Higher Lie algebra actions on Lie algebroids

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Abstract

We consider a simple instance of action up to homotopy. More precisely, we consider non-strict actions of DGLAs in degrees -1 and 0 on degree 1 NQ-manifolds. In a more conventional language this means: non-strict actions of Lie algebra crossed modules on Lie algebroids.

When the action is strict, we show that it integrates to group actions in the categories of Lie algebroids and Lie groupoids (i.e. actions of $\mathcal{LA}$-groups and 2-groups). We perform the integration in the framework of Mackenzie's doubles.

We study quotients of degree 1 NQ-manifolds by actions up to homotopy and by distributions, recovering Mackenzie's notion of Lie algebroid ideal system.

Contents

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Introduction

In recent years there has been an intense activity integrating certain infinitesimal structures, such as Lie algebroids [7][9] and L_∞-algebras [10][11]. Here "integration" is meant in the same sense in which a Lie algebra is integrated to a corresponding Lie group. Both Lie algebroids and (non-positively graded) L_∞-algebras are instances of *NQ-manifolds* – non-negatively graded manifolds equipped with a homological vector field (Def. 1.9). In this paper we focus on *actions*: our aim is to study infinitesimal actions on NQ-manifolds and their integrations to global actions.

Part of our motivation comes from the theory of Courant algebroids, which plays an important role in Hitchin and Gualtieri's generalized complex geometry. Indeed Courant algebroids are equivalent to a special class of NQ-manifolds (symplectic NQ-2 manifolds [23]), and the work [5] shows that actions on Courant algebroids are realized by more data than just a Lie group action.

Indeed, the symmetries of a NQ-manifold $\mathcal{M}$ are given by more data than simply a Lie group, by the following elementary but important observation: the vector fields $\chi(\mathcal{M})$ form a *differential* graded Lie algebra (DGLA). We say that $\mathcal{M}$ is a NQ-n manifold if its coordinates are concentrated in degrees $0, \dots, n$. In this case the DGLA $\chi(\mathcal{M})$ is generated as a $C(\mathcal{M})$-module by its elements in degrees $-n, \dots, 0$ (Lemma 1.11 and Remark 1.12). Thus the infinitesimal symmetries of $\mathcal{M}$ are controlled by an $(n+1)$-term DGLA – a special kind of Lie $(n+1)$-algebra –, just as the infinitesimal symmetries of an ordinary manifold are controlled by a Lie algebra. This implies that some kind of Lie $(n+1)$-group naturally plays the role of the global symmetries.

In this paper we consider the simplest case $n = 1$, that is, NQ-1 manifolds, which in ordinary differential geometry language are just *Lie algebroids*. The main purpose of this paper is to study infinitesimal actions on NQ-1 manifolds and to integrate them to global actions. In a future work we plan to extend our study to NQ-2 manifolds and to Courant algebroids. We saw above that the infinitesimal symmetries of NQ-1 manifolds are given by 2- term DGLAs. The latter are also known as *strict Lie 2-algebras*[1] (Def. 1.17), and they are equivalent to crossed modules of Lie algebras (Lemma 1.22). The integration of a strict Lie 2-algebra is a strict Lie 2-group. Hence strict Lie 2-groups control the global symmetries of an NQ-1 manifold.

An *infinitesimal action* on an NQ-1 manifold $\mathcal{M}$ is a suitable morphism from a Lie 2-algebra L into the strict Lie-2 algebra $\chi(\mathcal{M})$ controlling the infinitesimal symmetries of $\mathcal{M}$ (Def. 1.15). Even when L is strict, the action itself might be non-strict. We focus on strict actions, which we integrate explicitly in §2 (the main section of this paper). A strict action of L on $\mathcal{M}$ gives rise to a "action double Lie algebroid", which we integrate to an "action double Lie groupoid". This is a major step, as there are no general statements in the literature which allow to integrate double Lie algebroids to double groupoids. From the "action double Lie groupoid" we extract an action $\mathcal{G} \times \Gamma \to \Gamma$ of a strict Lie 2-group $\mathcal{G}$ on the Lie groupoid Γ integrating the Lie algebroid corresponding to $\mathcal{M}$ (Prop. 2.14, Thm. 2.16 and Prop. 2.18). This gives a conceptual explanation for the results obtained in a special case by Cattaneo and the first author [8, Thm. 14.1]. Notice that the integrated action is

[1] For a general study of Lie 2-algebras please see Baez et al. [2].

not on $\mathcal{M}$, but rather on the its integration Γ. When the action or the Lie 2-algebra are not necessarily strict, we expect to obtain an "action Lie 2-algebroid" (Remark 2.11), whose study we postpone to future work. In the strict action case we discuss the Lie 2-groupoid integrating the "action Lie 2-algebroid"; we obtain it applying the Artin-Mazur construction [1] [22, Chap.12.5.2] to the "action double Lie groupoid" (Remark 2.20).

In §3 we give examples of Lie 2-algebra actions on tangent bundles, cotangent bundles of Poisson manifolds and Lie algebras.

Finally in §4 we discuss quotients of NQ-1 manifolds by the above higher actions. We build on §1.3, where we discuss distributions on NQ-1 manifolds and rediscover Mackenzie's notion of Lie ideal system on Lie algebroids [18]. As such distributions can be thought of as the images of Lie 2-algebra actions, this serves as some evidence for our considerations on higher Lie algebras. What is special in the higher action case is that a *regular* action (not necessarily strict) of a strict Lie 2-algebra may *not* have an involutive image. We need to make an additional assumption to achieve involutivity (Prop. 4.1) and obtain a quotient NQ-1 manifold, which can also be realized via ideal systems.

Notation and conventions: M always denotes a smooth manifold. For any vector bundle E, we denote by $E[1]$ the N-manifold obtained from E by declaring that the fiber-wise linear coordinates on E have degree one. If $\mathcal{M}$ is an N-manifold, we denote by $C(\mathcal{M})$ the graded commutative algebras of "functions on $\mathcal{M}$". By $\chi(\mathcal{M})$ we denote graded Lie algebra of vector fields on $\mathcal{M}$ (i.e., graded derivations of $C(\mathcal{M})$).
The symbol A always denotes a Lie algebroid over M. When A is integrable, we denote the corresponding source simply connected Lie groupoid by Γ, and its source and target maps by $\mathbf{s}$ and $\mathbf{t}$. We adopt the convention that two elements $x, y \in \Gamma$ are composable to $x \circ y$ iff $\mathbf{s}(x) = \mathbf{t}(y)$. We identify $A \cong (ker\ \mathbf{s}_*)|_M$.

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1 Infinitesimal actions on NQ-manifolds

This section introduces NQ-manifolds (§1.1) and infinitesimal actions of Lie-2 algebras on them (§1.2). The integration of these infinitesimal actions is the main object of this paper. Further we study distributions on NQ-manifolds (§1.3).

1.1 Background on graded geometry

We start by proving some background material on graded geometry. In §1.1.1 we consider degree 1 N-manifolds, which correspond simply to vector bundles. Then §1.1.2 we endow them with a homological vector field, obtaining Lie algebroids.

1.1.1 N-manifolds

The notion of N-manifold was introduced by Ševera in [25] and appeared even earlier in [24]. We recall some basic notions about N-manifolds ("N" stands for non-negative). See also [23, §2], [6, §2].

If $V = \oplus_{i\in\mathbb{Z}} V_i$ is a finite dimensional $\mathbb{Z}$-graded vector space, recall that V^* is the $\mathbb{Z}$-graded vector space defined by $(V^*)^i = (V_{-i})^*$, and for any integer k, $V[k]$ denotes the $\mathbb{Z}$-graded vector space obtained from V shifting the degree by k: $(V[k])_i = V_{k+i}$.

Ordinary manifolds are modeled on open subsets of $\mathbb{R}^n$, and N-manifolds modeled on the following graded charts:

Definition 1.1. Let $U \subset \mathbb{R}^n$ open subset and $V = \oplus_{i<0} V_i$ a finite dimensional $\mathbb{Z}_{<0}$-graded vector space.
The *local model for an N-manifold* consists of the pair

- U (the "body")
- $C^\infty(U) \otimes S^\bullet(V^*)$ (the graded commutative algebra of "functions").

Notice that here $S^\bullet(V^*)$ denotes the *graded* symmetric algebra over V^*, so its homogeneous elements anti-commute if they both have odd degree.

Definition 1.2. An *N-manifold* $\mathcal{M}$ consists of a pair as follows:

- a topological space M (the "body")
- a sheaf $\mathcal{O}_M$ over M of graded commutative algebras, locally isomorphic to the above local model (the sheaf of "functions").

We use the notation $C(\mathcal{M})$ for $\mathcal{O}_M(M)$, the space of "functions on $\mathcal{M}$". It is a graded commutative algebra, and it is concentrated in non-negative degrees (as a consequence of the fact that in the local model V is concentrated in negative degrees and therefore V^* in positive degrees). For any non-negative k, $C_k(\mathcal{M})$ denotes the degree k component of $C(\mathcal{M})$. In the following we will work mainly with the algebra $C(\mathcal{M})$, even though strictly speaking all our considerations should be carried out for the sheaf $\mathcal{O}_M$ instead. When we work explicitly with the sheaf $\mathcal{O}_M$, we also use the notation $C(\mathcal{M})_U := \mathcal{O}_M(U)$ for all open subsets $U \subset M$.

The largest i such that $V_{-i} \neq \{0\}$ is called the *degree* of the graded manifold. Degree zero graded manifolds are just ordinary manifolds : $V = \{0\}$, and all functions have degree zero.

Remark 1.3. A graded vector bundle $E = \oplus_{i<0} E_i \to M$ can be viewed as an N-manifold with body M and functions $\Gamma(S^\bullet E^*)$. (To be more precise, the sheaf of "functions" is given by the sheaf of sections of $S^\bullet E^*$.) It can be shown that every finite dimensional N-manifold is (non-canonically) isomorphic to a graded vector bundle as above.

To study the symmetries of a N-manifold $\mathcal{M}$, we look at the vector fields on $\mathcal{M}$.

Definition 1.4. A *vector field* on $\mathcal{M}$ is a graded derivation of the algebra[2] $C(\mathcal{M})$.

Since $C(\mathcal{M})$ is a graded commutative algebras, the space of vector fields $\chi(\mathcal{M})$, equipped with the graded commutator $[-,-]$, is a graded Lie algebra (see Def. 1.10).

In this note we will consider mainly degree 1 N-manifolds, which we now describe in more detail. To do so we recall first

Definition 1.5. Given a vector bundle E over M, a *covariant differential operator*[3] *(CDO)* is a linear over $Y : \Gamma(E) \to \Gamma(E)$ such that there exists a vector field $\underline{Y}$ on M (called *symbol*) with

$$Y(f \cdot e) = \underline{Y}(f)e + f \cdot Y(e), \quad \text{for } f \in C^\infty(M),\ e \in \Gamma(E). \tag{1}$$

We denote the set of CDOs on E by $CDO(E)$. If $Y \in CDO(E)$, then the dual $Y^* \in CDO(E^*)$ is defined by

$$\langle Y^*(\xi), e\rangle + \langle \xi, Y(e)\rangle = \underline{Y}(\langle \xi, e\rangle), \quad \text{for all } e \in \Gamma(E),\ \xi \in \Gamma(E^*). \tag{2}$$

Recall that if $E \to M$ is a (ordinary) vector bundle, $E[1]$ denotes the graded vector bundle whose fiber over $x \in M$ is $(E_x)[1]$ (a graded vector space concentrated in degree -1).

Proposition 1.6. *If $E \to M$ is a vector bundle, then $\mathcal{M} := E[1]$ is a degree 1 N-manifold with body M, and conversely all degree 1 N-manifolds arise this way.*

The algebra of functions $C(\mathcal{M})$ is generated by

$$C_0(\mathcal{M}) = C^\infty(M) \text{ and } C_1(\mathcal{M}) = \Gamma(E^*).$$

The $C(\mathcal{M})$-module of vector fields is generated by elements in degrees -1 and 0. We have identifications

$$\chi_{-1}(\mathcal{M}) = \Gamma(E) \text{ and } \chi_0(\mathcal{M}) = CDO(E^*)$$

induced by the actions on functions. Further the map $\chi_0(\mathcal{M}) \cong CDO(E)$ obtained dualizing CDOs is just $X_0 \mapsto [X_0, \cdot]$ (using the identification $\chi_{-1}(\mathcal{M}) = \Gamma(E)$).

Proof. By Remark 1.3

$$\Gamma(S^\bullet(E^*[-1])) = \Gamma(\wedge^\bullet E^*) \tag{3}$$

is a sheaf over M making $\mathcal{M} := E[1]$ into a degree 1 N-manifold. On the right hand side appears the ordinary exterior product of the vector bundle E^*, and elements of $\wedge^k E^*$ are assigned degree k.

Conversely, let M be a topological space and $\mathcal{O}_M$ a sheaf of graded commutative algebras over M as in Def. 1.2, defining a graded manifold $\mathcal{M}'$. Then M must be a smooth manifold

[2] Strictly speaking one should define vector fields in terms of the sheaf $\mathcal{O}_M$ over M. However we will work only with objects defined on the whole of the body M, hence the above definition will suffice for our purposes.

[3] Also known as derivative endomorphism, see [13, §1].

and the degree 1 elements of $\mathcal{O}_M$ form a locally free module over $C^\infty(M)$, hence sections of a vector bundle, whose dual we denote by E. From Def. 1.1 we conclude that $\mathcal{M}' \cong E[1]$.

From Def. 1.1 it is clear that $C(\mathcal{M})$ is generated (as a graded commutative algebra) by its elements of degree 0 and 1. By eq. (3) we have $C_0(\mathcal{M}) = C^\infty(M)$ and $C_1(\mathcal{M}) = \Gamma(E^*)$.

A vector field on $\mathcal{M}$, since it is a graded derivation, is determined by its action on functions of degree 0 and 1. Let $f_0, g_0 \in C_0(\mathcal{M})$ and $f_1, g_1 \in C_1(\mathcal{M})$. If $X_{-1} \in \chi_{-1}(\mathcal{M})$, it maps $C_0(\mathcal{M})$ to zero and maps $C_1(\mathcal{M})$ to $C_0(\mathcal{M})$. Hence the graded derivation property is simply

$$X_{-1}(f_0 g_1) = f_0 X_{-1}(g_1), \tag{4}$$

so the action of X_{-1} on $\Gamma(E^*)$ is $C^\infty(M)$-linear, i.e. given pairing with a section of E, and we conclude that $\chi_{-1}(\mathcal{M}) = \Gamma(E)$.

If $X_0 \in \chi_0(\mathcal{M})$, then the action of X_0 preserves $C_0(\mathcal{M})$ as well as $C_1(\mathcal{M})$. The graded derivation property on generators reads

$$X_0(f_0 g_0) = X_0(f_0) g_0 + f_0 X_0(g_0), \qquad X_0(f_0 g_1) = f_0 X_0(g_1) + X_0(f_0) g_1.$$

The first equation tells us that $X_0|_M$, defined restricting the action of X_0 to $C_0(\mathcal{M})$, is a vector field on M, and altogether we conclude that X_0 is a covariant differential operator on E^* with symbol $X_0|_M$. Hence $\chi_0(\mathcal{M}) = CDO(E^*)$.

The canonical identification $CDO(E^*) \cong CDO(E)$ obtained dualizing CDOs sends $X_0 \in \chi_0(\mathcal{M}) = CDO(E^*)$ to $[X_0, \cdot] \in CDO(E)$ (using the identification $\chi_{-1}(\mathcal{M}) = \Gamma(E)$). Indeed eq. (2) applied to $Y^* = X_0$ reads

$$X_{-1}(X_0(\xi)) + [X_0, X_{-1}](\xi) = X_0(X_{-1}(\xi))$$

for all $X_{-1} \in \chi_{-1}(\mathcal{M}) = \Gamma(E)$ and $\xi \in C_1(\mathcal{M}) = \Gamma(E^*)$. $\square$

Remark 1.7. Consider again a vector bundle $E \to M$ and $\mathcal{M} = E[1]$. In Lemma 1.6 we saw that $C_0(\mathcal{M}) = C^\infty(M)$ and that $C_1(\mathcal{M})$ agrees with the fiber-wise linear functions on E. [8, Lemma 8.7] states that this induces a canonical, bracket preserving identification of vector fields

$$\chi_{-1}(E[1]) \cong \{\text{vertical vector fields on } E \text{ which are invariant under translations in the fiber direction }\}$$
$$\chi_0(E[1]) \cong \{\text{vector fields on } E \text{ whose flow preserves the vector bundle structure}\}.$$

Remark 1.8. Let us choose coordinates $\{x_i\}$ on an open subset $U \subset M$ and a frame $\{e_\alpha\}$ of sections of $E|_U$. Let $\{\xi^\alpha\}$ be the dual frame for $E^*|_U$, and assign degree 1 to its elements. Then $\{x_i, \xi^\alpha\}$ form a set of coordinates for $\mathcal{M} := E[1]$ (in particular they generate $C(\mathcal{M})$ over U). The coordinate expression of vector fields is as follows. $\chi_{-1}(\mathcal{M})$ consist of elements of the form $f_\alpha \frac{\partial}{\partial \xi^\alpha}$, and $\chi_0(\mathcal{M})$ of elements of the form $g_i \frac{\partial}{\partial x_i} + f_{\alpha\beta} \xi^\alpha \frac{\partial}{\partial \xi^\beta}$. Here $f_\alpha, g_i, f_{\alpha\beta} \in C^\infty(M)$, for $i \leq dim(M)$ and $\alpha, \beta \leq rk(E)$, and we adopt the Einstein summation convention.

