THE NEW ν-METRIC INDUCES THE CLASSICAL GAP TOPOLOGY

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Abstract. Let $\mathcal{A}_+$ denote the set of Laplace transforms of complex Borel measures μ on $[0, +\infty)$ such that μ does not have a singular non-atomic part. In [1], an extension of the classical ν-metric of Vinnicombe was given, which allowed one to address robust stabilization problems for unstable plants over $\mathcal{A}_+$. In this article, we show that this new ν-metric gives a topology on unstable plants which coincides with the classical gap topology for unstable plants over $\mathcal{A}_+$ with a single input and a single output.

1. Introduction

We recall the general *stabilization problem* in control theory. Suppose that R is a commutative integral domain with identity (thought of as the class of stable transfer functions) and let $\mathbb{F}(R)$ denote the field of fractions of R. Then the stabilization problem is:

Given $p \in \mathbb{F}(R)$ (an unstable plant transfer function),
find $c \in \mathbb{F}(R)$ (a stabilizing controller transfer function),
such that (the closed loop transfer function)

$$H(p, c) := \begin{bmatrix} p \\ 1 \end{bmatrix} (1 - cp)^{-1} \begin{bmatrix} -c & 1 \end{bmatrix}$$

belongs to $R^{2\times 2}$ (that is, it is stable).

In the *robust stabilization problem*, one goes a step further. One knows that the plant is just an approximation of reality, and so one would really like the controller c to not only stabilize the *nominal* plant p, but also all sufficiently close plants p' to p. The question of what one means by "closeness" of plants thus arises naturally. So one needs a function d defined on pairs of stabilizable plants such that

(1) d is a metric on the set of all stabilizable plants,
(2) d is amenable to computation, and
(3) stabilizability is a robust property of the plant with respect to d.

Such a desirable metric, was introduced by Glenn Vinnicombe in [14] and is called the *ν-metric*. In that paper, essentially R was taken to be the

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rational functions without poles in the closed unit disk, and it was also shown that the topology obtained was equivalent to the one obtained from the gap-metric (introduced by Zames and El-Sakkary [15],[5], which in turn is equivalent to the graph metric of Vidyasagar [13]).

The problem of what happens when R is some other ring of stable transfer functions of infinite-dimensional systems was left open in [14]. This problem of extending the ν-metric from the rational case to transfer function classes of infinite-dimensional systems was addressed in [1]. There the starting point in the approach was abstract. It was assumed that R is any commutative integral domain with identity which is a subset of a Banach algebra S satisfying certain assumptions, and then an "abstract" ν-metric was defined in this setup, and it was shown in [1] that it does define a metric on the class of all stabilizable plants. It was also shown there that stabilizability is a robust property of the plant. In particular, this gave a metric on unstable plants over $\mathcal{A}_+$, where $\mathcal{A}_+$ denotes the set of Laplace transforms of complex Borel measures μ on $[0, +\infty)$ such that μ does not have a singular non-atomic part.

One can also define a gap-metric for unstable plants over $\mathcal{A}_+$, and so it is natural to ask if the ν-metric and the gap-metric induce the same topologies on unstable plants over $\mathcal{A}_+$. In this article we address this issue, and prove the following result.

Theorem 1.1. *On the set $\mathbb{S}(\mathcal{A}_+)$, the topologies induced by the ν-metric d_ν and the gap-metric d_g are identical.*

The notation $\mathbb{S}(\mathcal{A}_+)$ will be explained carefully in Section 3, but roughly speaking, it is to be thought of as the class of unstable plants over $\mathcal{A}_+$ with a single input and a single output. Owing to a technical difficulty, we restrict ourselves to single input and single output systems. We end this article with an open problem, namely the validity of our main result for systems with multiple inputs and multiple outputs, while pointing out the precise nature of the technical difficulty.

The paper is organized as follows:

(1) In Section 3, we recall from [1] the ν-metric in the context of unstable plants over $\mathcal{A}_+$, and also derive an alternative expression for it in Proposition 3.6, reminiscent of Georgiou's formula for the gap-metric from [6].
(2) In Section 4, we give the definition of the gap-metric in the context of unstable plants over $\mathcal{A}_+$. An alternative expression for the gap-metric is given in Proposition 4.9, which will be used in order to show the equivalence of d_ν and d_g.
(3) Finally, in Section 5, we will prove our main result (Theorem 1.1). At the end of this section, we also highlight the main obstacle towards extending Theorem 1.1 to systems with multiple inputs and outputs.

