DISPERSIVE ESTIMATES AND NLS ON PRODUCT MANIFOLDS

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Abstract. We prove a general dispersive estimate for a Schrödinger type equation on a product manifold, under the assumption that the equation restricted to each factor satisfies suitable dispersive estimates. Among the applications are the two-particle Schrödinger equations

$$iu_t - \Delta_{x,y}u + V(x-y)u = 0$$

on $\mathbb{R}^{2n}$, and the nonlinear Schrödinger equation on the product of two real hyperbolic spaces $\mathbb{H}^m \times \mathbb{H}^n$.

1. Introduction

Let $X = M \times N$ be a product of oriented riemannian manifolds, each endowed with its canonical volume form, and let us consider three unbounded selfajoint operators, L on $L^2(X)$, H on $L^2(M)$ and K on $L^2(N)$. We shall assume that the operator L is the *sum* of H and K, in the following sense: we assume that, for every couple of functions f, g, with $f(x)$ in a dense subset of the domain of H and $g(y)$ in a dense subset of the domain of K, we have that the function $f(x)g(y)$ is in the domain of L and

$$L(f(x)g(y)) = Hf \cdot g + f \cdot Kg. \tag{1.1}$$

This situation is quite common and occurs in a number of interesting and natural examples. We mention a few:

Example 1.1. The simplest case of course is given by the standard Laplacian on $\mathbb{R}^n_x \times \mathbb{R}^m_y$; we have

$$\Delta_{x,y} = \Delta_x + \Delta_y$$

and the relation (1.1) is satisfied with the choices $L = -\Delta_{x,y}$, $H = -\Delta_x$, $K = -\Delta_y$. In greater generality, we can choose L, H and K to be the Laplace-Beltrami operators on the three manifolds X, M and N respectively. Indeed, in local coordinates the metric on X is given by a block matrix, with two blocks corresponding to the metrics of M and N; using the explicit representation of the Laplace-Beltrami operators it is easy to check that

$$\Delta_X(f(x)g(y)) = \Delta_M f \cdot g + f \cdot \Delta_N g.$$

Example 1.2. On $\mathbb{R}^m_x \times \mathbb{R}^n_y$, consider the Schrödinger operator

$$L = -\Delta_{x,y} + U(x,y), \qquad U(x,y) = V(x) + W(y) \tag{1.2}$$

where the potential $U(x,y)$ can be split in the sum of two potentials depending only on a group of variables each. Then we may choose

$$H = -\Delta_x + V(x), \qquad K = -\Delta_y + W(y).$$

Date: December 3, 2010.

2000 *Mathematics Subject Classification.* 35Q40, 35Q55, 58J50, 43A85, 22E30.

Key words and phrases. decay estimates, dispersive equations, hyperbolic space, nonlinear Schrödinger equation, scattering.

More generally, H and K can be two *electromagnetic* Schrödinger operators of the form

$$(i\nabla_x - A(x))^2 + V(x), \qquad (i\nabla_y - B(y))^2 + W(y)$$

with $A : \mathbb{R}^m \to \mathbb{R}^m$ and $B : \mathbb{R}^n \to \mathbb{R}^n$.

Example 1.3. The wave function $u(t, x, y)$ of two interacting particles is governed by a Schrödinger equation of the form

$$iu_t - \Delta_{x,y}u + V(x - y)u = 0, \qquad t \in \mathbb{R}, \quad (x, y) \in \mathbb{R}^{3+3}. \tag{1.3}$$

By the change of variables $x' = x + y$, $y' = x - y$, equation (1.3) reduces to the following equation for $v(t, x', y') = u(t, x, y)$:

$$iv_t - \Delta_{x',y'}v + V(y')v = 0.$$

We see that the Schrödinger operator here belongs to the class considered in Example 1.2.

The first goal of this paper is to show by an elementary abstract argument that the dispersive properties of the flows e^{itL}, e^{itH} and e^{itK} are related in a natural way. This approach allows to handle some cases when the usual methods to prove dispersive estimates can not be applied. Although the methods are completely elementary, the result has a number of interesting consequences. Our basic result is the following:

Theorem 1.4. *Assume the Schrödinger flows for H and K satisfy, for some real $a, b \geq 0$, and for t belonging to an interval $I \subseteq \mathbb{R}$, dispersive estimates of the form*

$$\|e^{itH}\phi\|_{L^r(M)} \lesssim |t|^{-a}\|\phi\|_{L^{\tilde{r}}(M)}, \qquad \|e^{itK}\psi\|_{L^r(N)} \lesssim |t|^{-b}\|\psi\|_{L^{\tilde{r}}(N)}. \tag{1.4}$$

for some exponents $\tilde{r} \leq r$ in $[1, \infty]$. Then the flow of L satisfies for $t \in I$ the estimate

$$\|e^{itL}f\|_{L^r(M\times N)} \lesssim |t|^{-a-b}\|f\|_{L^{\tilde{r}}(M\times N)}. \tag{1.5}$$