1.1.2 NQ-manifolds and Lie algebroids

We will be interested in N-manifolds equipped with extra structure:

Definition 1.9. An *NQ-manifold* is an N-manifold equipped with a *homological vector field*, i.e. a degree 1 vector field Q such that $[Q, Q] = 0$.

To shorten notation, we call a degree n NQ-manifold a *NQ-n manifold*.

We recall the notion of differential graded Lie algebra (DGLA):

Definition 1.10. A *graded Lie algebra* consists of a a graded vector space $L = \oplus_{i\in\mathbb{Z}} L_i$ together with a bilinear bracket $[\cdot,\cdot]\colon L \times L \to L$ such that

– the bracket is degree-preserving: $[L_i, L_j] \subset L_{i+j}$

– the bracket is graded skew-symmetric: $[a, b] = -(-1)^{|a||b|}[b, a]$

– the adjoint action $[a, \cdot]$ is a degree $|a|$ derivation of the bracket (Jacobi identity): $[a, [b, c]] = [[a, b], c] + (-1)^{|a||b|}[b, [a, c]]$.

A *differential graded Lie algebra* (DGLA) $(L, [\cdot,\cdot], \delta)$ is a graded Lie algebra together with a linear $\delta : L \to L$ such that

– δ is a degree 1 derivation of the bracket: $\delta(L_i) \subset L_{i+1}$ and $\delta[a, b] = [\delta a, b] + (-1)^{|a|}[a, \delta b]$

– $\delta^2 = 0$.

Above a, b, c are homogeneous elements of L of degrees $|a|, |b|, |c|$ respectively.

Lemma 1.11. *For a NQ-n manifold $\mathcal{M}$, the space of vector fields*

$$(\chi(\mathcal{M}), [Q, -], [-, -])$$

is a negatively bounded DGLA with lowest degree $-n$.

Proof. The fact that $[Q, -]$ squares to zero follows from $[Q, [Q, -]] = \frac{1}{2}[[Q, Q], -] = 0$. The fact that $[Q, -]$ is a degree 1 derivation of the Lie bracket follows from the Jacobi identity. So the above is a DGLA.

A vector field on $\mathcal{M}$ has local expression $\sum_i f_i \frac{\partial}{\partial y_i}$, where $f_i \in C(\mathcal{M})$ and y_i's are local coordinates on $\mathcal{M}$. The degree of $\frac{\partial}{\partial y_i}$ is $-deg(y_i)$. Since $deg(y_i) \in \{0, \ldots, n\}$ we are done. □

Remark 1.12. As a $C(\mathcal{M})$-module, $\chi(\mathcal{M})$ is generated by its elements in degrees $-n, \ldots, 0$. This suggests that the most important information is contained in the truncated DGLA

$$\chi_{-n}(\mathcal{M}) \oplus \cdots \oplus \chi_{-1}(\mathcal{M}) \oplus \chi_0^Q(\mathcal{M}), \tag{5}$$

where

$$\chi_0^Q(\mathcal{M}) := \{X \in \chi_0(\mathcal{M}) : [Q, X] = 0\}. \tag{6}$$

It is a sub-DGLA of $\chi(\mathcal{M})$.

A well-known example of NQ-manifolds is given by Lie algebroids [18].

Definition 1.13. A *Lie algebroid* A over a manifold M is a vector bundle over M, such that the global sections of A form a Lie algebra with Lie bracket $[\cdot,\cdot]_A$ and Leibniz rule holds

$$[X, fY]_A = f[X,Y]_A + \rho(X)(f)Y, \quad X, Y \in \Gamma(A), f \in C^\infty(M),$$

where $\rho : A \to TM$ is a vector bundle morphism called the *anchor*.

The following is well known ([28], see also [12]):

Lemma 1.14. *NQ-1 manifolds are in bijective correspondence with Lie algebroids.*

We describe the correspondence using the derived bracket construction. By Lemma 1.6 there is a bijection between vector bundles and degree 1 N-manifolds. If A is a Lie algebroid, then the homological vector field is just the Lie algebroid differential acting on $\Gamma(\wedge^\bullet A^*) = C(A[1])$. Conversely, if $(\mathcal{M} := A[1], Q_A)$ is an NQ-manifold, then the Lie algebroid structure on A can be recovered by the derived bracket construction [12, §4.3]: using the identification $\chi_{-1}(\mathcal{M}) = \Gamma(A)$ recalled in Lemma 1.6, we define

$$[a, a']_A = [[Q_A, a], a'], \qquad \rho(a)f = [[Q_A, a], f], \tag{7}$$

where $a, a' \in \Gamma(A)$ and $f \in C^\infty(M)$.

In coordinates the correspondence is as follows. Choose coordinates x_α on M and a frame of sections e_i of A, inducing (degree 1) coordinates ξ_i on the fibers of $A[1]$. Then

$$Q_A = \frac{1}{2}\xi^j\xi^i c_{ij}^k(x)\frac{\partial}{\partial \xi_k} + \rho_i^\alpha(x)\xi^i\frac{\partial}{\partial x_\alpha} \tag{8}$$

where $[e_i, e_j]_A = c_{ij}^k(x)e_k$ and the anchor of e_i is $\rho_i^\alpha(x)\frac{\partial}{\partial x_\alpha}$.

Viewing Lie algebroids as NQ-manifolds proves to be very valuable. For example, the definition of *Lie algebroid morphism* $A \to A'$ is quite involved, but in terms of NQ-manifolds it is simply a morphism of N-manifolds from $A[1]$ to $A'[1]$ (i.e., a morphism of graded commutative algebras $C(A'[1]) \to C(A[1])$) which respects homological vector field. Similarly, the notion of double Lie algebroid is quite involved, but it simplifies once expressed in terms of homological vector fields (Def. 2.5).

1.2 Lie 2-algebra actions on NQ-1 manifolds

In this subsection we define Lie 2-algebra actions on NQ-1 manifolds. We first give a general definition, but then restrict to a very special case (§1.2.1). Then in §1.2.2 we study actions from an ordinary differential geometry view point, that is, we view NQ-1 manifolds as Lie algebroids and describe the action by ordinary differential geometry data.

1.2.1 Definition of L_∞-action on a NQ-manifold

Recall that an L_∞-*algebra*[4] is a graded vector space $L = \bigoplus_{i\in\mathbb{Z}} L_i$ endowed with a sequence of multi-brackets ($n \geq 1$)

$$[\dots]_n \colon \wedge^n L \to L$$

[4]More precisely, this is a now-so-called *flat* L_∞-algebra, and is also the original definition. Flat means that the 0th-bracket (or curvature) vanishes. All L_∞-algebras appearing in this paper (for example DGLAs of vector fields on NQ-manifolds) are of this kind.

of degree $2-n$, satisfying the quadratic relations specified in [14, Def. 2.1][5]. Here $\wedge^n L$ denotes the n-th graded skew-symmetric product of L. When $[\dots]_n = 0$ for $n \geq 3$ we recover the notion of DGLA (Def. 1.10), which is the one of interest in this note. An *L_∞-morphism* $\mu\colon L \rightsquigarrow L'$ between L_∞-algebras is a sequence of maps ($n \geq 1$)

$$\mu^n\colon \wedge^n L \to L'$$

of degree $1-n$, satisfying certain relations (see [14, Def. 5.2] in the case when L' is a DGLA).

Definition 1.15. Let L be a L_∞-algebra and $\mathcal{M}$ be an NQ-manifold. An *action* of L on $\mathcal{M}$ is an L_∞-morphism

$$\mu : L \rightsquigarrow (\chi(\mathcal{M}), d_Q := [Q, -], [-, -])$$

where the right hand side is the DGLA of vector fields on $\mathcal{M}$ as in Lemma 1.11.

Remark 1.16. Even for L a DGLA, such action μ might not be a strict DGLA-morphism. There are important instances of this. For example, given a (ordinary) Lie algebra $\mathfrak{g}$ and a Poisson manifold (M, π), Ševera [26] defines an *up to homotopy Poisson action* as an L_∞-morphism $\mathfrak{g} \rightsquigarrow (C(T^*[1]M)[1], [\cdot,\cdot]_S, [\pi, \cdot]_S)$, where $[\cdot,\cdot]_S$ denotes the Schouten bracket. This is equivalent to the special case of Def. 1.15 in which $(\mathcal{M}, Q) = (T^*[1]M, [\pi, \cdot]_S)$ and the action is Hamiltonian.

We will be concerned with

Definition 1.17. A *Lie 2-algebra* is an L_∞-algebra concentrated in degrees -1 and 0.

A *strict Lie 2-algebra* in the sense of [2] is a DGLA (see Def. 1.10) concentrated in degrees -1 and 0, or equivalently a Lie 2-algebra for which $[\dots]_3 = 0$.

Consider the case that $\mathcal{M}$ is an NQ-1 manifold and $(L, \delta := [\cdot]_1, [\cdot,\cdot] := [\cdot,\cdot]_2)$ is a *strict* Lie 2-algebra. We write $L = \mathfrak{h}[1] \oplus \mathfrak{g}$ for the underlying graded vector space, where $\mathfrak{h}$ and $\mathfrak{g}$ are just ordinary vector spaces. (Notice that since the homogeneous vector fields on $\mathcal{M}$ which do not vanish identically along the body M must have degrees -1 or 0, "almost free" actions (meaning that μ is injective) can be achieved only when L is a Lie 2-algebra.)

The action

$$\mu : \mathfrak{h}[1] \oplus \mathfrak{g} \rightsquigarrow \chi(\mathcal{M}). \tag{9}$$

is an L_∞-morphisms between DGLAs. We spell out its components. Since $\mu^i : \wedge^i L \to \chi(\mathcal{M})$ has degree $1-i$ the components of μ are

$$\begin{aligned} &\mu^1\colon \mathfrak{g} \to \chi_0(\mathcal{M}) \\ &\mu^1\colon \mathfrak{h}[1] \to \chi_{-1}(\mathcal{M}) \\ &\mu^2\colon \mathfrak{g} \wedge \mathfrak{g} \to \chi_{-1}(\mathcal{M}), \end{aligned}$$

subject to the constraints

$$d_Q \mu^1 = \mu^1 \delta \tag{10}$$

$$\mu^1[x,y] - [\mu^1 x, \mu^1 y] = d_Q(\mu^2(x \wedge y)) \qquad \forall x, y \in \mathfrak{g}, \tag{11}$$

$$\mu^1[w,x] - [\mu^1 w, \mu^1 x] = \mu^2(\delta w \wedge x) \qquad \forall w \in \mathfrak{h}[1], x \in \mathfrak{g}, \tag{12}$$

[5] [14] uses a grading opposite to ours.

as well as an equation for $x \wedge y \wedge z \in \wedge^3\mathfrak{g}$:

$$\begin{aligned} 0 =& \mu^2(x,[y,z]) - \mu^2(y,[x,z]) + \mu^2(z,[x,y]) \\ &+[\mu^1(x),\mu^2(y,z)] - [\mu^1(y),\mu^2(x,z)] + [\mu^1(z),\mu^2(x,y)]. \end{aligned} \tag{13}$$

Notice that condition (10) says that μ^1 is a map of complexes, (12) says that $\mu^1|_{\mathfrak{g}}$ is a morphism of Lie algebras up to homotopy, and (11) says that $\mu^1|_{\mathfrak{h}[1]}$ is a morphism of Lie modules up to homotopy.

Remark 1.18. By eq. (10), the image of the action map μ will be contained in the truncated DGLA $\chi_{-1}(\mathcal{M})\oplus\chi_0^Q(\mathcal{M})$ (see eq. (5)). Hence Lie 2-algebra actions on $\mathcal{M}$ can be formulated using only the truncated DGLA.

Now we restrict ourselves even further, and consider actions (9) for which $\mu^2 = 0$:

Definition 1.19. A *strict* action of a strict Lie 2-algebra $\mathfrak{h}[1]\oplus\mathfrak{g}$ on an NQ-1 manifold $\mathcal{M}$ is a morphism of DGLAs

$$\mu : \mathfrak{h}[1]\oplus\mathfrak{g} \to \chi(\mathcal{M}),$$

i.e., a degree-preserving linear map preserving the differentials and Lie brackets.

The NQ-1 manifold $\mathcal{M}$ is equal to $A[1]$ for some Lie algebroid A, by Lemma 1.14. We spell out what Def. 1.19 means: we have maps

$$\begin{aligned} \mu|_{\mathfrak{h}[1]} \colon \mathfrak{h}[1] \to& \chi_{-1}(A[1]) \\ \mu|_{\mathfrak{g}} \colon \mathfrak{g} \to& \chi_0(A[1]) \end{aligned}$$

such that

$$\begin{aligned} \mu(\delta w) =& d_Q(\mu(w)) && \text{for all } w \in \mathfrak{h}[1], && (14)\\ 0 =& d_Q(\mu(v)) && \text{for all } v \in \mathfrak{g}, && (15)\\ \mu[v,w] =& [\mu(v),\mu(w)] && \text{for all } v \in \mathfrak{g}, w \in \mathfrak{h}[1], && (16)\\ \mu[v_1,v_2] =& [\mu(v_1),\mu(v_2)] && \text{for all } v_i \in \mathfrak{g}. && (17) \end{aligned}$$

1.2.2 DGLAs and Lie algebra crossed modules

Let $\mathfrak{h}[1]\oplus\mathfrak{g}$ be a strict Lie 2-algebra. Recall that $[w,w']_\delta := [\delta w, w']$ makes $\mathfrak{h}$ into a Lie algebra, which we denote by $\mathfrak{h}_\delta$. Let $A[1]$ be an NQ-1 manifold. Using the identifications $\chi_{-1}(A[1]) = \Gamma(A)$ and $\chi_0(A[1]) \cong CDO(A)$ given in Lemma 1.6, we obtain a characterization of strict actions (Def. 1.19) in terms of classical geometric objects.

Lemma 1.20. *Let $\mu : \mathfrak{h}[1]\oplus\mathfrak{g} \to \chi(\mathcal{M})$ be a linear map. Then μ is a morphism of DGLAs iff*

- *$\mu|_{\mathfrak{g}} \colon \mathfrak{g} \to CDO(A)$ is an infinitesimal action of $\mathfrak{g}$ on A by infinitesimal Lie algebroid automorphisms, with the property that $\delta(w)$ acts as $[\mu(w),\cdot]_A$ for all $w \in \mathfrak{h}$*

- *$\mu|_{\mathfrak{h}} \colon \mathfrak{h}_\delta \to \Gamma(A)$ is a Lie algebra homomorphism which is equivariant w.r.t. the representation of $\mathfrak{g}$ on $\mathfrak{h}$ by $v \mapsto [v,\cdot]$ and the representation $\mu|_{\mathfrak{g}}$ of $\mathfrak{g}$ on $\Gamma(A)$.*

Proof. Assume that μ is a strict action. (17) means that $\mu|_{\mathfrak{g}}$ is an infinitesimal action of $\mathfrak{g}$ on the vector bundle A, and (15) means that the action is by infinitesimal Lie algebroid automorphisms.

(14) means that δw acts by $[\mu(w),\cdot]_A$ for all $w \in \mathfrak{h}$, by the derived bracket construction.

(16) means that $\mu|_{\mathfrak{h}}$ is an equivariant map.

Further $\mu|_{\mathfrak{h}}$ is a Lie algebra morphism:

$$\mu[w_1,w_2]_\delta = \mu[\delta w_1, w_2] = [\mu(\delta w_1),\mu(w_2)] = [[Q,\mu(w_1)],\mu(w_2)] = [\mu(w_1),\mu(w_2)]_A,$$

where the second equality holds by (16) and the third equality by (14).

The converse implication is obtained reversing the argument. □

The idea behind Lemma 1.20 is the concept of crossed module, which might be more familiar to the reader than strict Lie 2-algebras, even though they are equivalent concepts (see Prop 1.22).

Definition 1.21. A *crossed module of Lie algebras* $(\mathfrak{h},\mathfrak{g},\delta,\alpha)$ consists of a Lie algebra morphism $\delta:\mathfrak{h}\to\mathfrak{g}$ and an (left) action of $\mathfrak{g}$ on $\mathfrak{h}$ by derivations, i.e. $\alpha:\mathfrak{g}\to Der(\mathfrak{h})$, such that

$$\delta(\alpha(v)(w)) = [v,\delta(w)], \qquad \alpha(\delta(w))w' = [w,w'].$$

where $v\in\mathfrak{g}$, $w,w'\in\mathfrak{h}$.