2. Notation index

For the convenience of the reader, we have included a table here which shows the page numbers of the places where the corresponding notation is first defined.

Notation	Page number
$\widehat{\cdot}$	Laplace transform (page 4) or Fourier transform (page 4)
$\cdot^*$	pages 4, 6, 9
$\mathcal{A}$	page 4
$\mathcal{A}_+$	page 4
AP	almost periodic functions (page 4)
C_0	functions vanishing at $\pm\infty$ (page 11)
$\mathbb{C}_+$	right half of the complex plane (page 4)
$\vec{\delta}$	directed gap (page 9)
d_g	gap-metric (page 9)
d_ν	ν-metric (page 6)
$\mathbb{F}(\mathcal{A}_+)$	field of fractions over $\mathcal{A}_+$ (page 5)
$\mathcal{G}$	graph of a system (page 8)
$G, \widetilde{G}, K, \widetilde{K}$	matrices built from coprime factorizations (page 6)
inv $\cdot$	invertible elements of a ring (page 3)
$P_{\mathcal{G}}$	projection onto $\mathcal{G}$ (page 8)
$P_{\mathcal{G}_1}\vert_{\mathcal{G}_2}$	restriction of $P_{\mathcal{G}_1}$ to $\mathcal{G}_2$ (page 10)
$\mathbb{S}(\mathcal{A}_+)$	plants with a normalized coprime factorization (page 6)
T_X	Toeplitz operator (page 11)
$\mathtt{w}$	winding number for continuous closed curves avoiding 0 (page 5)
w	average winding number for invertible AP functions (page 5)
W	index for invertible elements in $\mathcal{A}$ (page 5)

3. The ν-metric

In this section we will recall the new ν-metric for unstable plants over the ring $\mathcal{A}_+$ (defined below), which was listed as a particular example in [1, Subsection 5.3] of the abstract ν-metric introduced in that paper. At the end of this section, we will also give an alternate expression for the ν-metric, which will be used later in order to show the equivalence of the ν-metric topology with the classical gap topology.

If R is a commutative integral domain with identity 1, we use the symbol inv R for the set of invertible elements of R.

We denote by $\mathcal{A}_+$ the set of Laplace transforms of complex Borel measures μ on $[0, +\infty)$ such that μ does not have a singular non-atomic part. A more explicit description of the elements of $\mathcal{A}_+$ can be given as follows. Let

$$\mathbb{C}_+ := \{s \in \mathbb{C} : \text{Re}(s) \geq 0\}.$$

Then

$$\mathcal{A}_+ = \left\{ s(\in \mathbb{C}_+) \mapsto \widehat{f_a}(s) + \sum_{k\geq 0} f_k e^{-st_k} \;\middle|\; \begin{array}{l} f_a \in L^1(0,\infty),\ (f_k)_{k\geq 0} \in \ell^1, \\ 0 = t_0 < t_1, t_2, t_3, \dots \end{array} \right\},$$

and equipped with pointwise operations and the norm:

$$\|F\| = \|f_a\|_{L^1} + \|(f_k)_{k\geq 0}\|_{\ell^1}, \quad F(s) = \widehat{f_a}(s) + \sum_{k\geq 0} f_k e^{-st_k} \quad (s \in \mathbb{C}_+),$$

$\mathcal{A}_+$ is a Banach algebra. Here $\widehat{f_a}$ denotes the *Laplace transform* of f_a:

$$\widehat{f_a}(s) = \int_0^\infty e^{-st} f_a(t)dt, \quad s \in \mathbb{C}_+.$$

Similarly, define $\mathcal{A}$ as follows:

$$\mathcal{A} = \left\{ iy(\in i\mathbb{R}) \mapsto \widehat{f_a}(iy) + \sum_{k\in\mathbb{Z}} f_k e^{-iyt_k} \;\middle|\; \begin{array}{l} f_a \in L^1(\mathbb{R}),\ (f_k)_{k\in\mathbb{Z}} \in \ell^1, \\ \dots, t_{-2}, t_{-1} < 0 = t_0 < t_1, t_2, \dots \end{array} \right\}.$$