It is always possible to interpolate the previous dispersive estimate with the conservation of energy

$$\|e^{itL}f\|_{L^2} \equiv \|f\|_{L^2},$$

which follows from the selfadjointness of L. In particular, if the assumptions of Theorem 1.4 hold with $r = \infty$, $\tilde{r} = 1$, we obtain the complete set of dispersive $L^{q'} - L^q$ estimates

$$\|e^{itL}f\|_{L^q(M\times N)} \lesssim |t|^{-(a+b)\left(1-\frac{2}{q}\right)}\|f\|_{L^{q'}(M\times N)}, \qquad 2 \leq q \leq \infty. \tag{1.6}$$

Following the methods of [9], [6], [10], it is then possible to deduce in a standard way the corresponding *Strichartz estimates*. We use the notation, for any finite or infinite interval $I \subseteq \mathbb{R}$,

$$\|F(t, x, y)\|_{L^p_I L^q} = \left(\int_I \left(\int_{M\times N} |F(t, x, y)|^q dV_{x,y}\right)^{\frac{p}{q}} dt\right)^{\frac{1}{p}}.$$

We also define an *admissible couple*, associated to the index $a + b$, as follows: when $a + b > 1$, the couple (p, q) is admissible if it satisfies the conditions

$$\frac{1}{p} + \frac{a+b}{q} = \frac{a+b}{2}, \qquad 2 \leq p \leq \infty, \qquad \frac{2(a+b)}{a+b-1} \geq q \geq 2; \tag{1.7}$$

when $0 < a + b \leq 1$, the conditions are

$$\frac{1}{p} + \frac{a+b}{q} = \frac{a+b}{2}, \qquad \frac{2}{a+b} < p \leq \infty, \qquad \infty > q \geq 2. \tag{1.8}$$

We also denote with q' the dual exponent to q. The value

$$(p, q) = \left(2, \frac{2(a+b)}{a+b-1}\right)$$

(when $a+b \geq 1$) is the *endpoint*; notice that $q \neq \infty$ in all cases considered here.

Then we have:

Proposition 1.5. *Assume X, M, N and L, H, K are as in Theorem 1.4 with $r = \infty$, $\widetilde{r} = 1$. Then the following estimates hold:*

$$\|e^{itL} f\|_{L^p_I L^q} \lesssim \|f\|_{L^2(M \times N)}, \tag{1.9}$$

$$\left\| \int_0^t e^{i(t-s)L} F(s, x, y) ds \right\|_{L^p_I L^q} \lesssim \|F\|_{L^{\widetilde{p}'}_I L^{\widetilde{q}'}} \tag{1.10}$$

for all admissible couples (p, q) and $(\widetilde{p}, \widetilde{q})$.

Remark 1.1. It is evident that the result generalizes to a finite product of manifolds $X = M_1 \times \cdots \times M_k$ and an operator L on X decomposable as the sum $L = H_1 + \cdots + H_k$ where each H_j acts on M_j only.

Remark 1.2. If the flows satisfy estimates with loss of derivatives of the form

$$\|e^{itH}\phi\|_{L^\infty(M)} \lesssim |t|^{-a} \|H^r \phi\|_{L^1(M)}, \qquad \|e^{itK}\psi\|_{L^\infty(N)} \lesssim |t|^{-b} \|H^r \psi\|_{L^1(N)}$$

then it is easy to extend the result of Theorem 1.4 and obtain the estimate

$$\|e^{itL} f\|_{L^\infty(M \times N)} \lesssim |t|^{-a-b} \|H^r K^s f\|_{L^1(M \times N)}.$$

To this end, it is sufficient to apply the argument in the proof to the modified flows

$$H^{-r} e^{itH} \quad \text{and} \quad K^{-s} e^{itK}$$

instead of e^{itH}, e^{itK}.

Despite its simplicity, Theorem 1.4 has several applications. We begin by studying the case of Schrödinger operators with potential perturbations on $\mathbb{R}^n$. A first example is based on the 1D decay results of [5], [7]:

Corollary 1.6. *Let $V(x) \geq 0$ be a real valued function such that*

$$(1 + |x|)^2 V(x) \in L^1(\mathbb{R}). \tag{1.11}$$

Then, for all $n \geq 1$, the solution of the Schrödinger equation on $\mathbb{R}^n$

$$iu_t - \Delta u + (V(x_1) + \cdots + V(x_n))u = 0, \qquad u(0, x) = f(x)$$

satisfies the estimate

$$|u(t, x)| \lesssim |t|^{-n/2} \|f\|_{L^1(\mathbb{R}^n)}.$$

We notice that in dimension $n = 2$ this gives a classes of potentials for which a sharp $L^1 - L^\infty$ estimate is true; no other classes are known to our knowledge (the only known estimates are of type $L^p - L^{p'}$ with $2 \leq p < \infty$, see [14]).