A classical result (see [2, Thm. 36]) is that

Proposition 1.22. *There is a one-to-one correspondence between strict Lie 2-algebras and crossed modules of Lie algebra.*

Since a similar construction will show up again in §2.2, we recall the correspondence: A strict Lie 2-algebras $(L_{-1}[1]\oplus L_0,\delta,[\cdot,\cdot])$ gives rise to a Lie algebra crossed module with $\mathfrak{h}=L_{-1}$ and $\mathfrak{g}=L_0$ where

$$\begin{aligned}
[w,w']_{\mathfrak{h}} &:= [\delta(w),w'],\\
\alpha(v)(w) &:= [v,w],\\
[v,v']_{\mathfrak{g}} &:= [v,v']
\end{aligned}$$

for $v,v'\in\mathfrak{g}$, $w,w'\in\mathfrak{h}$. The Jacobi identity of $[\cdot,\cdot]$ gives the Jacobi identities of $[\cdot,\cdot]_{\mathfrak{h}}$ and $[\cdot,\cdot]_{\mathfrak{g}}$ and the remaining conditions for crossed modules.

On the other hand, a crossed module $(\mathfrak{h},\mathfrak{g},\delta,\alpha)$ gives rise to a strict Lie-2 algebra with $L_{-1}=\mathfrak{h}$, $L_0=\mathfrak{g}$, δ as a differential, and

$$\begin{aligned}
[w,w'] &:= 0,\\
[v,v'] &:= [v,v']_{\mathfrak{g}},\\
[v,w] &= -[w,v] := \alpha(v)(w)
\end{aligned}$$

for $v,v'\in L_0$, $w,w'\in L_{-1}$. The Jacobi identity of $[\cdot,\cdot]$ is implied by the Jacobi identities of $[\cdot,\cdot]_{\mathfrak{h}}$ and $[\cdot,\cdot]_{\mathfrak{g}}$ and various conditions for crossed modules.

Consequently, there is a 1-1 correspondence between DGLA morphisms and crossed module morphisms. Thus Lemma 1.20 basically explicitly tells us that given $\mu:\mathfrak{h}[1]\oplus$

$\mathfrak{g} \to \chi(\mathcal{M})$ a linear map, then μ is a morphism of DGLAs iff $(\mu|_{\mathfrak{h}}, \mu|_{\mathfrak{g}})$ is a morphism of Lie algebra crossed modules from the crossed module associated to $\mathfrak{h}[1] \oplus \mathfrak{g}$ to the one associated to the the truncated DGLA $\chi_{-1}(A[1]) \oplus \chi_0(A[1])^Q$. For this, we only need to notice that the crossed module associated to $\chi_{-1}(A[1]) \oplus \chi_0(A[1])^Q$ by Lemma 1.22 is the quadruple given by $(\Gamma(A), [\cdot,\cdot]_A)$, the subset of $CDO(A)$ consisting of infinitesimal Lie algebroid automorphisms, the morphism $\delta(a) = [a, \cdot]_A$ and the natural action of $CDO(A)$ on $\Gamma(A)$.

1.3 Distributions on NQ-1 manifolds

In this subsection A is a Lie algebroid over M and $\mathcal{M} := A[1]$ the corresponding NQ-1 manifold (see Lemma 1.14). We consider the notion of involutive distribution on $\mathcal{M}$, and show that it allows us to construct a quotient of $\mathcal{M}$ which is again an NQ-1 manifold provided some regularity condition is satisfied. Expressing our results in classical terms, what we recover as a quotient is hence a Lie algebroid.

Distributions are of interest to us because they are related to Lie 2-algebras actions, exactly as in ordinary geometry a distribution can be thought of as the image of a Lie algebra action. The precise relation between distributions on $\mathcal{M}$ and Lie 2-algebras actions on $\mathcal{M}$ is somehow subtle and is studied in §4. The material developed in the present subsection supports the *integration* procedure of Lie 2-algebra actions we will carry out in §2.1-2.3. Indeed in Prop. 4.3 we will see that, when the distribution is induced by a Lie 2-algebra action satisfying certain regularity conditions, the Lie algebroid obtained quotienting $\mathcal{M}$ by the distribution agrees with the one obtained by Stefanini [27] quotienting A by the integration of the Lie 2-algebra action.

At first we will only use the fact that $A \to M$ is a vector bundle.

Definition 1.23. A *distribution* $\mathcal{D}$ on an NQ-1 manifold $\mathcal{M} := A[1]$ is a graded $C(\mathcal{M})$-submodule of $\chi(\mathcal{M})$ such that for any $x \in M$ there exists a neighborhood $U \subset M$ and homogeneous generators of $\mathcal{D}$ over U such that their evaluations at every point of U are $\mathbb{R}$-linearly independent. A distribution is *involutive* if it is closed under the Lie bracket of vector fields on $\mathcal{M}$.

Here the evaluation on M of a vector field on $\mathcal{M}$ is its image in $\chi(\mathcal{M})/C_{\geq 1}(\mathcal{M})\chi(\mathcal{M})$, the space of sections of the graded vector bundle $T\mathcal{M}|_M = A[1] \oplus TM \to M$.

Remark 1.24. A graded $C(\mathcal{M})$-submodule $\mathcal{D}$ of $\chi(\mathcal{M})$ is a distribution iff there exist $l \leq dim(M)$, $\lambda \leq rk(A)$ and, for any $m \in M$, coordinates $\{x_i, \xi_\alpha\}$ on $\mathcal{M} = A[1]$ defined in a neighborhood U of m (see Remark 1.8) such that, over U, $\mathcal{D}$ is generated as a $C(\mathcal{M})$-module by

$$\left\{\frac{\partial}{\partial \xi_\alpha}\right\}_{\alpha \leq \lambda} \text{ and } \left\{f_i(x)\frac{\partial}{\partial x_i} + \sum_{\beta,\gamma \leq rk(A)} f^i_{\beta\gamma}(x)\xi_\beta \frac{\partial}{\partial \xi_\gamma}\right\}_{i \leq l}$$

and the $\{f_i \frac{\partial}{\partial x_i}\}_{i \leq l}$ are $\mathbb{R}$-linearly independent at every point of U. Notice that there is no condition on the coefficients $f^i_{\beta\gamma}$.

Let $\mathcal{D}$ be a distribution on $\mathcal{M}$. Denote by $\mathcal{D}_i$ the degree i component of $\mathcal{D}$, consisting of the degree i vector fields on $\mathcal{M}$ which lie in $\mathcal{D}$ ($i \geq -1$).

Recall that for any vector bundle A, there is an associated Lie algebroid $\mathfrak{D}(A)$ whose sections are exactly $CDO(A)$ (see Def. 1.5) endowed with the commutator bracket, and whose anchor $s\colon \mathfrak{D}(A) \to TM$ is given by the symbol [13, §1]. $\mathfrak{D}(A)$ fits in an exact sequence of vector bundles

$$0 \to End(A) \to \mathfrak{D}(A) \xrightarrow{s} TM \to 0 \tag{18}$$

where $End(A)$ denotes the vector bundle endomorphisms of A that cover Id_M. The following lemma gives a characterization of distributions on $\mathcal{M}$ in classical terms.

Lemma 1.25. *There is a bijection between distributions $\mathcal{D}$ on $\mathcal{M} := A[1]$ and the following data:*

- *subbundles $B \to M$ of A,*
- *subbundles $C \to M$ of $\mathfrak{D}(A)$ for which $ker(s) \cap C = \{\phi \in End(A) : \phi(A) \subset B\}$.*

The correspondence is

$$\mathcal{D} \mapsto \begin{cases} B \text{ such that } \Gamma(B) = \mathcal{D}_{-1} \\ C \text{ such that } \Gamma(C) = \mathcal{D}_0, \end{cases} \tag{19}$$

where we identify $\Gamma(A) = \chi_{-1}(\mathcal{M})$ and $\Gamma(\mathfrak{D}(A)) = CDO(A) \cong \chi_0(\mathcal{M})$ as in Lemma 1.6.

$\mathcal{D}$ is involutive iff $C \subset \mathfrak{D}(A)$ is a Lie subalgebroid and the action of sections of C preserves $\Gamma(B)$.

Remark 1.26. Recall from Lemma 1.6 that the identification $\chi_0(\mathcal{M}) \cong CDO(A)$ is given by $X_0 \to [X_0, \cdot]$.

Proof. Let $\mathcal{D}$ be a distribution. By definition, locally there exist integers $l \leq dim(M)$, $\lambda \leq rk(A)$ as well as homogeneous generators $\{X_{-1}^{\alpha}\}_{\alpha \leq \lambda}$ and $\{X_0^i\}_{i \leq l}$ of $\mathcal{D}$ whose evaluations at points of M, which are given by $\{X_{-1}^{\alpha}\}_{\alpha \leq \lambda}$ and $\{s(X_0^i)\}_{i \leq l}$, are linearly independent.

$$\mathcal{D}_{-1} = C^{\infty}(M) \cdot \{X_{-1}^{\alpha}\}_{\alpha \leq \lambda}$$

hence consists of sections of a subbundle $B \subset A$. We have

$$\mathcal{D}_0 = C^{\infty}(M) \cdot \{X_0^i\}_{i \leq l} + C_1(\mathcal{M}) \cdot \{X_{-1}^{\alpha}\}_{\alpha \leq \lambda}.$$

The linear independence condition on the generators implies that $ker(s|_{\mathcal{D}_0}) = C_1(\mathcal{M}) \cdot \{X_{-1}^{\alpha}\}_{\alpha \leq \lambda}$ is the space of sections of a subbundle of $End(A)$, and that $s(\mathcal{D}_0)$ is the space of sections of a subbundle of TM. Hence $\mathcal{D}_0$ is the space of sections of a subbundle $C \subset \mathfrak{D}(A)$, and $ker(s) \cap C = \{\phi \in End(A) : \phi(A) \subset B\}$.

Conversely, given a pair (B, C) as in the statement, $\mathcal{D}_{-1} := \Gamma(B)$ and $\mathcal{D}_0 := \Gamma(C)$ define a distribution.

Since a distribution $\mathcal{D}$ is generated (as a $C(\mathcal{M})$-module) by $\mathcal{D}_{-1}$ and $\mathcal{D}_0$, the involutivity of $\mathcal{D}$ is equivalent to $[\mathcal{D}_0, \mathcal{D}_0] \subset \mathcal{D}_0$ and $[\mathcal{D}_0, \mathcal{D}_{-1}] \subset \mathcal{D}_{-1}$. The first condition means that C is a subalgebroid of $\mathfrak{D}(A)$, the second (by Remark 1.26) that the sections of C preserve $\Gamma(B)$. □

By Lemma 1.25

$$F := s(C) = \cup_{x\in M}\{(\underline{X_0})_x : X_0 \in \mathcal{D}_0\} \tag{20}$$

is a distribution on M. Here and in the following, if $X_0 \in \mathcal{D}_0 = CDO(A)$, we denote its symbol by $\underline{X_0} := s(X_0) \in \chi(M)$. Notice that if $\mathcal{D}$ is involutive then F is an involutive distribution.

From now on we let $A \to M$ be a Lie algebroid (and not just a vector bundle). We recall the definition of ideal system on A [18, Def. 4.4.2].

Definition 1.27. An *ideal system* for the Lie algebroid $A \to M$ consists of

- a Lie subalgebroid $B \to M$ of A,
- a closed, embedded wide[6] subgroupoid of the pair groupoid $M \times M$ of the form $R = \{(x, x') : \pi(x) = \pi(x')\}$ for some surjective submersion $\pi\colon M \to N$,
- a linear action Θ of R on the vector bundle $A/B \to M$,

such that, referring to a section $Y \in \Gamma(A)$ as Θ-stable whenever $\Theta(Y) \in \Gamma(B)$,

(i) if $Y, Z \in \Gamma(A)$ are Θ-stable then $[Y, Z]_A$ is also Θ-stable,

(ii) if $X \in \Gamma(B)$, and $Y \in \Gamma(A)$ is Θ-stable, then $[X, Y]_A \in \Gamma(B)$,

(iii) the anchor ρ maps B into $F := ker(\pi_*)$,

(iv) the map $A/B \to TM/F$ induced by the anchor ρ is R-equivariant w.r.t. the action Θ of R on A/B and the canonical action of R on TM/F.

When A is a Lie algebra, an ideal system is simply an ideal of A.

By [18, Thm. 4.4.3], an ideal system always induces a Lie algebroid structure on the quotient of A/B by the action Θ (a vector bundle over N) such that the natural projection is a Lie algebroid morphism.

Under certain conditions, an involutive distribution on $A[1]$ preserved by the homological vector field Q gives an ideal system on A. Let us first understand this in the case of a Lie algebra $A = \mathfrak{g}$, then we give a proof for all Lie algebroids in Prop. 1.29.

Example 1.28. [Distributions on Lie algebras] Let $A = \mathfrak{g}$ be a Lie algebra. The degree -1 part of a distribution $\mathcal{D}$ on $\mathfrak{g}[1]$ corresponds to a subspace $B \subset \mathfrak{g}$. In local coordinates on $\mathfrak{g}[1]$, we have

$$\mathcal{D}_{-1} = span_{\mathbb{R}}\left\{\frac{\partial}{\partial \xi_\alpha}\right\}_{\alpha\leq \dim B}, \quad \mathcal{D}_0 = span_{\mathbb{R}}\left\{\xi_{\alpha'}\frac{\partial}{\partial \xi_\alpha}\right\}_{\alpha\leq \dim B, \alpha'\leq \dim \mathfrak{g}}.$$

This can be seen directly from the definition of distribution or from Lemma 1.25. The distribution $\mathcal{D}$ is automatically involutive.

[6] This means that the subgroupoid has the same base M.

Further, if $[Q,\cdot]$ preserves $\mathcal{D}$, then B is an ideal in $\mathfrak{g}$. Indeed, for any index $\gamma \leq dim(B)$ we have $[Q, \frac{\partial}{\partial \xi_\gamma}] \in \mathcal{D}_0$. Hence for any $\beta \leq dim(\mathfrak{g})$, using the identification $\mathfrak{g} \cong \chi_{-1}(\mathfrak{g}[1])$, we have

$$\left[\frac{\partial}{\partial \xi_\gamma}, \frac{\partial}{\partial \xi_\beta}\right]_{\mathfrak{g}} = \left[\left[Q, \frac{\partial}{\partial \xi_\gamma}\right], \frac{\partial}{\partial \xi_\beta}\right] = \left[\sum_{\alpha=1}^{\dim B} \sum_{\alpha'=1}^{\dim \mathfrak{g}} C_{\alpha'\alpha} \xi_{\alpha'} \frac{\partial}{\partial \xi_\alpha}, \frac{\partial}{\partial \xi_\beta}\right] = -\sum_{\alpha=1}^{\dim B} C_{\beta\alpha} \frac{\partial}{\partial \xi_\alpha} \in B$$

for some constants $C_{\alpha'\alpha}$.

Proposition 1.29. *Let $A \to M$ be a Lie algebroid, and denote by Q the homological vector field on $\mathcal{M} := A[1]$ as in Lemma 1.14. Let $\mathcal{D}$ be an involutive distribution on $\mathcal{M}$ and assume that $[Q, \mathcal{D}] \subset \mathcal{D}$.*

Denote by $B \subset A$ and $C \subset \mathfrak{D}(A)$ the subbundles associated to $\mathcal{D}$ as in Lemma 1.25, and denote $F := s(C)$. Consider the flat F-connection $\nabla_{\underline{X_0}} = ([X_0, \cdot]$ mod $B)$ on A/B, where $X_0 \in \mathcal{D}_0$ and the bracket denotes the Lie bracket of vector fields on $\mathcal{M}$.

If M/F is a smooth manifold such that $\pi\colon M \to M/F$ is a submersion and if ∇ has no holonomy then the following is an ideal system for the Lie algebroid A:

- *the Lie subalgebroid B of A,*
- *the Lie subgroupoid R of $M \times M$ associated to the submersion $\pi\colon M \to M/F$,*
- *the linear action of R on the vector bundle A/B given by parallel translation w.r.t. ∇.*

Proof. $B \to M$ is a subbundle of A by Lemma 1.25. We show that ∇ is an F-connection on A/B and that it is flat. Let $X_0 \in \mathcal{D}_0$. Then $[X_0, \cdot]$ sends $\mathcal{D}_{-1} = \Gamma(B)$ to itself since $\mathcal{D}$ is involutive. If $\underline{X_0} = 0$ then by Lemma 1.25 $[X_0, \cdot]$ sends $\Gamma(A)$ into $\Gamma(B)$. Altogether we have a well-defined map $\nabla\colon \Gamma(F) \times \Gamma(A/B) \to \Gamma(A/B)$, which is easily seen to be a connection. The flatness of ∇ follows from

$$\nabla_{[\underline{X_0}, \underline{X_0'}]} = [[X_0, X_0'], \cdot] \text{ mod } B = [\nabla_{\underline{X_0}}, \nabla_{\underline{X_0'}}]$$

where in the first equality we used the integrability of $\mathcal{D}$ and in the second the Jacobi identity for the bracket of vector fields on $\mathcal{M}$.

B is a Lie subalgebroid of A: if $X_{-1}, X'_{-1} \in \Gamma(B)$ then

$$[X_{-1}, X'_{-1}]_A = [[Q, X_{-1}], X'_{-1}] \in \mathcal{D}_{-1} = \Gamma(B)$$

using $[Q, \mathcal{D}] \subset \mathcal{D}$ and $[\mathcal{D}_0, \mathcal{D}_{-1}] \subset \mathcal{D}_{-1}$. If M/F is a smooth manifold then R is a closed, embedded wide subgroupoid of $M \times M$. If ∇ has no holonomy then the groupoid action of R on A/B is well-defined.