Then, equipped with pointwise operations and the norm:

$$\|F\| = \|f_a\|_{L^1} + \|(f_k)_{k\in\mathbb{Z}}\|_{\ell^1}, \quad F(iy) := \widehat{f_a}(iy) + \sum_{k\in\mathbb{Z}} f_k e^{-iyt_k} \quad (y \in \mathbb{R}),$$

$\mathcal{A}$ is a unital commutative complex semisimple Banach algebra. Here $\widehat{f_a}$ is the *Fourier transform* of f_a,

$$\widehat{f_a}(iy) = \int_{-\infty}^\infty e^{-iyt} f_a(t)dt \quad (y \in \mathbb{R}).$$

One can also define an involution $\cdot^*$ on $\mathcal{A}$, given by

$$F^*(iy) = \overline{F(iy)}, \quad y \in \mathbb{R},$$

for $F \in \mathcal{A}$. Clearly, $\mathcal{A}_+ \subset \mathcal{A}$.

The algebra AP of complex valued (uniformly) *almost periodic functions* is the smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains all the functions $e_\lambda := e^{i\lambda y}$. Here the parameter λ belongs to $\mathbb{R}$. For any $f \in AP$, its *Bohr-Fourier series* is defined by the formal sum

$$\sum_\lambda f_\lambda e^{i\lambda y}, \quad y \in \mathbb{R}, \tag{3.1}$$

where

$$f_\lambda := \lim_{N\to\infty} \frac{1}{2N} \int_{[-N,N]} e^{-i\lambda y} f(y)dy, \quad \lambda \in \mathbb{R},$$

and the sum in (3.1) is taken over the set $\sigma(f) := \{\lambda \in \mathbb{R} \mid f_\lambda \neq 0\}$, called the *Bohr-Fourier spectrum* of f. The Bohr-Fourier spectrum of every

$f \in AP$ is at most a countable set. For each $f \in \text{inv } AP$, we can define the *average winding number* $w(f) \in \mathbb{R}$ of f as follows [8, Theorem 1, p. 167]:

$$w(f) = \lim_{T\to\infty} \frac{1}{2T}\Big(\arg(f(T)) - \arg(f(-T))\Big).$$

We set

$$F_{AP}(iy) = \sum_{k\in\mathbb{Z}} f_k e^{-iyt_k} \quad (y \in \mathbb{R}) \quad \text{for} \quad F = \widehat{f}_a + \sum_{k\in\mathbb{Z}} f_k e^{-i\cdot t_k} \in \mathcal{A}.$$

If $F = \widehat{f}_a + F_{AP} \in \text{inv } \mathcal{A}$, then it can be shown that ([1, Subsection 5.3]) $F_{AP}(i\cdot) \in \text{inv } AP$. Moreover, $F = \widehat{f}_a + F_{AP} \in \mathcal{A}$ is invertible if and only if for all $y \in \mathbb{R}$, $F(iy) \neq 0$ and $\inf_{y\in\mathbb{R}} |F_{AP}(iy)| > 0$.

Since $\widehat{L^1(\mathbb{R})}$ is an ideal in $\mathcal{A}$, it follows that $F_{AP}^{-1}\widehat{f}_a$ is the Fourier transform of a function in $L^1(\mathbb{R})$, and so the map

$$y \mapsto 1 + (F_{AP}(iy))^{-1}\widehat{f}_a(iy) = \frac{F(iy)}{F_{AP}(iy)}$$

has a well-defined winding number $\mathtt{w}$ around 0. Geometrically, $\mathtt{w}(f)$ is the number of times the curve $t \mapsto f(t)$ winds around the origin in a counterclockwise direction.