In dimension $n \geq 3$, it is not known what are the optimal conditions on a potential $V(x)$ such that the flow $i\partial_t - \Delta + V(x)$ satisfies a dispersive estimate. However there are several sufficient conditions due to different authors. We mention for instance the following (see [13]): $n \geq 3$, $p_0 > n/2$, $\delta > 3n/2 + 1$, $\ell_0 = 0$ if $n = 3$ and $\ell_0 = [(n-1)/2]$ if $n \geq 4$, and $V : \mathbb{R}^n \to \mathbb{R}$ satisfies

$$\|D^\alpha V\|_{L^{p_0}(|x-y| \leq 1)} \leq \frac{C}{(1 + |x|)^\delta} \qquad \forall |\alpha| \leq \ell_0; \tag{1.12}$$

Corollary 1.7. *Let $m, n \geq 3$, assume the potentials $V : \mathbb{R}^m \to \mathbb{R}$ and $W : \mathbb{R}^n \to \mathbb{R}$ satisfy condition (1.12) (in dimension m and n respectively). Then the solution $u(t, x, y)$ of the Schrödinger equation on $\mathbb{R}^{m+n}$*

$$iu_t - \Delta_{x,y}u + V(x)u + W(y)u = 0, \qquad u(0, x, y) = f(x, y)$$

satisfies the dispersive estimate

$$|u(t, x, y)| \lesssim |t|^{-\frac{m+n}{2}} \|f\|_{L^1(\mathbb{R}^{m+n})}.$$

For the Schrödinger equation describing the interaction of two particles, which was examined in Example 1.3, we can prove the following:

Corollary 1.8. *Let $n \geq 3$ and $V \colon \mathbb{R}^n \to \mathbb{R}$ satisfying condition (1.12). Then the solution of the Schrödinger equation on $\mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_y$*

$$iu_t - \Delta_{x,y}u + V(x - y)u = 0, \qquad u(0, x, y) = f(x, y)$$

satisfies the dispersive estimate

$$|u(t, x, y)| \lesssim |t|^{-n} \|f\|_{L^1(\mathbb{R}^{2n})}.$$

As a final application, we consider a manifold X which is the product of two hyperbolic spaces

$$X = \mathbb{H}^m \times \mathbb{H}^n, \qquad m, n \geq 2.$$

The Schrödinger equation on hyperbolic spaces was investigated in several papers; in particular, weighted Strichartz estimates were proved in [11] while sharp dispersive estimates were obtained in [1]. We recall the main result of [1] for $e^{it\Delta_{\mathbb{H}^n}}$: for all $r, \widetilde{r} \in (2, \infty]$ we have

$$\|e^{it\Delta_{\mathbb{H}^n}} f\|_{L^r(\mathbb{H}^n)} \lesssim \begin{cases} |t|^{-\max\{\frac{1}{2}-\frac{1}{r}, \frac{1}{2}-\frac{1}{\widetilde{r}}\}n} \|f\|_{L^{\widetilde{r}'}} & \text{if } 0 < |t| \leq 1, \\ |t|^{-\frac{3}{2}} \|f\|_{L^{\widetilde{r}'}} & \text{if } |t| \geq 1. \end{cases} \tag{1.13}$$

Using Theorem 1.4 we obtain:

Corollary 1.9. *Consider the Schrödinger equation $iu_t - \Delta_X u = 0$, $u(0) = f$ on the product manifold $X = \mathbb{H}^m \times \mathbb{H}^n$, $m, n \geq 2$, where Δ_X is the Laplace-Beltrami operator on X. Then the solution $u(t, x, y)$ satisfies, for all $r, \widetilde{r} \in (2, \infty]$, the dispersive estimate*

$$\|u(t)\|_{L^r} \lesssim \begin{cases} |t|^{-\max\{\frac{1}{2}-\frac{1}{r}, \frac{1}{2}-\frac{1}{\widetilde{r}}\}(n+m)} \|f\|_{L^{\widetilde{r}'}} & \text{if } 0 < |t| \leq 1, \\ |t|^{-3} \|f\|_{L^{\widetilde{r}'}} & \text{if } |t| \geq 1. \end{cases} \tag{1.14}$$

Analogous estimates hold for the product of k hyperbolic spaces, and more general estimates can be obtained in general for products of Damek-Ricci spaces; this will be the object of future work.