To check that we indeed have an ideal system we need to check (i)-(iv) in Def. 1.27.

(ii) *If $X_{-1} \in \Gamma(B)$, and $Y \in \Gamma(A)$ satisfies $[\mathcal{D}_0, Y] \in \Gamma(B)$, then $[X_{-1}, Y]_A \in \Gamma(B)$*

Indeed $[[Q, X_{-1}], Y] \in \Gamma(B)$ because Q preserves $\mathcal{D}$.

(iii) *If $X_{-1} \in \Gamma(B)$ then $\rho(X_{-1}) = \underline{[Q, X_{-1}]} \in \Gamma(F)$.*

This is clear because Q preserves $\mathcal{D}$.

(iv) *If p, q are points of M lying in the same fiber of π and $Y_p \in A_p$, then $\pi_*(\rho(Y_p)) = \pi_*(\rho(Y_q))$, where $Y_q \in A_q$ is a lift of the ∇-parallel translation of (Y_p mod B) from p to q.*

It is enough to show that if $Y \in \Gamma(A)$ satisfies $[\mathcal{D}_0, Y] \in \Gamma(B)$ then $\rho(Y) = \underline{[Q,Y]}$ descends to a vector field on M/F, i.e. $[\underline{\mathcal{D}_0}, \underline{[Q,Y]}] \in \Gamma(F)$. To show this we proceed as follows. For any $X_0 \in \mathcal{D}_0$ consider the r.h.s. of

$$[X_0, [Q, Y]] = [[X_0, Q], Y] + [Q, [X_0, Y]].$$

$[X_0, Q] \in \mathcal{D}_1 = C_1(\mathcal{M}) \cdot \mathcal{D}_0$, so using the assumption on Y we get $[[X_0, Q], Y] \in \mathcal{D}_0$. The second term on the r.h.s.al so lies in $\mathcal{D}_0$, since $[X_0, Y] \in \Gamma(B)$ by assumption and $[Q, \mathcal{D}] \subset \mathcal{D}$. Hence $[\underline{\mathcal{D}_0}, \underline{[Q,Y]}] = \underline{[\mathcal{D}_0, [Q,Y]]} \in s(\mathcal{D}_0) = \Gamma(F)$.

(i) *If $Y, Z \in \chi_{-1}(\mathcal{M}) = \Gamma(A)$ satisfy $[\mathcal{D}_0, Y] \in \Gamma(B)$, $[\mathcal{D}_0, Z] \in \Gamma(B)$ then $[\mathcal{D}_0, [Y, Z]_A] \in \Gamma(B)$.*

Let $X_0 \in \mathcal{D}_0$. Using repeatedly the Jacobi identity we have

$$\begin{aligned}[X_0, [[Q, Y], Z]] =& [[X_0, [Q, Y]], Z] + [[Q, Y], [X_0, Z]] \\ =& [[[X_0, Q], Y], Z] + [[X_0, Y], Z]_A + [Y, [X_0, Z]]_A.\end{aligned}$$

Since $[X_0, Y], [X_0, Z] \in \Gamma(B)$ by assumption, (ii) implies that the second and third term on the r.h.s. lie in $\Gamma(B)$. The computation in (iv) shows that $[[X_0, Q], Y] \in \mathcal{D}_0$, so by the assumption on Z the first term on the r.h.s. also lies in $\Gamma(B)$.

□

We conclude showing that the quotient – which we now define – of $\mathcal{M} = A[1]$ by an involutive distribution $\mathcal{D}$ agrees with the natural quotient of the Lie algebroid A by the corresponding ideal system ([18, Thm. 4.4.3]). Consider $C(\mathcal{M})^{\mathcal{D}}$, the sheaf (over the body M) of $\mathcal{D}$-invariant functions on $\mathcal{M}$, defined assigning to $U \subset M$ the algebra

$$C(\mathcal{M})^{\mathcal{D}}_U := \{f \in C(\mathcal{M})_U : X(f) = 0 \text{ for all } X \in \mathcal{D}_U\}.$$

Assume that the quotient M/F is a smooth manifold such that $\pi \colon M \to M/F$ is a submersion, where the involutive distribution F on M was defined in eq. (20). Then

$$V \mapsto C(\mathcal{M})^{\mathcal{D}}_{\pi^{-1}(V)} \tag{21}$$

is a presheaf of graded commutative algebras over M/F. When it defines an N-manifold, we say that the quotient of $\mathcal{M}$ by $\mathcal{D}$ is *smooth*, and denote the quotient N-manifold by $\mathcal{M}/\mathcal{D}$.

Proposition 1.30. *Let $A \to M$ be a Lie algebroid. Let $\mathcal{D}$ be an involutive distribution on $\mathcal{M} := A[1]$ such that $[Q, \mathcal{D}] \subset \mathcal{D}$. Let F and ∇ be as in Prop.1.29.*

The quotient $\mathcal{M}/\mathcal{D}$ is a smooth iff M/F is smooth and the connection ∇ has no holonomy. In this case

$$\mathcal{M}/\mathcal{D} \cong \tilde{A}[1]$$

as NQ-manifolds, where $\tilde{A} \to M/F$ is the Lie algebroid obtained as the quotient of A by the ideal system defined in Prop. 1.29.

Proof. We compute $C(\mathcal{M})^{\mathcal{D}}$. In degree zero we simply have $C_0(\mathcal{M})^{\mathcal{D}} = C^\infty(M)^F$. In degree 1 we have that the subset of $C_1(\mathcal{M})$ annihilated by $\mathcal{D}_{-1}$ is exactly $\Gamma(B^\circ)$. Now consider the F-connection ∇^* on $B^\circ = (A/B)^*$ dual to ∇, defined by

$$\langle \nabla^* \xi, V\rangle = -\langle \xi, \nabla V\rangle + d\langle \xi, V\rangle$$

for sections ξ of B° and V of A/B. It is given by $\nabla^*_{\underline{X_0}}\xi = X_0(\xi)$ where $X_0 \in \mathcal{D}_0$ and where we identify $\Gamma(A^*) = C_1(\mathcal{M})$ as in Lemma 1.6. Hence we conclude that

$$\begin{aligned} C_1(\mathcal{M})^{\mathcal{D}} =& \{\xi \in \Gamma(B^\circ) : \nabla^*\xi = 0\} \\ =& \{\xi \in \Gamma(B^\circ) : \langle \xi, \Gamma_\nabla(A/B)\rangle \subset C^\infty(M)^F\} \end{aligned}$$

where $\Gamma_\nabla(A/B)$ denotes the space of ∇-parallel sections of A/B.

Now suppose that M/F is smooth and ∇ has no holonomy. On one hand, this assures that the quotient of A by the ideal system defined in Prop. 1.29 is a Lie algebroid $\tilde{A} \to M/F$, in particular a smooth vector bundle. On the other hand this implies that the technical conditions i) and ii) of [8, Lemma 5.12] are satisfied and hence eq. (21) gives a sheaf of graded commutative algebras generated by its elements in degrees 0 and 1. Since by the above $C_0(\mathcal{M}) \cong C^\infty(M/F)$ and $C_1(\mathcal{M})^{\mathcal{D}} \cong \Gamma(\tilde{A}^*)$, this implies that the sheaf given by eq. (21) corresponds to the N-manifold $\tilde{A}[1]$. Hence $\mathcal{M}/\mathcal{D} \cong \tilde{A}[1]$ as N-manifolds.

Conversely, if $C_0(\mathcal{M})^{\mathcal{D}}$ is isomorphic to the functions on a smooth manifold and $C_1(\mathcal{M})^{\mathcal{D}}$ to sections of a the dual of a smooth vector bundle, then necessarily M/F is smooth and ∇ has no holonomy.

The assumption $[Q, \mathcal{D}] \subset \mathcal{D}$ implies that Q preserves $C(\mathcal{M})^{\mathcal{D}}$: if f is $\mathcal{D}$-invariant, then $Q(f)$ is also $\mathcal{D}$-invariant, because for all $X \in \mathcal{D}$ we have

$$X(Q(f)) = \pm Q(X(f)) + [X, Q](f) = 0.$$

Hence, by restricting the action of Q to $C(\mathcal{M})^{\mathcal{D}} \cong C(\tilde{A}[1])$, we obtain a homological vector field $\tilde{Q}$ on $\tilde{A}[1]$. By construction the inclusion $C(\mathcal{M})^{\mathcal{D}} \to C(\mathcal{M})$ respects the action of the homological vector fields, so that the quotient Lie algebroid structure on $\tilde{A}$ obtained via the derived bracket construction using $\tilde{Q}$ has the property that the projection $A \to \tilde{A}$ is a Lie algebroid morphism. Hence it agrees with the Lie algebroid structure obtained by the ideal system. □

We present an example where $\mathcal{D}$ is singular and the quotient is nevertheless a smooth NQ-manifold (even though not concentrated in degrees 0 and 1).

Example 1.31. [Singular quotient] Let A be the Lie algebra $\mathfrak{su}(2, \mathbb{R})$, so that in a suitable basis we have $[e_1, e_2] = e_3$, $[e_2, e_3] = e_1$,$[e_3, e_1] = e_2$. Denote by ξ_i the coordinates in $A[1]$ dual to the basis e_i. The homological vector field on $A[1]$ is $Q = \xi_2\xi_1\frac{\partial}{\partial \xi_3} + \xi_1\xi_3\frac{\partial}{\partial \xi_2} + \xi_3\xi_2\frac{\partial}{\partial \xi_1}$.

On $A[1]$ consider the span $\mathcal{D}$ of $\frac{\partial}{\partial \xi_1}$ and $[Q, \frac{\partial}{\partial \xi_1}] = -\xi_3\frac{\partial}{\partial \xi_2} + \xi_2\frac{\partial}{\partial \xi_3}$. It is *not* a distribution (compare also to Ex. 1.28). The set of invariant functions $C(A[1])^{\mathcal{D}}$ is $\{a + b\xi_2\xi_3 : a, b \in \mathbb{R}\}$, so it is isomorphic to the functions on $\mathbb{R}[2]$ (with vanishing homological vector field).

2 Integration to global actions

The purpose of this section is to integrate the infinitesimal actions introduced in §1. We do so using the framework of Mackenzie's doubles, which we review in §2.1. In §2.2 we display the objects integrating strict Lie 2-algebras, and in §2.3 – the heart of this paper – we integrate the corresponding strict actions.

Recall that one can differentiate a Lie groupoid $G_1 \rightrightarrows G_0$ to obtain its Lie algebroid. This defines a functor from the category of Lie groupoids to the category of Lie algebroids, called Lie functor. We refer to the inverse process as "integration".

2.1 Background on Mackenzie's doubles

In this subsection we recall the formalism of Mackenzie doubles and its extension by Mehta. We will use it in §2.2 and §2.3 to integrate strict Lie 2-algebras and their actions.

Recall that one can apply the Lie functor to any Lie groupoid to obtain its Lie algebroid. Further, given a Lie algebroid, applying the degree shifting functor [1] one obtains an NQ-1 manifold (Lemma 1.14). The formalism of Mackenzie doubles relates *double Lie groupoids* (see Def. 2.2) to three other structures obtained applying (horizontally or vertically) the Lie functor. This was extended by Mehta [19] who applied (horizontally or vertically) the degree shifting functor [1] to obtain NQ-manifolds with additional structures. The situation is summarized in the following diagram taken from [19]:

$$\begin{tikzcd}
\text{Double Lie groupoids} \arrow[r, "\mathrm{Lie}_H"] \arrow[d, "\mathrm{Lie}_V"] & \mathcal{LA}\text{-groupoids} \arrow[r, "{[1]}"] \arrow[d, "\mathrm{Lie}"] & Q\text{-groupoids} \arrow[d, "\mathrm{Lie}"] \\
\mathcal{LA}\text{-groupoids} \arrow[r, "\mathrm{Lie}"] \arrow[d, "{[1]}"] & \text{Double Lie algebroids} \arrow[r, "{[1]_H}"] \arrow[d, "{[1]_V}"] & Q\text{-algebroids} \arrow[d, "{[1]}"] \\
Q\text{-groupoids} \arrow[r, "\mathrm{Lie}"] & Q\text{-algebroids} \arrow[r, "{[1]}"] & \text{Double } Q\text{-manifolds}
\end{tikzcd} \tag{22}$$

Remark 2.1. In general it is not known whether the Lie functors appearing in the above diagram can be inverted. For instance, given a double Lie algebroid whose vertical Lie algebroids are integrable to Lie groupoids, it is not known if the integrating Lie groupoids form an $\mathcal{LA}$-groupoid. The following question is also open: does a $\mathcal{LA}$-groupoid for which the Lie algebroid structures are integrable arise from a double groupoid? Partial answers to this problem were worked out in [27].

We define the objects appearing in diagram (22) which are relevant to us. We point out that Mehta [19] works entirely in the category of graded manifolds, whereas we want to assume that the double Lie groupoids appearing in (22) consist of ordinary manifolds. This explain why our definitions below are more restrictive than those of [19].

Definition 2.2. Let **StrLgd** be the category of Lie groupoids with strict morphisms[7]. A *double Lie groupoid* is a groupoid object in **StrLgd**. A *strict Lie 2-group* is a group object in **StrLgd**. A *strict Lie 2-group action* is a group action in **StrLgd**.

[7]That is, usual Lie groupoid morphisms. We will not make use of the notion of generalized morphism, i.e. Hilsum-Skandalis bimodule.

Definition 2.3. Let **LA** be the category of Lie algebroid and Lie algebroid morphisms. A *$\mathcal{LA}$-groupoid* is a groupoid object in **LA**. An *$\mathcal{LA}$-group* is a group object in **LA**. An *$\mathcal{LA}$-group action* on a Lie algebroid is a group action in **LA**.

In plain English, an $\mathcal{LA}$-group is a Lie algebroid C endowed with an additional group structure. For example, there is a multiplication

$$m : C \times C \to C$$

which is a Lie algebroid morphism and satisfies the (strict) associativity diagram. There are also an identity morphism and an inverse morphism

$$e : pt \to C, \quad i : C \to C,$$

which satisfy the group axioms. If C is a Lie algebroid over N, then these axioms say exactly that both C and N are Lie groups, and $C \to N$ is a group morphism, i.e. N has an induced group structure from C.

Remark 2.4. A Q-groupoid is a groupoid object in the category of NQ-1 manifolds. We will not make use of this notion. By the correspondence between Lie algebroids and NQ-1 manifolds (see Lemma 1.14) it is clear that $\mathcal{LA}$-group(oid)s correspond to Q-group(oid)s.

Definition 2.5. A *double Lie algebroid* [16] is a double vector bundle

$$\begin{tikzcd} D \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & M \end{tikzcd} \tag{23}$$

such that both vertical sides and both horizontal sides are Lie algebroids, subject to certain compatibility conditions (see for instance [30, §1]). By [30, Thm. 1], the compatibility conditions are equivalent to the following condition. If we apply the [1]-functor to the vertical sides, to obtain a vector bundle of graded manifolds $D[1]_A \to B[1]$, and then we apply again the [1]-functor to it, the resulting degree 2 graded manifold $(D[1]_A)[1]$ will be endowed with homological vector fields encoding the Lie algebroid structures on $D \to A$ and $D \to B$; the condition is that these two vector fields commute.

Definition 2.6. A *Q-algebroid* [20, Def. 4.22] is a **N1**-algebroid[8] [20, Def. 4.1] $\mathcal{A} \to \mathcal{M}$ with a homological vector field Q which is *morphic* [20, Def. 4.14]. A Q-algebroid for which $\mathcal{M}$ is a point is called a *Q-algebra*.

Q-algebras are exactly the same thing as strict Lie 2-algebras, as we will show in Lemma 2.9.

Definition 2.6 needs some explanation. An **N1**-vector bundle[9] [20, Def. 2.1] $\mathcal{E} \to \mathcal{M}$ consists of two degree 1 N-manifolds and a surjection between them, subject to a trivialization condition. A section [20, Def. 2.4] is just a map of N-manifolds $s\colon \mathcal{M} \to \mathcal{E}$ (not

[8]Mehta refers to it as "superalgebroid" and allows for $\mathbb{Z}$-graded manifolds.

[9]Mehta [20] refers to it as "super vector bundle" and allows for arbitrary $\mathbb{Z}$-graded manifolds. In our notation, the prefix "**N1**" refers to the category of degree 1 N-manifolds.

necessarily degree-preserving) which composed with the projection equals $Id_{\mathcal{M}}$. A **N1**-vector bundle $\mathcal{E} \to \mathcal{M}$, endowed with a (degree preserving) morphism from $\Gamma(\mathcal{E}) \to \chi(\mathcal{M})$ and bracket $\Gamma(\mathcal{E}) \times \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ satisfying the graded Leibniz rule and Jacobi identity, forms a **N1**-algebroid [20, Def. 4.1]. An example [19, Ex. 2.4.4] is given by the action algebroid induced from a morphism of graded Lie algebras into $\chi(\mathcal{M})$, for $\mathcal{M}$ a degree 1 N-manifold. Graded Lie algebras concentrated in degrees -1 and 0 are exactly the **N1**-algebroids for which the base $\mathcal{M}$ is a point.

