mathbb{Z}}(V,Z,\beta))$ equals $\mathrm{Aut}(\mathbb{K}) \ltimes (\mathrm{Aut}(\mathfrak{gh}(V,Z,\beta)) \cdot H_{\mathbb{Z}})$?

A positive partial answer is known: according to [6, 1.1.1], the assertion is true for $\mathfrak{gh}(V,Z,\beta) = \mathfrak{gh}(\mathbb{K}^2, \mathbb{K}, \det)$.

8.2 Problem. A deeper understanding of the action $\#\colon \mathrm{SL}_2\mathbb{L} \times \mathbb{K}^2 \to \mathbb{K}^2$ in 5.6 would be very welcome.

8.3 Problem. Our results allow to identify the cases where $\omega(\mathfrak{gh}(V,Z,\beta))$ is finite (depending on the ground field $\mathbb{K}$), and then those where the invariant ω takes on values that are particularly small. It appears that these algebras merit deeper study.

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