Remark 1.3. We recall that the decay rate $\sim |t|^{-\frac{3}{2}}$ on $\mathbb{H}^n$ for large times is sharp (see [1]). Thus we notice a new phenomenon, indeed, the decay for large t on $\mathbb{H}^m \times \mathbb{H}^n$ is faster than on $\mathbb{H}^{m+n}$. More generally, we can consider the product of k real hyperbolic spaces ($m_j \geq 2$)

$$X = \mathbb{H}^{m_1} \times \cdots \times \mathbb{H}^{m_k}$$

and we obtain a decay of order

$$|u| \lesssim |t|^{-\frac{1}{2}\sum m_j}$$

for small times, while the decay for large times is

$$|u| \lesssim |t|^{-\frac{3}{2}k}.$$

In particular, for spaces of the same dimension $m \geq 2$

$$X = \mathbb{H}^m \times \cdots \times \mathbb{H}^m$$

the total dimension is mk but we get a decay rate $\sim |t|^{-\frac{3}{2}k}$ for large times. Thus if $m > 3$, the decay rate is slower than in the euclidean case of the same dimension $\mathbb{R}^{mk}$, where one has $\sim |t|^{-\frac{mk}{2}}$. On the other hand, if $m = 3$ we obtain exactly the same decay as in the euclidean case, and if $m = 2$ a better decay.

As already revealed in [1], the range of exponents allowed in the dispersive estimate on $\mathbb{H}^n$ is wider than in the euclidean case. This reflects in a much wider range for the Strichartz admissible indices. Indeed, for the nonhomogeneous equation on $X = \mathbb{H}^m \times \mathbb{H}^n$

$$iu_t - \Delta_X u = F(t,x,y), \qquad u(0,x,y) = f(x,y) \tag{1.15}$$

we have the following result:

Corollary 1.10. *Let* $\left(\frac{1}{p},\frac{1}{q}\right)$ *and* $\left(\frac{1}{\tilde{p}},\frac{1}{\tilde{q}}\right)$ *belong to the triangle*

$$T = \left\{\left(\frac{1}{p},\frac{1}{q}\right) \in \left(0,\frac{1}{2}\right] \times \left(0,\frac{1}{2}\right] \ s.t.\ \frac{2}{p} + \frac{m+n}{q} \geq \frac{m+n}{2}\right\} \cup \left\{\left(0,\frac{1}{2}\right)\right\}. \tag{1.16}$$

Then the solution $u(t,x,y)$ of equation (1.15) on $X = \mathbb{H}^m \times \mathbb{H}^n$, $m,n \geq 2$ satisfies the estimate

$$\|u\|_{L_t^p L^q(X)} \lesssim \|f\|_{L^2(X)} + \|u\|_{L_t^{\tilde{p}'} L^{\tilde{q}'}(X)}. \tag{1.17}$$

We recall that in the euclidean case the range of admissible indices is limited to the lower side of the triangle T defined in (1.16). It is not difficult to extend Corollary 1.10 to the product of k real hyperbolic spaces.

To conclude the paper we apply our estimates to the nonlinear Schrödinger equation on $X = \mathbb{H}^m \times \mathbb{H}^n$

$$iu_t - \Delta_X u = F(u). \tag{1.18}$$

We shall limit ourself here to the L^2 well posedness, but an analogous H^1 theory with scattering holds for suitable gauge invariant or defocusing type nonlinearity. We recall that on the hyperbolic spaces $\mathbb{H}^n$, under the additional assumptions of radial symmetry on the data and gauge invariance or defocusing type, scattering properties were studied in [3], using the weighted radial Strichartz estimates obtained in [2] for $n = 3$ and in [12] for $n \geq 3$. Scattering for general power nonlinearities without gauge invariance, and with small L^2 or H^1 data, was obtained in [1], using suitable generalized Strichartz estimates.

Here we consider a nonlinear term satisfying

$$|F(u)| \leq C|u|^\gamma, \qquad |F(u) - F(v)| \leq C(|u| + |v|)^{\gamma-1}|u - v| \tag{1.19}$$

for some real $\gamma \geq 1$, without gauge invariance or sign assumptions. Then we have:

Theorem 1.11. *Let $X = \mathbb{H}^m \times \mathbb{H}^n$, $m,n \geq 2$. Assume $\gamma \leq 1 + \frac{4}{m+n}$. Then, for all small data $f \in L^2(X)$, equation (1.18) has a global unique solution, continuous with values in L^2, which in addition has the scattering property: there exist $u_\pm \in L^2$ such that*

$$\|u - e^{it\Delta_X} u_\pm\|_{L^2(X)} \to 0 \quad as \qquad t \to \pm\infty. \tag{1.20}$$

For large L^2 data and $\gamma < 1 + \frac{4}{m+n}$ the Cauchy problem is locally well posed.

2. Proof of Theorem 1.4

As mentioned in the Introduction, the proof of the Theorem is completely elementary and is based on the factorization

$$e^{itL} = e^{itH} e^{itK}$$

with the two flows acting on independent variables $x \in M$ and $y \in N$. Thus we can write

$$\|e^{itL} f\|_{L^r_{x,y}} = \big\| \|e^{itH} e^{itK} f\|_{L^r_x} \big\|_{L^r_y} \leq C_0 |t|^{-a} \|e^{itK} f\|_{L^{\widetilde{r}}_x L^r_y} \tag{2.1}$$

by the first part of assumption (1.4). Now we notice that

$$\|g(x,y)\|_{L^{\widetilde{r}}_x L^r_y} \leq \|g(x,y)\|_{L^r_y L^{\widetilde{r}}_x} \qquad \text{provided } 1 \leq \widetilde{r} \leq r \leq \infty. \tag{2.2}$$

Inequality (2.2) is obvious in the endpoint cases $r = \widetilde{r} = 1$ and $r = \widetilde{r} = \infty$, and in the case $\widetilde{r} = 1$, $r = \infty$ it reduces to

$$\sup_{y \in N} \int_M |g(x,y)| dx \leq \int_M \sup_{y \in N} |g(x,y)| dx$$

which is also obvious. The remaining cases follow by complex interpolation.