A vector field on a **N1**-algebroid $\mathcal{A} \to \mathcal{M}$ is *morphic* [20, Def. 4.14] if it is linear in on the fibers (in the sense that its action on a function linear on the fibers is linear again) and, viewed as a vector field on $\mathcal{A}[1]$, it commutes with $Q_{\mathcal{A}}$. Here $Q_{\mathcal{A}}$ is the homological vector field on $\mathcal{A}[1]$ which, by virtue of [20, Thm. 4.6], encodes the **N1**-algebroid structure on $\mathcal{A} \to \mathcal{M}$.

2.2 Integrating strict Lie 2-algebras: $\mathcal{LA}$-groups and 2-groups

In this subsection we describe the objects that integrate a strict Lie-2 algebra (Def. 1.17), namely strict 2-groups (Def. 2.2) and $\mathcal{LA}$-groups (Def. 2.3). The idea is

- §2.2.1: show that strict Lie-2 algebras fit in the framework of diagram (22)
- §2.2.2: chase back in diagram (22) to obtain the integrating objects.

2.2.1 What is an integration of a strict Lie 2-algebra?

A classical point of view is the following: strict Lie 2-algebras correspond bijectively to crossed modules of Lie algebras (Lemma 1.22), which integrate to crossed modules of Lie groups, which in turn correspond bijectively to strict 2-groups [3]. Therefore the latter are regarded as the integration of the strict Lie 2-algebra.

We will adopt a different point of view, which fits into the formalism of Mackenzie doubles described in §2.1. The outcome will be the same, but this point of view has the advantage of providing a hint for what the right notion of integration of a strict action of a strict Lie 2-algebra should be (see Def. 1.19).

More precisely, in this subsection we show that strict Lie 2-algebras can be viewed as a Q-algebroids[10] (Lemma 2.9). We then *define* their integration to be a double groupoid or $\mathcal{LA}$-groupoid which, upon applying the relevant functors in the diagram (22), give back the original Q-algebroid. We will identify the integrations in the next subsection by "chasing back" in diagram (22).

Remark 2.7. As mentioned above, a strict Lie 2-algebra can be viewed as a Q-algebroid, or equivalently as a double Lie algebroid, and the integration we perform in §2.2.2 is its integration to a strict Lie 2-group.

Another instance of Lie 2-algebroid that fits into the framework of double Lie algebroids is given by a Courant algebroid $A \oplus A^*$ arising from a Lie bialgebroid (A, A^*). In this case, however, the integration of the corresponding double Lie algebroid via diagram (22) does not coincide with the integration of the Courant algebroid $A \oplus A^*$. The relation between the two integrations is not yet clear to us.

[10] Equivalently we could view strict Lie 2-algebras as double Q-manifolds or as double Lie algebroids.

We now describe how to view certain kinds of L_∞-algebras – among them strict Lie 2-algebras – as Q-algebroids. Given a finite dimensional $\mathbb{Z}$-graded vector space L, there is a bijection between L_∞-algebra structures on L (see §1.2) and homological vector fields Q on $L[1]$ (see Def. 1.9). We recall the procedure to recover the L_∞-algebra structure on L from the corresponding homological vector field on $L[1]$; it is a special case of Voronov's *higher derived brackets* construction [29, Ex. 4.1]. Let Q be a degree 1 vector field on $L[1]$. The higher derived brackets induced by Q are the maps

$$\{\dots\}_n \colon S^n(L[1]) \to L[1], \;\; (v_1, \dots, v_n) \mapsto [[[Q, \iota_{v_1}], \dots], \iota_{v_n}]|_0,$$

where a vector $v \in L[1]$ is also viewed as the constant vector field ι_v on $L[1]$, which acts on linear functions by contraction with v, and "$|_0$" denotes the evaluation at the origin in $L[1]$.

Composing with

$$(\wedge^n L)[n] \cong S^n(L[1]), \;\; v_1 \dots v_n \mapsto v_1 \dots v_n \cdot (-1)^{|v_{n-1}|+2|v_{n-2}|+\dots+(n-1)|v_1|}, \tag{24}$$

where $|v|$ denotes the degree of $v \in L$, we can view the higher derived brackets as maps $[\dots]_n \colon \wedge^n L \to L$. By [29, Thm. 1], Q satisfies $[Q, Q] = 0$ iff the $[\dots]_n$ make L into an L_∞-algebra.

Remark 2.8. To make things more explicit, we express the above construction in coordinates, for the case that the L_∞-structure on L induced by Q is a DGLA. If $\{e_i\}$ is a basis of L and ξ^i are the corresponding coordinates on $L[1]$, then Q is at most quadratic:

$$Q = Q_\delta + Q_{br} = \left(\xi^i Q_i^k + \frac{1}{2}\xi^j \xi^i Q_{ij}^k\right) \frac{\partial}{\partial \xi^k} \tag{25}$$

where we denote by Q_δ and Q_{br} respectively the linear and quadratic component of Q. One has[11]

$$\delta e_i = (-1)^{|e_i|} Q_i^k e_k \qquad \text{and} \qquad [e_i, e_j]_L = (-1)^{|e_j|} Q_{ij}^k e_k. \tag{26}$$

Lemma 2.9. *There is a bijection between strict Lie 2-algebras (that is, DGLAs concentrated in degrees -1 and 0) and Q-algebras, given by $(L, [\cdot,\cdot]_L, \delta) \mapsto (L, [\cdot,\cdot]_L, -Q_\delta)$.*

Proof. Recall from Def. 2.6 that a Q-algebra is a graded Lie algebra concentrated in degrees -1 and 0, together with a morphic vector field.

Let $(L, [\cdot,\cdot]_L, \delta)$ be a strict Lie 2-algebra. All three of Q_{br}, Q_δ and $Q_{br} + Q_\delta$ are self-commuting, because by the derived bracket construction they define L_∞-structures on L (namely the graded Lie bracket $[\cdot,\cdot]_L$, the differential δ, and the DGLA structure). In particular Q_δ is a linear homological vector field on $L[1]$. Further

$$[Q_{br} + Q_\delta, Q_{br} + Q_\delta] = [Q_{br}, Q_{br}] + [Q_\delta, Q_\delta] + 2[Q_{br}, Q_\delta]$$

implies that $[Q_{br}, Q_\delta] = 0$, that is, Q_δ is a morphic vector field. Clearly $-Q_\delta$ is also a morphic vector field, so $(L, [\cdot,\cdot]_L, -Q_\delta)$ is a Q-algebra. Reversing the argument we see that all Q-algebras arise this way. $\square$

[11] These are the formulas in [29, §4] after application of the isomorphism (24).

2.2.2 Integrating the strict Lie 2-algebra

Let $\mathfrak{h}[1] \oplus \mathfrak{g}$ be a strict Lie 2-algebra. By Lemma 2.9 we can view it as a Q-algebroid. We now argue that it lies in the image of the functors appearing in the diagram (22), as follows[12]:

$$\begin{tikzcd} (H \rtimes G) \rightrightarrows G \arrow[r, "\mathrm{Lie}_H"] & (\mathfrak{h} \rtimes G) \to G \arrow[d, "\mathrm{Lie}"] & \\ & (\mathfrak{h} \rtimes \mathfrak{g}) \to \mathfrak{g} \arrow[r, "{[1]_H}"] & (\mathfrak{h}[1] \rtimes \mathfrak{g}, Q_\delta). \end{tikzcd} \tag{27}$$

We *define* the integration of the strict Lie 2-algebra $\mathfrak{h}[1] \oplus \mathfrak{g}$ to be the strict Lie 2-group in the upper left corner.

The bottom horizontal sides of the double Lie groupoid, $\mathcal{LA}$-groupoid, etc appearing in diagram (27) are just points, so they are omitted.

We describe the structures appearing in diagram (27), in particular the strict Lie 2-group in the upper left corner. The strict Lie 2-algebra $\mathfrak{h}[1] \oplus \mathfrak{g}$ corresponds to the crossed module of Lie algebras $(\mathfrak{g}, \mathfrak{h}_\delta, \delta, \alpha)$ (Lemma 1.22), where $\mathfrak{h}_\delta$ denotes the Lie algebra structure on the vector space $\mathfrak{h}$ with bracket $[w_1, w_2]_\delta := [\delta w_1, w_2]$ and $\alpha(v) = [v, \cdot]$ for $v \in \mathfrak{g}$. Consider the quadruple[13] $(G, H, \mathbf{t}, \phi)$. Here H and G are the simply connected Lie groups integrating $\mathfrak{h}_\delta$ and $\mathfrak{g}$, the map $\mathbf{t}\colon H \to G$ is the Lie group morphism integrating δ, and the left action ϕ of G on H (by automorphisms of H) is obtained integrating the infinitesimal action (by Lie algebra derivations) α.

- **Strict Lie 2-group** The Lie 2-group[14] $G_\bullet = G_1 \rightrightarrows G_0$ is as follows. Its Lie groupoid structure is the action groupoid of the H action on G via $hg := \mathbf{t}(h) \cdot g$ (so the target of (g, h) is given by $\mathbf{t}(h) \cdot g$ and its source by g). The group structure on $H \times G$ is the semidirect product structure by the action ϕ, i.e., it is given by

  $$m : G_1 \times G_1 \to G_1, \quad m((h_1, g_1), (h_2, g_2)) = (h_1 \cdot \phi(g_1)(h_2), g_1 g_2)$$

  over the base map $m(g_1, g_2) = g_1 g_2$.

- **$\mathcal{LA}$-group** Its Lie algebroid structure is obtained by differentiating the Lie groupoid structure of the strict Lie 2-group $G_\bullet$, hence it is the transformation algebroid of the infinitesimal action of $\mathfrak{h}_\delta$ on G by $w \mapsto \overrightarrow{\delta w}$ (the right-invariant vector field whose value at the identity is δw).

  The group multiplication on $\mathfrak{h} \times G$ is the Lie algebroid morphism corresponding to $m : G_1 \times G_1 \to G_1$, i.e., $(w_1, g_1)(w_2, g_2) = (w_1 + g_1 w_2, g_1 g_2)$. In other words, the group structure on $\mathfrak{h} \times G$ is the semidirect product by the action of G on the vector space $\mathfrak{h}$ obtained integrating α.

[12] the notation is chosen so to describe the *vertical* (Lie group or Lie algebra) structures

[13] This quadruple forms what is known as the *crossed module of Lie groups* integrating the above crossed module of Lie algebras.

[14] This is the Lie 2-group associated to a crossed module of Lie groups [3].

- **Double Lie algebroid** Notice first that applying Lie_V to the strict Lie 2-group above one obtains the Lie groupoid in the category of Lie algebras $(\mathfrak{h}_\delta \rtimes \mathfrak{g}) \rightrightarrows \mathfrak{g}$. The Lie groupoid structure is the transformation groupoid of the action of the abelian Lie algebra $\mathfrak{h}$ on $\mathfrak{g}$ which sends $w \in \mathfrak{h}$ to the translation by δw.

  Differentiating this Lie groupoid structure we obtain the Lie algebroid structure of our double Lie algebroid, which therefore[15] is the transformation algebroid of the infinitesimal action of the abelian Lie algebra $\mathfrak{h}$ on $\mathfrak{g}$ which sends w to the constant vector field δw.

  The Lie algebra structure of our double Lie algebroid is obtained differentiating the Lie group structure of the $\mathcal{LA}$-group, so it is the semidirect product $\mathfrak{h} \rtimes \mathfrak{g}$ of the action α of $\mathfrak{g}$ on the vector space (abelian Lie algebra) $\mathfrak{h}$. Explicitly: $[(w_1, v_1), (w_2, v_2)] = ([v_1, w_2] - [v_2, w_1]\ ,\ [v_1, v_2])$.

- **Q-algebra** Applying the functor $[1]_H$ to the double Lie algebroid we obtain the NQ-manifold $(\mathfrak{h}[1] \oplus \mathfrak{g}, -Q_\delta)$. (To see this, recall that the anchor of a transformation Lie algebroid is given by the corresponding infinitesimal action, and use eq. (8) and eq. (26).) It has the graded Lie algebra structure $\mathfrak{h}[1] \rtimes \mathfrak{g}$.

The latter Q-algebra structure is exactly the one corresponding to our original strict Lie-2 algebra by Lemma 2.9, hence the strict Lie 2-group $(H \rtimes G) \rightrightarrows G$ described above is the integration of the strict Lie-2 algebra.

2.3 Integrating strict actions: $\mathcal{LA}$-group actions on Lie algebroids and 2-group actions on Lie groupoids

This subsection is the heart of the paper: we define the notion of global action integrating a strict action μ, and show that the global action exists. The idea is

- §2.3.1: show that the transformation algebroid of the action μ fits in the framework of diagram (22)
- §2.3.2: chase back in the diagram to obtain certain transformation groupoids, and describe the corresponding actions.

The diagram of Mackenzie's doubles (22), in the setup at hand, is displayed in (39) (it extends our previous diagram (27)). The various actions involved are displayed just before Thm. 2.17.

We end this subsection with remarks on the integrated actions (§2.3.3).

As earlier, we let $\mathfrak{h}[1] \oplus \mathfrak{g}$ be a strict Lie-2 algebra and $A \to M$ a Lie algebroid (so $A[1]$ is a NQ-1 manifold).

---

[15] Here we are using [17, Thm 2.3], which states that starting from a double Lie groupoid and applying the functors $Lie_H \circ Lie_V$ or $Lie_V \circ Lie_H$, one obtains the same double algebroid *up to* a canonical isomorphism. The canonical isomorphism in our case is the identity on $\mathfrak{h} \times \mathfrak{g}$. This follows from a simple computation (see the proof of [17, Thm 2.3]) representing each element $(w, v) \in \mathfrak{h} \times \mathfrak{g}$ as second derivative of the map $\gamma\colon \mathbb{R}^2 \to H \times G, \gamma(t, u) = (exp_{\mathfrak{h}}(tuw), exp_{\mathfrak{g}}(tv))$.

2.3.1 What is an integration of a strict Lie-2 algebra action?

The following proposition, which is an analogue to Lemma 2.9, associates a Q-algebroid to the strict action μ.

Proposition 2.10. *Let $\mathfrak{h}[1] \oplus \mathfrak{g}$ be a strict Lie-2 algebra, A be a Lie algebroid. We consider a* morphism of graded Lie algebras $\mu \colon \mathfrak{h}[1] \oplus \mathfrak{g} \to \chi(A[1])$. *Then μ is a morphism of DGLAs (i.e. it respects differentials) iff the transformation algebroid of the action μ*

$$\begin{array}{c} T_\mu := (\mathfrak{h}[1] \oplus \mathfrak{g}) \times A[1] \\ \big\downarrow \\ A[1], \end{array} \tag{28}$$

endowed with the homological vector field $-Q_\delta + Q_A$, is a Q-algebroid. Recall that Q_δ was defined in Remark 2.8, and Q_A encodes the Lie algebroid structure on A as in Lemma 1.14.

Proof. Denote $L := \mathfrak{h}[1] \oplus \mathfrak{g}$. The vector field $-Q_\delta + Q_A$ is homological and it is linear in the fibers. So, in order to show that it is morphic, we just have to show that, once we view it as a vector field on $L[1] \times A[1]$, it commutes with the homological vector field encoding the Lie algebroid structure of (28). The latter is $Q_{br} + Q_{action}$, where Q_{br} was defined in Remark 2.8 and Q_{action} is defined in eq. (30) below and encodes the action μ.

To show

$$[-Q_\delta + Q_A, Q_{br} + Q_{action}] = 0. \tag{29}$$

we proceed as follows.

Take bases $\{v_i\}$ of $\mathfrak{g}$ and $\{w_\alpha\}$ of $\mathfrak{h}[1]$, whose dual bases induce coordinates η^i on $\mathfrak{g}[1]$ and P^α of $\mathfrak{h}[2]$ (of degrees 1 and 2 respectively).

If the differential on L is given by $\delta w_\alpha = D_{\alpha i} v_i$, then in coordinates

$$Q_\delta = -P^\alpha D_{\alpha k} \frac{\partial}{\partial \eta^k}.$$

Further

$$Q_{action} := \eta^i \mu(v_i) - P^\alpha \mu(w_\alpha) \tag{30}$$

encodes the L-action on $A[1]$

We have $[Q_A, Q_{br}] = 0$, since the two vector fields are defined on different manifolds, and $[-Q_\delta, Q_{br}] = 0$ as shown in Lemma 2.9. Since

$$[Q_A, Q_{action}] = -\eta^i [Q_A, \mu(v_i)] - P^\alpha [Q_A, \mu(w_\alpha)]$$

and

$$[-Q_\delta, Q_{action}] = P^\alpha D_{\alpha k} \mu(v_k) = P^\alpha \mu(\delta w_\alpha)$$

we conclude that (29) holds iff $[Q_A, \mu(w_\alpha)] = \mu(\delta w_\alpha)$ for all w_α and $[Q_A, \mu(v_i)] = 0$ for all v_i, which means that μ respects differentials. $\square$

Remark 2.11. 1) Prop. 2.10 together with diagram (22) imply that

$$\big((\mathfrak{h}[1] \times \mathfrak{g})[1] \times A[1]\ ,\ Q_\delta + Q_A\ ,\ Q_{br} + Q_{action}\big)$$

is a double Q-manifold [30].

When the action μ is not necessarily strict but rather as in Def. 9, the above is no longer a double Q-manifold. However in that case one can show [21] that the sum of the four above vector fields is still a homological vector field. This makes (28) into an "action" Lie 2-algebroid. Thus the integration of an action as in Def. 9 should be encoded by the Lie 2-groupoid integrating this action Lie 2-algebroid.

In the strict case, we will see below that the integration of the action is given by a double Lie groupoid (T_Φ in diagram (39)). Moreover in this case, unlike the case of Courant algebroids (see Remark 2.7), there is a Lie 2-groupoid, obtained applying the Artin-Mazur's codiagonal construction [1] (see Remark 2.20) to T_Φ, which is also to be considered an integration of the action. These two integrating objects are not exactly equivalent: the double Lie groupoid contains more information because the double Lie algebroid contains more information than the Lie 2-algebroid.

2) [20, Thm. 6.2], which is proved without the explicit use of coordinates, is a special case of Prop. 2.10 (namely, the special case we consider in Ex. 3.1).

Now assume that

$$\boxed{\mu\colon \mathfrak{h}[1] \oplus \mathfrak{g} \quad\to\quad \chi(A[1])}$$

is a strict action (Def. 1.19), i.e. a morphism of DGLAs. Prop. 2.10 allows to view the action μ in the framework of diagram (22).

We make the following definitions of integration of μ.

Definition 2.12. The *integration* of the strict action μ above is either of the following actions:

I) An $\mathcal{LA}$-group action Ψ of $(\mathfrak{h} \rtimes G) \to G$ on $A \to M$ such that applying the functor Lie_V to its transformation groupoid one obtains the Q-algebroid (28).

II) A strict Lie 2-group action Φ of $(H \rtimes G) \rightrightarrows G$ on $\Gamma \rightrightarrows M$ such that applying the functors Lie_H and Lie_V to its transformation groupoid one obtains the Q-algebroid (28). (Here $\Gamma \rightrightarrows M$ denotes the source simply connected Lie groupoid integrating A.)

Def. 2.12 requires some explanation. The transformation groupoid appearing in I) is the one corresponding to the group action Ψ. As Ψ is a group action in the category **LA**, the transformation groupoid is a groupoid in **LA**, (an $\mathcal{LA}$-groupoid), and hence fits in diagram (22). This follows taking $\mathcal{C} = \mathbf{LA}$ in [27, Prop. 3.0.15], which we now reproduce. The same reasoning, applied to the category **StrLgd**, holds for II).

Proposition 2.13. *Let $\mathcal{C}$ be a small category with direct products, G a group*[16] *object in $\mathcal{C}$ and N an object in $\mathcal{C}$. Then a group action $G \times N \to N$ is a morphism in $\mathcal{C}$ iff the corresponding transformation groupoid $G \times N \rightrightarrows N$ is a groupoid object in $\mathcal{C}$.*

[16] In [27, Prop. 3.0.15] G is allowed to be a *groupoid* object and $\mathcal{C}$ is additionally required to have fiber products.

The situation is summarized in diagram (39) in §2.3.2, which is the diagram of Mackenzie's doubles (22) for the above transformation groupoids.

Applying the vertical integration functor to a double Lie algebroid whose vertical structures are *transformation* algebroids (such as the one associated to (28) above), one expects to obtain an $\mathcal{LA}$-groupoid whose vertical structures are *transformation* groupoids.

Unfortunately there are no general enough statements about the integration of double algebroids to $\mathcal{LA}$-groupoids or $\mathcal{LA}$-groupoid to double groupoids (see Remark 2.1), so we can not "chase back" in diagram (22), but rather in §2.3.2 we have to check explicitly that the expected fact mentioned above is true for the double algebroid (28).

Moreover our method has the advantage that it provides explicit formulae.

2.3.2 Integrating the strict action

Now we integrate the strict Lie 2-algebra action μ on a Lie algebroid A to both a $\mathcal{LA}$-group action on A and a Lie 2-group action on the source-simply connected Lie groupoid of A.

By virtue of the identification of Remark 1.7, μ induces a Lie algebra morphism

$$\boxed{\tilde{\mu}\colon \mathfrak{h} \rtimes \mathfrak{g} \to \chi(A).}$$

Here $\mathfrak{h} \rtimes \mathfrak{g}$ is the semidirect product of $\mathfrak{g}$ and the abelian Lie algebra $\mathfrak{h}$, and agrees with the Lie algebra structure induced by the original graded Lie algebra structure on $\mathfrak{h}[1] \oplus \mathfrak{g}$. In other words, we have an infinitesimal action of the Lie algebra $\mathfrak{h} \rtimes \mathfrak{g}$ on A. Using this identification, $w \in \mathfrak{h}$ maps to $\mu(w)$ seen as a constant vertical vector field on A and $v \in \mathfrak{g}$ maps to the infinitesimal vector bundle automorphism given by $\mu(v)$.

Assume that the infinitesimal action $\tilde{\mu}|_{\mathfrak{g}}\colon \mathfrak{g} \to \chi(A)$ is *complete* in the sense that the image of $\tilde{\mu}$ consists of complete vector fields.

Then the infinitesimal action $\tilde{\mu}|_{\mathfrak{g}}$ can be integrated to a Lie group action

$$\psi\colon G \times A \to A$$

of G on A, where G denotes the simply connected Lie group integrating $\mathfrak{g}$. Notice that ψ acts by Lie algebroid automorphisms of A, as a consequence of Lemma 1.20.

Proposition 2.14. *Consider the Lie algebroids $A \to M$ and $\mathfrak{h} \rtimes G \to G$ (as in §2.2.2). The Lie group action on A obtained integrating the infinitesimal action $\tilde{\mu}$ is*

$$\boxed{\begin{array}{rcl} \Psi\colon (\mathfrak{h} \rtimes G) \times A & \to & A \\ (w,g), a_x & \mapsto & \psi(g,a_x) + \mu(w)|_{gx} \end{array}}$$

where $x \in M$ and $a_x \in A_x$.

Further Ψ is a Lie algebroid morphism. In other words, Ψ is an $\mathcal{LA}$-group action.

Proof. Checking that Ψ is really a group action is a straight-forward computation that uses

$$\psi(g,(\mu(w))_x) = (\mu(gw))_{gx} \quad \text{for all } g \in G, w \in \mathfrak{h}, x \in M,$$

which is just the equivariance of $\mu|_{\mathfrak{h}}\colon \mathfrak{h} \to \chi_{-1}(A) = \Gamma(A)$ with respect to the G actions on $\mathfrak{h}$ and on the vector bundle A. The infinitesimal $\mathfrak{g}$-equivariance holds by Lemma 1.20, and as G is connected the global G-equivariance also holds. A simple computation also shows that differentiating the group action Ψ one obtains $\tilde{\mu}$. This proves the first part of the proposition.

Now we check that Ψ preserves the anchor maps. Fix $(w, g) \in \mathfrak{h} \rtimes G$ and $a_x \in A_x$ (the fiber of A over $x \in M$). Applying the anchor to $((w, g), a_x)$ we obtain $(\overrightarrow{\delta w}(g), \rho_A(a_x))$, and applying the derivative of the action map $G \times M \to M$ gives

$$(\delta w)_M(gx) + g \cdot \rho_A(a_x), \tag{31}$$

where $(\delta w)_M$ denotes the vector field on M induced by the infinitesimal action of $\delta w \in \mathfrak{g}$ on M, the dot denotes tangent lift of the action of G on M, and ρ_A is the anchor of A. Now

$$(\delta w)_M = (\mu(\delta w))|_M = [Q, \mu w]|_M = \rho_A(\mu w),$$

where the second equality (between vector fields on $A[1]$) holds because μ respects differentials, or alternatively by Lemma 1.20. We saw that the G-action ψ on A is by Lie algebroid automorphism, so in particular $\rho_A\colon A \to TM$ is G-equivariant and $g \cdot \rho_A(a_x) = \rho_A(g \cdot a_x)$. Hence (31) is equal to $\rho_A(\Psi((w, g), a_x))$, proving that Ψ respects the anchor maps.

Checking that Ψ maps the bracket $[\cdot,\cdot]_E$ on the product Lie algebroid $E := (\mathfrak{h} \rtimes G) \times A$ to the bracket $[\cdot,\cdot]_A$ on A is more involved. First we remark that E, as a vector bundle over $G \times M$, is a Whitney sum of pullback vector bundles $\pi_G^*(\mathfrak{h} \rtimes G) \oplus \pi_M^* A$, where π_G and π_M are the obvious projections of $G \times M$ onto G and M. We define the vector bundle automorphism

$$\varphi\colon \pi_M^* A \to \pi_M^* A\ ,\ a_{(g,x)} \mapsto g^{-1}(a_{(g,x)}) \tag{32}$$

over the base diffeomorphism $(g, x) \mapsto (g, g^{-1}x)$ of $G \times M$. It is actually a Lie algebroid automorphism, since each $g \in G$ acts by Lie algebroid automorphisms of A. Notice that any $a \in \Gamma(A)$ can be pulled back to a section of $\pi_M^* A \subset E$ (also denoted by a), and its image $\varphi(a)$ under φ is given by

$$(\varphi(a))_{(g,x)} = g^{-1} \cdot a_{gx} \tag{33}$$

The section $\varphi(a) \in \Gamma(E)$ is Ψ-projectable, and projects to $a \in \Gamma(A)$. Similarly, for any $w \in \mathfrak{h}$, the constant section $w \in \Gamma(\pi_G^*(\mathfrak{h} \times G)) \subset E$ projects to $\mu(w) \in \Gamma(A)$. Sections of the form $\varphi(a)$ and w span the whole of E, hence, by [18, Prop. 4.3.8], it suffices to consider such sections. We have

$$\Psi[\varphi(a_1), \varphi(a_2)]_E = \Psi(\varphi[a_1, a_2]_A) = [a_1, a_2]_A = [\Psi\varphi a_1, \Psi\varphi a_2]_A$$

where in the first equality we used that φ is a Lie algebroid automorphisms of $\pi_M^* A$. We have

$$\Psi[w_1, w_2]_E = \Psi([w_1, w_2]_\delta) = \mu([w_1, w_2]_\delta) = [\mu w_1, \mu w_2]_A = [\Psi w_1, \Psi w_2]_A$$

where in the third equality we used Lemma 1.20.

Next we show that

$$[w, \varphi(a)]_E = \varphi([\mu(w), a]_A) \text{ for all } a \in \Gamma(A), w \in \mathfrak{h}, \tag{34}$$

as it will imply that

$$\Psi[\varphi(a), w]_E = \Psi\varphi([a, \mu(w)]_A) = [a, \mu(w)]_A = [\Psi\varphi(a), \Psi w]_A$$

and thus conclude our proof.

To show (34) we choose, on an open set U of M, a local frame of sections a_i of $\Gamma(A)$. We have

$$(\varphi a_i)_{(g,x)} = f_i^j(g,x)a_j(x) \tag{35}$$

for functions f_i^j defined on $G \times U$. Here we use the Einstein summation convention. By the Leibniz rule we can write the l.h.s. of eq. (34) as

$$([w, \varphi(a_i)]_E)_{(g,x)} = \overrightarrow{\delta w}(f_i^k(g,x))a_k(x), \tag{36}$$

where $\overrightarrow{\delta w}$ denotes the right-invariant vector field on G whose value at the identity is δw.

Notice that this lies in $\pi_M^* A \subset E$. Further, using the identification $\Gamma(A) = \chi_{-1}(A[1])$, we have

$$\begin{aligned}([\mu w, a_i]_A)_x =& [[Q_A, \mu w], a_i]_x = [\mu(\delta w), a_i]_x = (\mathcal{L}_{\mu(\delta w)} a_i)_x = \frac{d}{dt}|_0 exp(-t\delta w) \cdot (a_i)_{exp(t\delta w)\cdot x} \\ \overset{(33)}{=}& \frac{d}{dt}|_0 (\varphi a_i)_{(exp(t\delta w),x)} = \frac{d}{dt}|_0 f_i^j(exp(t\delta w), x)a_j(x).\end{aligned} \tag{37}$$

We deduce that

$$\begin{aligned}(\varphi[\mu(w), a_i]_A)_{(g,x)} &\overset{(33)}{=} g^{-1} \cdot ([\mu(w), a_i]_A)_{gx} \\ &\overset{(37)}{=} \frac{d}{dt}|_0 f_i^j(exp(t\delta w), gx) \cdot g^{-1}a_j(gx) \\ &\overset{(33)}{=} \frac{d}{dt}|_0 f_i^j(exp(t\delta w), gx) \cdot (\varphi a_j)_{(g,x)} \\ &= \frac{d}{dt}|_0 f_i^j(exp(t\delta w), gx) \cdot f_j^k(g,x)a_k(x) \\ &= \frac{d}{dt}|_0 f_i^k(exp(t\delta w)g, x)a_k(x) \\ &= \overrightarrow{\delta w}(f_i^k(g,x))a_k(x) \\ &\overset{(36)}{=} ([w, \varphi(a_i)]_E)_{(g,x)}.\end{aligned}$$

Here in the third last equality we used the "multiplicativity formula"

$$f_i^j(gh, x) = f_i^k(g, hx)f_k^j(h, x) \tag{38}$$

which can be checked writing out $(\varphi a_i)_{(gh,x)}$ by means of eq. (33). Hence eq. (34) is proved and we are done. □

Remark 2.15. It can be checked that the Lie algebroid automorphism φ of $\pi_M^* A$ in (32) can be extended to a Lie algebroid automorphism of E by asking that it maps the constant section w to $w - \varphi(\mu w) \in \Gamma(E)$ for all $w \in \mathfrak{h}$.

Integrating the Lie algebroid $\mathfrak{h} \rtimes G \to G$ we obtain the strict Lie 2-group $(H \rtimes G) \rightrightarrows G$ described in §2.2.2, where H is simply connected. Assume that the Lie algebroid $A \to M$ is integrable to a source simply connected Lie groupoid Γ.

Theorem 2.16. *Consider the Lie groupoids $\Gamma \rightrightarrows M$ and the Lie 2-group $(H \rtimes G) \rightrightarrows G$. The Lie groupoid morphism*

$$\boxed{\Phi\colon (H \rtimes G) \times \Gamma \quad \to \quad \Gamma}$$

integrating the Lie algebroid morphism Ψ is a Lie 2-group action.

In Prop. 2.18 we will give a description of the action Φ.

Proof. Notice that Φ is a well-defined Lie groupoid morphism as its domain is source simply connected.

We show that Φ is a group action. Recall from §2.2.2 that the group multiplication $m\colon (H \rtimes G) \times (H \rtimes G) \to (H \rtimes G)$ is the Lie groupoid morphism which integrates the multiplication $\tilde{m}$ on $\mathfrak{h} \rtimes G$. Hence integrating both sides of the equality of Lie algebroid morphisms

$$\Psi \circ (\tilde{m} \times Id_A) = \Psi \circ (Id_{\mathfrak{h} \rtimes G} \times \Psi),$$

which holds because Ψ is a group action, one obtains

$$\Phi \circ (m \times Id_\Gamma) = \Phi \circ (Id_{H \rtimes G} \times \Phi).$$

This shows that Φ is a group action both at the level of objects and of morphisms, hence it is a Lie 2-group action. $\square$

The various actions appearing in this subsection can are summarized in the following diagram:

$$\boxed{\begin{array}{rc} & \mu\colon \mathfrak{h}[1] \oplus \mathfrak{g} \to \chi(A[1]) \\ \rightsquigarrow & \tilde{\mu}\colon \mathfrak{h} \rtimes \mathfrak{g} \to \chi(A) \\ \underset{\rightsquigarrow}{\text{Integrate the Lie algebra action}} & \Psi\colon (\mathfrak{h} \rtimes G) \times A \to A \\ \underset{\rightsquigarrow}{\text{Integrate the Lie algebroid morphism}} & \Phi\colon (H \rtimes G) \times \Gamma \to \Gamma \end{array}}$$

Theorem 2.17. *The actions Φ and Ψ defined above are really integrations of μ in the sense of Def. 2.12 I) and II) respectively.*

Proof. We consider the double Lie groupoid T_Φ obtained by taking the transformation groupoid of the action Φ and the $\mathcal{LA}$-groupoid T_Ψ obtained by taking the transformation groupoid of the action Ψ. Notice that T_Φ is really a double Lie groupoid by Prop. 2.13 together with Thm. 2.16, and T_Ψ is really an $\mathcal{LA}$-groupoid because of Prop. 2.13 together with Prop. 2.14.

Further we consider the double algebroid $T_{\tilde{\mu}}$ consisting of the transformation algebroid of the action $\tilde{\mu}\colon \mathfrak{h} \rtimes \mathfrak{g} \to \chi(A)$ and of the $\mathfrak{g}$ action on M obtained by restricting $\tilde{\mu}$ (vertically), together with $A \to M$ and the transformation algebroid of the action of the abelian Lie algebra $\mathfrak{h}$ on $\mathfrak{g}$ which sends $w \in \mathfrak{h}$ to the constant vector field δw (vertically):

$$
\begin{tikzcd}
T_{\tilde{\mu}} = (\mathfrak{h} \rtimes \mathfrak{g}) \times A \arrow[r] \arrow[d] & \mathfrak{g} \times M \arrow[d] \\
A \arrow[r] & M.
\end{tikzcd}
$$

Applying to $T_{\tilde{\mu}}$ the horizontal degree shifting functor $[1]_H$ we obtain (28) together with the homological vector field $-Q_\delta + Q_A$ (see §2.2.2). Since by Prop. 2.10 it is a Q-algebroid, from Voronov's work ([30, Thm. 1], see also [30, §3]) it follows in particular that $T_{\tilde{\mu}}$ is a really a double Lie algebroid.