Now we can continue (2.1) using the second part of assumption (1.4) and we obtain

$$\leq C_0 |t|^{-a} \|e^{itK} f\|_{L^r_y L^{\widetilde{r}}_x} \leq C_0 |t|^{-a} \cdot C_0 |t|^{-b} \|f\|_{L^{\widetilde{r}}_{x,y}}$$

and we obtain the estimate (1.5).

3. Sketch of the proof of Proposition 1.5

We follow the standard strategy developed by Kato, Ginibre–Velo and Keel–Tao. For simplicity, we give the argument for the case $I = \mathbb{R}$ and $a+b > 1$. the remaining cases are analogous.

Consider the operator

$$Tf(t,x) = e^{itL} f$$

and its formal L^2 adjoint

$$T^* F = \int_{-\infty}^{+\infty} e^{-isL} F(s)\, ds, \qquad F : \mathbb{R} \times X \to \mathbb{C}.$$

The first step of the method consists in proving the $L^{p'}_t L^{q'}_X \to L^p_t L^q_X$ boundedness of the operator

$$TT^* F = \int_{-\infty}^{+\infty} e^{i(t-s)L} F(s)\, ds \tag{3.1}$$

and of its truncated version

$$\widetilde{TT^*} F = \int_0^t e^{i(t-s)L} F(s)\, ds\,, \tag{3.2}$$

for every admissible pair (p,q). The endpoint $(\frac{1}{p}, \frac{1}{q}) = (0, \frac{1}{2})$ is settled by L^2 conservation and the endpoint $(\frac{1}{p}, \frac{1}{q}) = (\frac{1}{2}, \frac{1}{2} - \frac{1}{2(a+b)})$ will be handled at the end. Thus we are left with the pairs (p,q) such that $\frac{1}{2} - \frac{1}{2(a+b)} < \frac{1}{q} < \frac{1}{2}$. According to the dispersive estimates in Theorem 1.4, the $L^p_t L^q_x$ norms of (4.1) and (4.2) are bounded above by

$$\Big\| \int |t-s|^{-\sigma(q)} \, \|F(s)\|_{L^{q'}_x} \Big\|_{L^{p'}_t}, \qquad \sigma(q) = (a+b)\left(1 - \frac{2}{q}\right). \tag{3.3}$$

The convolution kernel $|t-s|^{-\sigma(q)}$ on $\mathbb{R}$ defines a bounded operator from $L^{p_1}_s$ to $L^{p_2}_t$, for p the first element of the admissible couple (p,q), and this proves the estimate in the non-endpoint case. Consider eventually the endpoint $(\frac{1}{p},\frac{1}{q})=(\frac{1}{2},\frac{1}{2}-\frac{1}{2(a+b)})$; then we can proceed exactly as in [10] by splitting the time integral in dyadic regions $|t-s|\sim 2^j$, $j\in\mathbb{Z}$. Indices are finally decoupled, using the TT^* argument.

4. Proof of the corollaries

The proof of Corollaries 1.6–1.9 is a direct application of Theorem 1.4, combined with dispersive estimates from different papers. More precisely:

(1) For Corollary 1.6, we use n times the dispersive estimate
$$|P_{ac}e^{itH}f|\lesssim |t|^{-1/2}\|f\|_{L^1(\mathbb{R})}$$
where H is the Schrödinger operator on $\mathbb{R}$
$$H=-\frac{d^2}{dx^2}+V(x),\qquad (1+|x|)^2V\in L^1(\mathbb{R})$$
and P_{ac} is the projection on the absolutely continuous space associated to H (see [7] and [5]). From the general theory it is known that all eigenvalues (if present) must be nonnegative. The additional assumption $V\geq 0$ ensures that no eigenvalues exist so that the projection P_{ac} is not necessary.

(2) For Corollaries 1.7 and 1.8, we use the results of [13], where it is proved that under assumption (1.12) the wave operator associated to $H=-\Delta+V$ is bounded on L^p. In particular, this gives dispersive estimates for the Schrödinger equation of the form
$$|e^{itH}f|\lesssim |t|^{-n/2}\|f\|_{L^1}.$$
To our knowledge, Yajima's conditions are the best known for large space dimension $n\geq 4$.