T_Φ, T_Ψ and $T_{\tilde{\mu}}$ fit into the upper row of the following table.

$$
\begin{tikzcd}
T_\Phi = (H \rtimes G) \times \Gamma \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & G \times M \arrow[d, shift left] \arrow[d, shift right] \arrow[r, "\mathrm{Lie}_H"] & T_\Psi = (\mathfrak{h} \rtimes G) \times A \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & G \times M \arrow[d, shift left] \arrow[d, shift right] & \\
\Gamma \arrow[r, shift left] \arrow[r, shift right] & M & A \arrow[r] \arrow[d, "\mathrm{Lie}_V"] & M & \\
 & & T_{\tilde{\mu}} = (\mathfrak{h} \rtimes \mathfrak{g}) \times A \arrow[r] \arrow[d] & \mathfrak{g} \times M \arrow[d] \arrow[r, "{[1]_H}"] & T_\mu = (\mathfrak{h}[1] \rtimes \mathfrak{g}) \times A[1] \arrow[d] \\
 & & A \arrow[r] & M & A[1]
\end{tikzcd}
\tag{39}
$$

We check that applying the functor Lie_H to T_Φ we obtain T_Ψ. Clearly the horizontal Lie algebroid structures obtained this way are $A \to M$ and its product with the Lie algebroid structure of the $\mathcal{LA}$-group $\mathfrak{h} \rtimes G \to G$ (see §2.2.2). The left vertical Lie groupoid structure is obtained applying Lie_H to the maps defining the vertical Lie algebroid structures in T_Φ. It is the transformation groupoid for the action Ψ, since the Lie algebroid morphism Ψ is obtained differentiating the Lie groupoid morphism Φ.

Next we check that applying the functor Lie_V to T_Ψ we obtain $T_{\tilde{\mu}}$. Since the vertical groupoids of T_Ψ are transformation groupoids for group actions (of $\mathfrak{h} \rtimes G$ and G respectively), applying the functor Lie_V we obtain the transformation algebroids of the corresponding infinitesimal actions, which by Prop. 2.14 are $\tilde{\mu}$ and the $\mathfrak{g}$-action on M respectively. The horizontal Lie algebroids in T_Ψ are the Lie algebroid $A \to M$ and its product with the Lie algebroid structure on the $\mathcal{LA}$-group $\mathfrak{h} \rtimes G \to G$. Since the application of the Lie functor to the vertical groupoids of T_Ψ does not affect their spaces of units (A and M respectively), as horizontal Lie algebroid structures we obtain again $A \to M$ and its product with the Lie algebroid structure of the double Lie algebroid $\mathfrak{h} \rtimes \mathfrak{g} \to \mathfrak{g}$ (see §2.2.2). Altogether we obtain exactly $T_{\tilde{\mu}}$.

Finally, we saw in the first part of this proof that applying to $T_{\tilde{\mu}}$ the horizontal degree shifting functor $[1]_H$ we obtain the Q-algebroid T_μ given in (28).

From this we conclude that Ψ and Φ are integrations of the action μ in the sense of Def. 2.12 I) and II) respectively. $\square$

It is abstract to define the Lie 2-group action Φ in Thm. 2.16 as the integration of the strict action μ we started with. Now we give an explicit description of Φ.

Proposition 2.18. *The Lie 2-group action $\Phi\colon (H \rtimes G) \times \Gamma \to \Gamma$ can be described in terms of $\tilde{\mu}$ as follows:*

a) $g \in G$ acts by the Lie groupoid automorphism of Γ which integrates the Lie algebroid automorphism $\psi(g, \cdot)$ of A, where ψ denotes the Lie group action of G on A obtained by integrating the Lie algebra action $\tilde{\mu}|_{\mathfrak{g}}\colon \mathfrak{g} \to \chi(A)$.

b) the Lie group action of H on Γ is obtained integrating the Lie algebra action

$$\mathfrak{h}_\delta \to \chi^{right}(\Gamma), w \mapsto \overrightarrow{\tilde{\mu}(w)},$$

where for any section s of $A \cong (ker\ \mathbf{s}_*)|_M$ we denote by $\overrightarrow{s}$ its extension as a right invariant vector field on Γ.

Since a general element of $H \rtimes G$ can be written as $(h, g) = (h, e)(e, g)$, this gives a complete description of the action Φ.

Proof. a) Fix $g \in G$. Consider $\Phi((e, g), \cdot)\colon \Gamma \to \Gamma$. Its derivative restricted to $(ker\ \mathbf{s}_*)|_M \cong A$ is $\Psi((0, g), \cdot) = \psi(g, \cdot)\colon A \to A$, since Φ is the Lie groupoid morphism integrating the Lie algebroid morphism Ψ.

b) Let $w \in \mathfrak{h}$, denote by W the corresponding vector field on Γ induced by the action Φ, and let $h(t) := exp_{\mathfrak{h}_\delta}(tw) \in H$. For any $m \in M$ we have

$$W_m = \frac{d}{dt}|_0 \Phi((h(t), e), m) = \Psi((w, e), 0_m) = \tilde{\mu}(w)_m,$$

so $W|_M = \tilde{\mu}(w)$. We want to show that W is a right-invariant vector field, i.e., that if $x, y \in \Gamma$ are composable elements then $(R_y)_* W_x = W_{x \circ y}$, where $\circ$ denotes the groupoid composition on Γ. Since Φ is a Lie groupoid morphism (Thm. 2.16) and the elements $(h(t), e)$ and (e, e) of the groupoid $H \rtimes G$ are composable, we have

$$\Phi(((h(t), e), x)) \circ \Phi((e, e), y) = \Phi((h(t), e), x \circ y),$$

and since $\Phi((e, e), y) = y$ applying the time derivative at $t = 0$ we obtain exactly $(R_y)_* W_x = W_{x \circ y}$. This shows that $W = \overrightarrow{\tilde{\mu}(w)}$ and concludes the proof. $\square$

Remark 2.19. Prop. 2.18 allows also to describe the Lie algebra action on Γ corresponding to the Lie group action Φ. It maps $v \in \mathfrak{g}$ to the multiplicative vector field on Γ which corresponds to the vector field $\tilde{\mu}(v)$ on A. It maps $w \in \mathfrak{h}$ to $\overrightarrow{\tilde{\mu}(w)}$. This shows that the action Φ coincides with the action constructed in [8, §11.2] in the special case where A is the Lie algebroid of a Poisson manifold. Notice that the action of [8, §11.2] was constructed "integrating" $T_{\tilde{\mu}}$ (see diagram in the proof of Thm. 2.17) first horizontally and then vertically, while we constructed Φ "integrating" $T_{\tilde{\mu}}$ in the opposite order.

2.3.3 Lie 2-groupoids and principality of the integrated action

First, we make a remark on the integration of the Lie 2-algebroid (28). This is the point of view that should be generalized to integrate non-strict actions.

Remark 2.20. We describe the strict Lie 2-groupoid integrating the Lie 2-algebroid (28), obtained by applying the Artin-Mazur's codiagonal construction to the double groupoid T_Φ appearing in (39). A strict Lie 2-groupoid is a groupoid object in the category of Lie groupoids whose space of objects (object-groupoid) is just a manifold. Our strict Lie 2-groupoid has as space of objects M (the base of A), as object space of the morphism-groupoid (space of 1-arrows) $G\times\Gamma$, and as morphism space of the morphism-groupoid (space of 2-arrows) $H\times G\times\Gamma$. The space of 2-arrows is a Lie groupoid over the space of 1-arrows with source and target defined by

$$\mathbf{s}(h,g,\gamma)=(g,\gamma),\quad \mathbf{t}(h,g,\gamma)=(\delta(h)g,\Phi(\phi(g^{-1})(h^{-1}),1,\gamma)),$$

and groupoid multiplication $\cdot_v$ (vertical product) defined by

$$(h,g,\gamma)\cdot_v(h',g',\gamma')=(hh',g',\gamma\gamma').$$

$$\begin{tikzcd}
y & & x \arrow[ll, bend right=40, "{(g,\gamma)}"'{name=U}] \arrow[ll, "{(g',\gamma')}"{name=M, description}] \arrow[ll, bend left=40, "{(\delta(h')g',\Phi(\phi(g'^{-1})(h'^{-1}),1,\gamma'))}"{name=D}]
\arrow[from=U, to=M, phantom, "\Downarrow (h,g,\gamma)=:a"]
\arrow[from=M, to=D, phantom, "\Downarrow (h',g',\gamma')=:a'"]
\end{tikzcd}
\longmapsto
\begin{tikzcd}
y & & x \arrow[ll, bend right=30, "{(g,\gamma)}"'{name=U}] \arrow[ll, bend left=30, "{(\delta(h')g',\Phi(\phi(g'^{-1})(h'^{-1}),1,\gamma'))}"{name=D}]
\arrow[from=U, to=D, phantom, "\Downarrow a\cdot_v a'"]
\end{tikzcd}$$

Moreover, there are two maps $\mathbf{t}_0,\mathbf{s}_0: G\times\Gamma\to M$ defined by $\mathbf{s}_0(g,\gamma)=\mathbf{s}_\Gamma(\gamma)$ and $\mathbf{t}_0(g,\gamma)=\psi(g,\mathbf{t}_\Gamma(\gamma))$ via the source and target of Γ (denoted as x and y respectively in the above picture), so that we define the horizontal multiplication $\cdot_h$

$$\begin{tikzcd}
(H\times G\times\Gamma)\times_{\mathbf{s}\circ\mathbf{s}_0,M,\mathbf{t}\circ\mathbf{t}_0}(H\times G\times\Gamma) \arrow[r, "\cdot_h"] \arrow[d, shift left] \arrow[d, shift right] & H\times G\times\Gamma\,, \arrow[d, shift left] \arrow[d, shift right] \\
(G\times\Gamma)\times_{\mathbf{s}_0,M,\mathbf{t}_0}(G\times\Gamma) \arrow[r, "\cdot_h"] & G\times\Gamma
\end{tikzcd}$$

by

$$\begin{aligned}
&(h_1,g_1,\gamma_1)\cdot_h(h_2,g_2,\gamma_2)=(h_1\phi(g_1)(h_2),g_1g_2,\Phi(1,g_2^{-1},\gamma_1)\gamma_2),\\
&(g_1,\gamma_1)\cdot_h(g_2,\gamma_2)=(g_1g_2,\Phi(1,g_2^{-1},\gamma_1)\gamma_2).
\end{aligned}$$

$$\begin{tikzcd}
z & & y \arrow[ll, bend right=30, "{(g_1,\gamma_1)}"'{name=U1}] \arrow[ll, bend left=30, "{\mathbf{t}(h_1,g_1,\gamma_1)}"{name=D1}] & & x \arrow[ll, bend right=30, "{(g_2,\gamma_2)}"'{name=U2}] \arrow[ll, bend left=30, "{\mathbf{t}(h_2,g_2,\gamma_2)}"{name=D2}]
\arrow[from=U1, to=D1, phantom, "\Downarrow (h_1,g_1,\gamma_1)=:a_1"]
\arrow[from=U2, to=D2, phantom, "\Downarrow (h_2,g_2,\gamma_2)=:a_2"]
\end{tikzcd}
\mapsto
\begin{tikzcd}
z & & x. \arrow[ll, bend right=30, "{(g_1g_2,\Phi(1,g_2^{-1},\gamma_1)\gamma_2)}"'{name=U}] \arrow[ll, bend left=30, "{(\delta(h_1)g_1\delta(h_2)g_2,\Phi(\phi((g_1g_2)^{-1})((h_1\phi(g_1)(h_2))^{-1}),1,\gamma)}"{name=D}]
\arrow[from=U, to=D, phantom, "\Downarrow a_1\cdot_h a_2"]
\end{tikzcd}$$

It is routine to verify that $\cdot_h$ is a groupoid morphism. Thus these data define a strict Lie 2-groupoid. It is obvious that this Lie 2-groupoid differentiates to the Lie 2-algebroid (28). This Lie 2-groupoid can thus be interpreted as the "action"-groupoid of the action Φ.

Next we remark that, under certain assumptions, the above action Φ in Theorem 2.16 defines a *principal 2-group bundle* (over a manifold N) in the sense of Wockel [31, Def. I.8]. One reason why principal 2-group bundles are interesting is the following [31, Rem. II.11]: when the Lie 2-group $\mathcal{G}$ corresponds to a crossed module of Lie groups of the form $(H, Aut(H), \partial, \phi)$, where H is a Lie group and $\partial\colon H \to Aut(H)$ is given by conjugation, then principal $\mathcal{G}$-2-bundles define gerbes over N [15].

Proposition 2.21. *Let $\Phi\colon (H \rtimes G) \times \Gamma \to \Gamma$ be a Lie 2-group action such that both the G-action on M and the $H \rtimes G$-action on Γ are free and proper with the same quotient $N := M/G = \Gamma/(H \rtimes G)$. Then the action Φ makes*

$$\Gamma \xrightarrow{\pi} N$$

into a principal $\mathcal{G}$-2-bundle.

Proof. A special case of the the above statement is [8, Prop. 4.2]. Its proof carries over literarily to the present case, after one checks that the projection $\Gamma \to \Gamma/(H \rtimes G) = N$ is a Lie groupoid morphism. The latter fact follows easily since the action map $\Phi\colon (H \rtimes G) \times \Gamma \to \Gamma$ is a Lie groupoid morphism. □

3 Examples

In this section we display classes of examples of Lie 2-algebra actions on NQ-1 manifolds (see §1.2) and of the integrated actions (see §2.3). More examples of Lie 2-algebra actions are given at the end of §4.

The starting data for the first example is just a bracket-preserving map of a Lie algebra into the sections of a Lie algebroid.

Example 3.1. [$\mathfrak{g}$ acting on A] Let $\mathfrak{g}$ be a Lie algebra, $A \to M$ a Lie algebroid and $\eta\colon \mathfrak{g} \to (\Gamma(A), [\cdot,\cdot]_A)$ a Lie algebra morphism[17]. Then $(A[1], Q := Q_A)$ is an NQ-1 manifold by Lemma 1.14, and

$$\begin{aligned} \mu\colon \mathfrak{g}[1] \oplus \mathfrak{g} &\to \chi(A[1]) \\ (w, 0) &\mapsto \eta(w) \\ (0, \delta w) &\mapsto [Q, \eta(w)] \end{aligned}$$

is a morphism a DGLAs. Here $\mathfrak{g}[1] \oplus \mathfrak{g}$ denotes the strict Lie-2 algebra with $\delta = Id_{\mathfrak{g}}$ and $[\cdot,\cdot]$ given by the Lie bracket on $\mathfrak{g}$ and its adjoint action. Further we view $\eta(w)$ as an element of $\chi_{-1}(A[1]) \cong \Gamma(A)$.

The image of μ generates a submodule $\mathcal{D}$ of $\chi(A[1])$ as in eq. (42). It is a distribution iff $\eta(\mathfrak{g}) \subset \Gamma(A)$ spans a subbundle $B \subset A$, $\rho(B)$ is a distribution on M, and $[\eta(w), A_x]_A \subset B_x$ whenever $\rho(\eta(w))_x = 0$ for some $w \in \mathfrak{g}$ and $x \in M$ (this follows from Lemma 1.25). Here $\rho\colon A \to TM$ is the anchor, which satisfies $\rho(\eta(w)) = [Q, \eta(w)]$.

In this case, since μ is a strict action, $\mathcal{D}$ is integrable as a distribution and it is preserved by $[Q,\cdot]$. Under the regularity assumptions of Prop. 1.29, the ideal system corresponding to $\mathcal{D}$ by Prop. 1.29 is given by

[17] In [20, Def. 6.1] this is called *A-action of $\mathfrak{g}$ on M*.

- the Lie subalgebroid B spanned by $\eta(\mathfrak{g})$
- the Lie subgroupoid of $M \times M$ given by the foliation integrating $\rho(B)$
- parallel translation by the flat $\rho(B)$-connection on A/B given by $\nabla_{\rho(s)} = ([s,\cdot]_A \text{ mod } B)$ for all $s \in \Gamma(B)$.

The following example is a special case of Ex. 3.1 for which we can write down explicitly the integrated action Φ.

Example 3.2. [$\mathfrak{g}$ acting on TM] Let G be a Lie group and $\hat{\eta}$ an action of G on a manifold M (which for simplicity we take to be simply connected). It is immediate that the product action

$$\begin{aligned} (G \times G) \times (M \times M) &\to (M \times M) \\ (g_1, g_2), (m_1, m_2) &\mapsto (g_1 m_1, g_2 m_2) \end{aligned} \tag{40}$$

is a Lie 2-group action, where all the Lie groupoids appearing are pair groupoids.