(3) Corollary 1.9 follows easily from (1.13). Notice that the wider range of exponents compared with the euclidean case is due to the Kunze-Stein phenomenon
$$\|f*g\|_{L^{q',\infty}(\mathbb{H}^n)}\lesssim \|f\|_{L^{q'}(\mathbb{H}^n)}\|g\|_{L^{q'}(\mathbb{H}^n)}\qquad \forall q>2$$
(see [4], [8]); however we need here only the endpoint case $q=\widetilde{q}=\infty$ in order to verify the assumptions of Theorem 1.4.

(4) Corollary 1.10 is proved by the same TT^* method as sketched in Section 3, however with an important difference since now the rate of decay for small and large times is different. As above, we consider the operators
$$TT^*F(t,x,y)=\int_{-\infty}^{+\infty}e^{i(t-s)\Delta_X}F(s,x,y)\,ds \tag{4.1}$$
and
$$\widetilde{TT^*}F(t,x,y)=\int_0^t e^{i(t-s)\Delta_X}F(s,x,y)\,ds\,, \tag{4.2}$$
for every admissible pair (p,q). The endpoint $(\frac{1}{p},\frac{1}{q})=(0,\frac{1}{2})$ is true by L^2 conservation. The endpoint $(\frac{1}{p},\frac{1}{q})=(\frac{1}{2},\frac{1}{2}-\frac{1}{m+n})$ for $m+n\geq 3$ is handled by the standard method of [10] applied to the truncated $\widetilde{TT^*}$ directly. Finally consider the pairs (p,q) such that $\frac{1}{2}-\frac{1}{m+n}<\frac{1}{q}<\frac{1}{2}$ and $(\frac{1}{2}-\frac{1}{q})\frac{m+n}{2}\leq\frac{1}{p}\leq\frac{1}{2}$, for which it is sufficient to study TT^*. According to the dispersive estimates in Corollary 1.9, the $L^p_tL^q(X)$ norms of (4.1) and (4.2) are bounded above by
$$\left\|\int_{|t-s|\geq 1}|t-s|^{-3}\,\|F(s)\|_{L^{q'}_x}\right\|_{L^p_t}+\left\|\int_{|t-s|\leq 1}|t-s|^{-(\frac{1}{2}-\frac{1}{q})(m+n)}\,\|F(s)\|_{L^{q'}_x}\right\|_{L^p_t}. \tag{4.3}$$

The convolution kernel in the first integral $|t-s|^{-3}\,\mathbb{1}_{\{|t-s|\geq 1\}}$ on $\mathbb{R}$ defines a bounded operator from $L^{p_1}_s$ to $L^{p_2}_t$, for all $1\leq p_1\leq p_2\leq\infty$, in particular from $L^{p'}_s$ to L^p_t, for all $2\leq p\leq\infty$. The convolution kernel in the second integral $|t-s|^{-(\frac{1}{2}-\frac{1}{q})(m+n)}\,\mathbb{1}_{\{|t-s|\leq 1\}}$ defines a bounded operator from $L^{p_1}_s$ to $L^{p_2}_t$, for all $1<p_1,p_2<\infty$ such that $0\leq\frac{1}{p_1}-\frac{1}{p_2}\leq 1-(\frac{1}{2}-\frac{1}{q})(m+n)$, in particular from $L^{p'}_s$ to L^p_t, for all $2\leq p<\infty$ such that $\frac{1}{p}\geq(\frac{1}{2}-\frac{1}{q})\frac{m+n}{2}$. This proves the result for all dual estimates with $(p,q)=(\widetilde{p},\widetilde{q})$. The standard TT^* argument allows to decouple the pairs and conclude the proof.

5. The nonlinear Schrödinger equation on $\mathbb{H}^m\times\mathbb{H}^n$

We follows the same strategy used in [1]. We resume the standard fixed point method based on Strichartz estimates. Define $u=\Phi(v)$ as the solution to the Cauchy problem

$$\begin{cases} i\,\partial_t u(t,x,y)+\Delta_X u(t,x,y)=F(v(t,x,y))\,,\\ u(0,x,y)=f(x,y)\,,\end{cases}\tag{5.1}$$

which is given by Duhamel's formula:

$$u(t,x,y)=\,e^{it\Delta_X}f(x,y)+\int_0^t e^{i(t-s)\Delta_X}F(v(s,x,y))\,ds\,.$$

According to (1.17), we have the following Strichartz estimate

$$\|u\|_{L^\infty_t L^2(X)}+\|u\|_{L^p_t L^q(X)}\leq C\,\|f\|_{L^2(X)}+C\,\|F(v)\|_{L^{\tilde{p}'}_t L^{\tilde{q}'}(X)}\tag{5.2}$$

for all $(\frac{1}{p},\frac{1}{q})$ and $(\frac{1}{\tilde{p}},\frac{1}{\tilde{q}})$ in the triangle T, which amounts to the conditions

$$\begin{cases} 2\leq p,q\leq\infty\ \text{ such that }\ \frac{\beta}{p}+\frac{m+n}{q}=\frac{m+n}{2}\ \text{ for some }\ 0<\beta\leq 2\,,\\ 2\leq\tilde{p},\tilde{q}\leq\infty\ \text{ such that }\ \frac{\tilde{\beta}}{\tilde{p}}+\frac{m+n}{\tilde{q}}=\frac{m+n}{2}\ \text{ for some }\ 0<\tilde{\beta}\leq 2\,.\end{cases}\tag{5.3}$$