We show how to recover this Lie 2-group action from infinitesimal data. Let $\mathfrak{g}$ be the Lie algebra of G, and $\eta \colon \mathfrak{g} \to \chi(M)$ the infinitesimal action corresponding to $\hat{\eta}$ (that is, a Lie algebra morphism $\mathfrak{g} \to (\Gamma(TM), [\cdot,\cdot]_{TM})$). By Ex. 3.1 we obtain a (strict) morphism of DGLAs $\mu \colon \mathfrak{g}[1] \oplus \mathfrak{g} \to \chi(T[1]M)$. It induces a Lie algebra morphism

$$\begin{aligned} \tilde{\mu} \colon \mathfrak{g}_{ab} \rtimes \mathfrak{g} &\to \chi(TM) \\ (w, 0) &\mapsto \eta(w) \\ (0, \delta w) &\mapsto (\eta(w))^T. \end{aligned}$$

Here $\mathfrak{g}_{ab}$ denotes the vector space underlying the Lie algebra $\mathfrak{g}$ and the first $\eta(w)$ is viewed a vector field tangent to the fibers of $TM \to M$. The tangent lift $(\eta(w))^T$ of the vector field $\eta(w)$ on M appears since it agrees with the element $[[Q, \eta(w)], \cdot] = [\eta(w), \cdot]_{TM}$ of $CDO(TM)$ (see Remark 1.26). By Prop. 2.18 this integrates to the Lie 2-group action

$$\begin{aligned} \Phi \colon (G \rtimes G) \times (M \times M) &\to (M \times M) \\ (h, g), (m_1, m_2) &\mapsto (hgm_1, gm_2). \end{aligned}$$

Under the isomorphism of Lie 2-groups (over Id_G)

$$G \rtimes G \cong G \times G, \quad (h, g) \mapsto (hg, g)$$

to the pair groupoid $G \times G \rightrightarrows G$, this action corresponds to (40).

Example 3.3. [Actions on T^*M] Let $\mathfrak{h}[1] \oplus \mathfrak{g}$ be a strict Lie-2 algebra and (M, π) a Poisson manifold. Since T^*M is a Lie algebroid, $(\mathcal{M}, Q) := (T^*[1]M, [\pi, \cdot]_S)$ is a NQ-1 manifold, where $[\cdot,\cdot]_S$ denotes the Schouten bracket of multivector fields.

In [8, §8,9] Cattaneo and the first author consider a morphism of DGLAs of the form

$$\begin{aligned} \mathfrak{h}[1] \oplus \mathfrak{g} &\to \chi_{-1}(\mathcal{M}) \oplus \chi_0(\mathcal{M}) \\ (w, v) &\mapsto \quad X_{J_0{}^* w} + X_{J_1{}^* v} \end{aligned} \tag{41}$$

where $(J_0, J_1) : \mathcal{M} \to (\mathfrak{h}[1] \oplus \mathfrak{g})^*[1]$ is a Poisson (moment) map, and discuss its reduction.

When $\mathfrak{h} = \mathfrak{g}$ and the differential is $Id_{\mathfrak{g}}$, the morphism (41) is recovered from our Ex. 3.1 with $A = T^*M$ and $\eta\colon \mathfrak{g} \to \Gamma(T^*M), w \to -d(J_0^* v)$. Notice that in this case, as pointed out in [8, Ex. 15.2], the morphism (41) is equivalent to an ordinary Hamiltonian action of $\mathfrak{g}$ on (M, π).

Given a Lie group, we see that the conjugation and adjoint actions fit into our framework:

Example 3.4. [Adjoint action] Let $\mathfrak{g}$ be a Lie algebra and consider the DGLA morphism

$$\mu\colon \mathfrak{g} \to \chi(\mathfrak{g}[1]), \quad v \mapsto [Q, \iota_v],$$

where ι_v is the (constant) vector field corresponding to v under $\mathfrak{g} \cong \chi_{-1}(\mathfrak{g}[1])$ and Q is the homological vector field on $\mathfrak{g}[1]$. Notice that $\mathcal{D}$ as in eq. (42) is usually not a distribution on $\mathfrak{g}[1]$ (it is iff $\mathfrak{g}$ is an abelian Lie algebra). We now show that diagram (39) about transformation groupoids/algebroids applied to this example is the following:

$$\begin{tikzcd}
\text{Transf. groupoid of conjugation} \arrow[r, "\mathrm{Lie}_H"] & \text{Transf. groupoid of } Ad \arrow[d, "\mathrm{Lie}"] \\
& \text{Transf. algebroid of } ad.
\end{tikzcd}$$

The Lie algebra action on $\mathfrak{g}$ corresponding to μ reads

$$\tilde{\mu} = ad\colon \mathfrak{g} \to End(\mathfrak{g}) \subset \chi(\mathfrak{g}), \quad v \mapsto ad_v$$

by Remark 1.26. Integrating this Lie algebra action we obtain essentially the adjoint action of G on $\mathfrak{g}$. More precisely, we obtain the Lie algebroid morphism (see Prop. 2.14)

$$\Psi = Ad\colon (G \to G) \times (\mathfrak{g} \to \{pt\}) \to (\mathfrak{g} \to \{pt\}).$$

Integrating this Lie algebroid morphism, by Prop. 2.18 a) we obtain the action of G on itself by conjugation, or more precisely, the Lie 2-group action

$$\Phi = \text{conjugation}\colon (G \rightrightarrows G) \times (G \rightrightarrows \{pt\}) \to (G \rightrightarrows \{pt\}).$$

4 Quotients by Lie 2-algebra actions

In this final section we consider again distributions, as in §1.3. We define the distribution associated to a Lie 2-algebra action, and study the corresponding quotient.

Let $L = L_{-1}[1] \oplus L_0$ be a Lie 2-algebra (Def. 1.17), $\mathcal{M}$ an NQ-1 manifold, and $\mu : L \rightsquigarrow \chi(\mathcal{M})$ a Lie 2-algebra action as in (9). The $C(\mathcal{M})$-submodule of $\chi(M)$ generated by $\mu^1(L)$ and $\mu^2(L_0 \wedge L_0)$ is usually not a distribution on $\mathcal{M}$. Moreover it has two defects: first it is not involutive in general (as suggested by eq. (11)), and second the operator $d_Q := [Q, \cdot]$ does not preserve its sections. (A counterexample for both defects is given in Ex. 4.4 below).

Therefore we are lead to consider the $C(\mathcal{M})$-module

$$\mathcal{D} := Span_{C(\mathcal{M})}\{\mu^1(L)\ ,\ \mu^2(L_0 \wedge L_0)\ ,\ d_Q(\mu^2(L_0 \wedge L_0))\}. \tag{42}$$

When $\mathcal{D}$ forms a distribution, we say the action of the Lie 2-algebra is *regular*.

Unfortunately, the fact that μ is an L_∞-morphism (the constraints (10)-(13)) do not imply that $\mathcal{D}$ is involutive. (A counterexample is given by Ex. 4.5 below.)

Making an additional assumption, we can achieve that $\mathcal{D}$ is involutive and Q-invariant:

Proposition 4.1. *If μ is a regular action of the Lie 2-algebra L on $\mathcal{M}$ and $[\mathcal{D}_0, \mathcal{D}_{-1}] \subset \mathcal{D}_{-1}$ then*
i) $[Q, \mathcal{D}] \subset \mathcal{D}$
ii) $\mathcal{D}$ is involutive

Proof. i) follows from the facts that μ^1 satisfies eq. (10) and that $d_Q^2 = 0$.

ii) Since we assume $[\mathcal{D}_0, \mathcal{D}_{-1}] \subset \mathcal{D}_{-1}$, we just need to check that $\mathcal{D}_0$ is closed under the bracket, so we just need to consider $\mu^1(L_0)$ and $d_Q(\mu^2(L_0 \wedge L_0))$. Let $l_0, l_0' \in L_0$ and $m, m' \in L_0 \wedge L_0$. We have $[\mu^1(l_0), \mu^1(l_0')] \in \mathcal{D}$ by eq. (11).

Further

$$[\mu^1(l_0), [Q, \mu^2(m)]] = [Q, [\mu^1(l_0), \mu^2(m)]] - [[Q, \mu^1(l_0)], \mu^2(m)]$$

also lies in $\mathcal{D}$. Indeed $[Q, \mu^1(l_0)] = 0$ by eq. (10), and the first term on the r.h.s. lies in $\mathcal{D}_0$ because of $[\mathcal{D}_0, \mathcal{D}_{-1}] \subset \mathcal{D}_{-1}$ and because of i).

Similarly, we have $[[Q, \mu_2(m)], [Q, \mu_2(m')]] = [Q, [[Q, \mu_2(m)], \mu_2(m')]] \in \mathcal{D}_0$. □

We summarize the conditions under which we can nicely quotient $\mathcal{M}$ by the Lie 2-algebra action μ:

Corollary 4.2. *Let μ be a regular action of the Lie 2-algebra L on $\mathcal{M} = A[1]$ such that $[\mathcal{D}_0, \mathcal{D}_{-1}] \subset \mathcal{D}_{-1}$ and the assumptions on smoothness and holonomy of Prop. 1.29 are satisfied. Then $\mathcal{M}/\mathcal{D}$ is an NQ-1 manifold. It corresponds to the quotient of A by the ideal system given by*

- *the Lie subalgebroid $B = span_{C^\infty(M)}\{\mu^1(L_{-1}), Im(\mu^2)\}$ of A*
- *the Lie subgroupoid R of $M \times M$ associated to the integrable distribution $F := span\{\underline{\mu^1(L_0)}, \rho(Im(\mu^2))\}$ on M*
- *the action of R on A/B induced by $span_{C^\infty(M)}\{\mu^1(L_0), d_Q(Im(\mu^2))\} \subset \chi_0(\mathcal{M}) \cong CDO(A)$ (where the identification is given in Remark 1.26).*

Proof. $\mathcal{M}/\mathcal{D}$ is an NQ-manifold by Prop. 4.1 and Prop. 1.30. By the same Prop. 1.30 $\mathcal{M}/\mathcal{D}$ corresponds to the quotient of A by the ideal system associated to $\mathcal{D}$ as in Prop. 1.29, which is the above. □

Let us specialize to the case when the action μ is strict.

In this case, we showed in Prop. 2.14 that $\Psi\colon (\mathfrak{h} \rtimes G) \times A \to A$ is an $\mathcal{LA}$-group action. The latter is called a morphic action [27, Def. 3.0.14]. From [27, Thm. 3.2.1] in Stefanini's thesis, it follows that, when the action is free and proper, $A/(\mathfrak{h} \rtimes G) \to M/G$ is again a Lie algebroid with the property[18] that the projection of A onto the quotient is a Lie algebroid morphism.

[18] Even more, it is a Lie algebroid fibration [4, Def. 1.1], which means that the projection $A/(\mathfrak{h} \rtimes G) \to M/G$ admit a complete Ehresmann connection. Such a map is called a "fibration" in the sense that it introduces the expected long exact sequence of homotopy groups of Lie algebroids.

Corollary 4.3. *Let L be a strict Lie-2 algebra, $A[1]$ a NQ-1 manifold, and let $\mu\colon L \to \chi(A[1])$ be a strict Lie 2-algebra action. Assume that the action is locally free, that is, for all $m \in M$, the map $L \to T_m\mathcal{M} = A_m[1] \oplus T_mM$ is injective. Further assume that the induced Lie group action $\psi\colon G \times A \to A$ defined just before Prop. 2.14 is free and proper.*

Then the Lie algebroid corresponding to the NQ-1 manifold $A[1]/\mathcal{D}$ agrees with Stefanini's quotient of A by the $\mathcal{LA}$-group action Ψ.

Proof. We have $\mathcal{D} = Span_{C(\mathcal{M})}\{\mu^1(L)\}$. By the local freeness assumption, the image under μ^1 of a basis of L provides a set of local homogeneous generators of $\mathcal{D}$ whose evaluations at points of M are linearly independent, hence $\mathcal{D}$ is a distribution and the action is regular. $\mathcal{D}$ is involutive since μ preserves brackets. The leaves of the distribution F on M are just the orbits of the free action $\psi|_M$, the restriction of the G-action ψ to M, so M/F is a smooth manifold. Let B be as in Cor. 4.2, then the holonomy of the partial connection ∇ is given by the action of G on A/B induced by ψ, so by the freeness of ψ the holonomy is trivial.

Hence we can apply Cor. 4.2. The vector bundle obtained quotienting A by the ideal system of Cor. 4.2 agrees with the quotient of A by the action Ψ, and the induced Lie algebroid structures agree because in both cases the projection map from A is a Lie algebroid morphism. ☐

The following is an example where the image of the action μ is a distribution, which however is neither involutive nor preserved by $[Q, \cdot]$.

Example 4.4. [The image of μ is not involutive] Let $L = \mathbb{R}^2$ be the abelian 2-dimensional Lie algebra (concentrated in degree zero), fix a basis l_0, l_0'. Let $A = T\mathbb{R}^3$, so $A[1] = T[1]\mathbb{R}^3$, on which we take the standard degree 0 coordinates x_1, x_2, x_3 and the corresponding degree 1 coordinates $\xi_i (= dx_i)$. The deRham vector field on $A[1]$ reads $Q = \sum_i \xi_i \frac{\partial}{\partial x_i}$. Define $\mu\colon L \rightsquigarrow \chi(T[1]\mathbb{R}^3)$ by

$$\begin{aligned} \mu^1(l_0) =& [Q, \frac{\partial}{\partial \xi_1}] \\ \mu^1(l_0') =& [Q, x_1\frac{\partial}{\partial \xi_3} - \frac{\partial}{\partial \xi_2}] \\ \mu^2(l_0 \wedge l_0') =& -\frac{\partial}{\partial \xi_3}. \end{aligned}$$

Notice that $\mu^1(l_0)$ acts on functions on $T[1]\mathbb{R}^3$ – which are just differential forms on $\mathbb{R}^3$ – by the Lie derivative $\mathcal{L}_{\frac{\partial}{\partial x_1}}$, and similarly for $\mu^1(l_0')$.

One checks that μ is an L_∞-morphism (conditions (10)-(13)). The span of the image of μ is $span\{\frac{\partial}{\partial x_1}, x_1\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_2}, \frac{\partial}{\partial \xi_3}\}$: it has constant rank 3, and is not involutive (because its restriction to the body $\mathbb{R}^3$ is not involutive) nor preserved by $[Q, \cdot]$.

On the other hand, the distribution $\mathcal{D}$ (defined in eq. (42)) on $\mathcal{M}$ is spanned by $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial \xi_3}$. $\mathcal{D}$ is involutive and its sections are preserved by $[Q, \cdot]$. The quotient $\mathcal{M}/\mathcal{D}$ is isomorphic to $\mathbb{R}^2[1]$ with vanishing homological vector field, which corresponds to the abelian 2-dimensional Lie algebra.

Next we display an example where the $C(\mathcal{M})$-module $\mathcal{D}$ is not involutive.

Example 4.5. [$\mathcal{D}$ is not involutive] Consider the strict Lie-2 algebra $L = \mathbb{R}[1] \oplus \mathbb{R}^2$, with zero bracket and differential $\delta\colon \mathbb{R}[1] \to \mathbb{R}^2, l_{-1} \mapsto l_0$, where $\{l_{-1}\}$ and $\{l_0, l_0'\}$ are bases of $\mathbb{R}$ and

$\mathbb{R}^2$ respectively. Let M be a manifold and X, Y vector fields such that $[[X,[X,Y]_M]_M$ does not lie in the the span of X and $[X,Y]_M$. Take $\mathcal{M} = (T[1]M, Q)$ where Q is the deRham vector field. By $[\cdot,\cdot]_M$ we denote the Lie bracket of vector fields on M, while by $[\cdot,\cdot]$ we denote the graded Lie bracket of vector fields on $\mathcal{M}$.

Consider the Lie 2-algebra action

$$\mu : \mathbb{R}[1] \oplus \mathbb{R}^2 \rightsquigarrow \chi(\mathcal{M})$$

given by

$$\begin{aligned}
\mu^1(l_{-1}) &= \bar{X} \\
\mu^1(l_0) &= [Q, \bar{X}] \\
\mu^1(l_0') &= [Q, \bar{Y}] \\
\mu^2(l_0 \wedge l_0') &= -\overline{[X,Y]_M}.
\end{aligned}$$

Here the overline denotes the identification $\Gamma(TM) \cong \chi_{-1}(\mathcal{M}), X \mapsto \bar{X}$ as in Prop. 1.6). It is easy to check that μ is an L_∞-morphism, i.e. that eq. (10)-(13) are satisfied. However

$$[\mu^1(l_0), \mu^2(l_0 \wedge l_0')] = -[[Q, \bar{X}], \overline{[X,Y]_M}] = -\overline{[X,[X,Y]_M]_M}$$

does not lie in $\mathcal{D}_{-1} = span_{C^\infty(M)}\{\mu^1(l_{-1}), \mu^2(l_0 \wedge l_0')\} = span_{C^\infty(M)}\{\bar{X}, \overline{[X,Y]_M}\}$. This shows that $\mathcal{D}$ is not involutive.

More concretely, we can take $M = \mathbb{R}^3$ with coordinates x_1, x_2, x_3 and $X = \frac{\partial}{\partial x_1}$, $Y = \frac{x_1^2}{2}\frac{\partial}{\partial x_3} - x_1 \frac{\partial}{\partial x_2}$, so that $[X,Y]_M = x_1 \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_2}$ and $[X,[X,Y]_M]_M = \frac{\partial}{\partial x_3}$.

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