Moreover

$$\|F(v)\|_{L^{\tilde{p}'}_t L^{\tilde{q}'}(X)}\leq C\,\|\,|v|^\gamma\|_{L^{\tilde{p}'}_t L^{\tilde{q}'}(X)}\leq C\,\|v\|^\gamma_{L^{\tilde{p}'\gamma}_t L^{\tilde{q}'\gamma}(X)}$$

by our nonlinear assumption. Thus

$$\|u\|_{L^\infty_t L^2(X)}+\|u\|_{L^p_t L^q(X)}\leq C\,\|f\|_{L^2(X)}+C\,\|v\|^\gamma_{L^{\tilde{p}'\gamma}_t L^{\tilde{q}'\gamma}(X)}\,.\tag{5.4}$$

In order to remain within the same function space, we require in addition

$$p=\tilde{p}'\gamma,\ q=\tilde{q}'\gamma.\tag{5.5}$$

It is easily checked that all these conditions are fulfilled if we take for instance

$$0<\beta=\tilde{\beta}\leq 2\quad\text{such that}\quad\gamma=1+\tfrac{2\beta}{m+n}\quad\text{and}\quad p=q=\tilde{p}=\tilde{q}=1+\gamma=2+\tfrac{2\beta}{m+n}\,.$$

For such a choice, Φ maps $L^\infty(\mathbb{R};L^2(X))\cap L^p(\mathbb{R};L^q(X))$ into itself, and actually $Y=C(\mathbb{R};L^2(X))\cap L^p(\mathbb{R};L^q(X))$ into itself. Since Y is a Banach space for the norm

$$\|u\|_Y=\|u\|_{L^\infty_t L^2}+\|u\|_{L^p_t L^q}\,,$$

it remains for us to show that Φ is a contraction in the ball

$$Y_\varepsilon=\{\,u\in Y\mid\|u\|_Y\leq\varepsilon\,\}\,,$$

provided $\varepsilon>0$ and $\|f\|_{L^2}$ are sufficiently small. Let $v,\tilde{v}\in X$ and $u=\Phi(v)$, $\tilde{u}=\Phi(\tilde{v})$. Arguying as above and using in addition Hölder's inequality, we estimate

$$\begin{aligned}\|u-\tilde{u}\|_Y &\leq C\,\|F(v)-F(\tilde{v})\|_{L_t^{\tilde{p}'}L^{\tilde{q}'}}\\ &\leq C\,\|\{\,|v|^{\gamma-1}+|\tilde{v}|^{\gamma-1}\}\,|v-\tilde{v}|\|_{L_t^{\tilde{p}'}L^{\tilde{q}'}}\\ &\leq C\,\left\{\|v\|_{L_t^pL^q}^{\gamma-1}+\|\tilde{v}\|_{L_t^pL^q}^{\gamma-1}\right\}\|v-\tilde{v}\|_{L_t^pL^q},\end{aligned}$$

hence

$$\|u-\tilde{u}\|_Y \leq C\left(\,\|v\|_Y^{\gamma-1}+\|\tilde{v}\|_Y^{\gamma-1}\,\right)\|v-\tilde{v}\|_Y\,. \tag{5.6}$$

If we assume $\|v\|_Y\leq\varepsilon$, $\|\tilde{v}\|_Y\leq\varepsilon$ and $\|f\|_{L^2}\leq\delta$, then (5.4) and (5.6) yield

$$\|u\|_Y\leq C\,\delta+C\,\varepsilon^\gamma\,,\ \|\tilde{u}\|_Y\leq C\,\delta+C\,\varepsilon^\gamma \quad\text{and}\quad \|u-\tilde{u}\|_Y\leq 2\,C\,\varepsilon^{\gamma-1}\,\|v-\tilde{v}\|_Y\,.$$

Thus

$$\|u\|_Y\leq\varepsilon\,,\ \|\tilde{u}\|_Y\leq\varepsilon \quad\text{and}\quad \|\,u-\tilde{u}\,\|_Y\leq\tfrac{1}{2}\,\|\,v-\tilde{v}\,\|_Y$$

if $C\,\varepsilon^{\gamma-1}\leq\frac{1}{4}$ and $C\,\delta\leq\frac{3}{4}\,\varepsilon$. We conclude by applying the fixed point theorem in the complete metric space Y_ε. Hence, for $1<\gamma\leq 1+\frac{4}{m+n}$ and small L^2 data, the Cauchy problem (1.19) has a unique solution $u(t,x,y)$ in $C(\mathbb{R};L^2(X))\cap L^p(\mathbb{R};L^q(X))$, for the above choice of a suitable pair (p,q). Scattering will follow from the Cauchy criterion:

If $\|z(t_1)-z(t_2)\|_{L^2(X)}\to 0$ as $t_1,t_2\to+\infty$, then there exists $z_+\in L^2$ such that $\|z(t)-z_+\|_{L^2(X)}\to 0$ as $t\to+\infty$.

In our case $z(t,x,y)=e^{-it\Delta_X}u(t,x,y)$. So if we prove that

$$\|e^{-it_2\Delta_X}u(t_2)-e^{-it_1\Delta_X}u(t_1)\|_{L^2(X)}\to 0 \quad\text{as}\quad t_1\leq t_2\to\pm\infty\,,$$

we can conclude that the global solution $u(t,x))$ has the scattering property stated above. Using our Strichartz estimates (1.17), we get

$$\begin{aligned}\left\|e^{-it_2\Delta_X}u(t_2)-e^{-it_1\Delta_X}u(t_1)\right\|_{L^2(X)} &= \left\|\int_{t_1}^{t_2}e^{-is\Delta_X}F(u(s))\,ds\right\|_{L^2(X)}\\ &\leq \|u\|_{L^p([t_1,t_2];L^q(X))}^\gamma\,.\end{aligned}$$

Since $u(t,x,y)\in L^p(\mathbb{R};L^q(X))$, the last expression vanishes as $t_1\leq t_2$ tend both to $+\infty$ or $-\infty$.

In the subcritical case $\gamma<1+\frac{4}{m+n}$, one can prove in a similar way local well–posedness in L^2 for arbitrary data f. Specifically, we restrict to a small time interval $I=[-T,+T]$ and proceed as above, except that we increase $\tilde{\beta}\in(\beta,2\,]$ and $\tilde{p}=\frac{\tilde{\beta}}{\beta}\,p$ accordingly, and that we apply in addition Hölder's inequality in time.

References

[1] Jean-Philippe Anker and Vittoria Pierfelice. Nonlinear Schrödinger equation on real hyperbolic spaces. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(5):1853–1869, 2009.

[2] Valeria Banica: *The nonlinear Schrödinger equation on the hyperbolic space*, Comm. P.D.E. 32 (2007), no. 10, 1643–1677 [arXiv:math/0406058]

[3] Valeria Banica, R. Carles, G. Staffilani: *Scattering theory for radial nonlinear Schrödinger equations on hyperbolic space*, Geom. Funct. Anal. (to appear) [arXiv:math/0607186]

[4] Michael Cowling. Herz's "principe de majoration" and the Kunze-Stein phenomenon. In *Harmonic analysis and number theory (Montreal, PQ, 1996)*, volume 21 of *CMS Conf. Proc.*, pages 73–88. Amer. Math. Soc., Providence, RI, 1997.

[5] Piero D'Ancona and Luca Fanelli. L^p-boundedness of the wave operator for the one dimensional Schrödinger operator. *Comm. Math. Phys.*, 268(2):415–438, 2006.

[6] Jean Ginibre and Giorgio Velo. Generalized Strichartz inequalities for the wave equation. In *Partial differential operators and mathematical physics (Holzhau, 1994)*, volume 78 of *Oper. Theory Adv. Appl.*, pages 153–160. Birkhäuser, Basel, 1995.

[7] Michael Goldberg and Wilhelm Schlag. Dispersive estimates for Schrödinger operators in dimensions one and three. *Comm. Math. Phys.*, 251(1):157–178, 2004.

[8] Alexandru D. Ionescu. An endpoint estimate for the Kunze-Stein phenomenon and related maximal operators. *Ann. of Math. (2)*, 152(1):259–275, 2000.

[9] Tosio Kato. On nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré Sect. A (N.S.)*, 46(1):113–129, 1987.

[10] Markus Keel and Terence Tao. Endpoint Strichartz estimates. *Amer. J. Math.*, 120(5):955–980, 1998.

[11] Vittoria Pierfelice. Weighted Strichartz estimates for the radial perturbed Schrödinger equation on the hyperbolic space. *Manuscripta Math.*, 120(4):377–389, 2006.

[12] Vittoria Pierfelice : *Weighted Strichartz estimates for the Schrödinger and wave equations on Damek–Ricci spaces*, Math. Z. 260 (2008) , no. 2, 377-392

[13] Kenji Yajima. The $W^{k,p}$-continuity of wave operators for schrödinger operators. iii. even-dimensional cases $m \geq 4$. *J. Math. Sci. Univ. Tokyo*, 2(2):311–346, 1995.

[14] Kenji Yajima. L^p-boundedness of wave operators for two-dimensional Schrödinger operators. *Comm. Math. Phys.*, 208(1):125–152, 1999.

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