Define the *index* $W : \text{inv } \mathcal{A} \to \mathbb{R} \times \mathbb{Z}$ by

$$W(F) = \Big(w(F_{AP}), \mathtt{w}(1 + F_{AP}^{-1}\widehat{f}_a)\Big), \tag{3.2}$$

where $F = \widehat{f}_a + F_{AP} \in \text{inv } \mathcal{A}$, and

$$w(F_{AP}) := \lim_{R\to\infty} \frac{1}{2R}\Big(\arg\big(F_{AP}(iR)\big) - \arg\big(F_{AP}(-iR)\big)\Big),$$
$$\mathtt{w}(1 + F_{AP}^{-1}\widehat{f}_a) := \frac{1}{2\pi}\Big(\arg\big(1 + (F_{AP}(iy))^{-1}\widehat{f}_a(iy)\big)\Big|_{y=-\infty}^{y=+\infty}\Big).$$

The map $W : \text{inv } \mathcal{A} \to \mathbb{R} \times \mathbb{Z}$ satisfies:

- (I1) $W(ab) = W(a) + W(b)$ $(a, b \in \text{inv } \mathcal{A})$.
- (I2) $W(a^*) = -W(a)$ $(a \in \text{inv } \mathcal{A})$.
- (I3) W is locally constant, that is, W continuous when $\mathbb{R}\times\mathbb{Z}$ is equipped with the discrete topology.
- (I4) $x \in \mathcal{A}_+ \cap (\text{inv } \mathcal{A})$ is invertible as an element of $\mathcal{A}_+$ if and only if $W(x) = (0, 0)$.

A consequence of (I3) is the following "homotopic invariance of the index" (see [1, Proposition 2.1]): if $H : [0, 1] \to \text{inv } \mathcal{A}$ is a continuous map, then $W(H(0)) = W(H(1))$.

We recall the following standard notation and definitions from the factorization approach to control theory.

3.1. **The notation $\mathbb{F}(\mathcal{A}_+)$:** $\mathbb{F}(\mathcal{A}_+)$ denotes the field of fractions of $\mathcal{A}_+$.

3.2. **The notation F^*:** If $F \in \mathcal{A}_+^{p\times m}$, then $F^* \in \mathcal{A}^{m\times p}$ is the matrix with the entry in the ith row and jth column given by F_{ji}^*, for all $1 \le i \le p$, and all $1 \le j \le m$.

3.3. **Coprime/normalized coprime factorization:** Given $p \in \mathbb{F}(R)$, a factorization $p = nd^{-1}$, where $n, d \in R$, is called a *coprime factorization of* P if there exist $x, y \in R$ such that $xn + yd = 1$. If moreover there holds that $n^*n + d^*d = 1$, then the coprime factorization is referred to as a *normalized* coprime factorization of p.

3.4. **The notation $G, \widetilde{G}, K, \widetilde{K}$:** Given $p \in \mathbb{F}(\mathcal{A}_+)$ with a normalized coprime factorization $p = nd^{-1}$, we introduce the following matrices with entries from $\mathcal{A}_+$:

$$G = \begin{bmatrix} n \\ d \end{bmatrix} \quad \text{and} \quad \widetilde{G} = \begin{bmatrix} -d & n \end{bmatrix}.$$

Similarly, given $c \in \mathbb{F}(\mathcal{A}_+)$ with normalized coprime factorization $c = xy^{-1}$, we introduce the following matrices with entries from $\mathcal{A}_+$:

$$K = \begin{bmatrix} y \\ x \end{bmatrix} \quad \text{and} \quad \widetilde{K} = \begin{bmatrix} -x & y \end{bmatrix}.$$

3.5. **The notation $\mathbb{S}(\mathcal{A}_+)$:** We denote by $\mathbb{S}(\mathcal{A}_+)$ the set of all elements $p \in \mathbb{F}(\mathcal{A}_+)$ that possess a normalized coprime factorization.

Remark 3.1.

(1) It can be shown (see for example [13, Chapter 8]) that if $p \in \mathbb{S}(\mathcal{A}_+)$, then p is a *stabilizable plant over* $\mathcal{A}_+$, that is, there exists a $c \in \mathbb{F}(\mathcal{A}_+)$ such that $H(p, c) \in R^{2\times 2}$.
(2) [2, Subsection 3.5] shows that every stabilizable plant $p \in \mathbb{F}(\mathcal{A}_+)$ admits a coprime factorization over $\mathcal{A}_+$.
(3) It follows from the proof of [9, Lemma 6.5.6.(e)] and [9, Theorem 5.2.8] that whenever $p \in \mathbb{F}(\mathcal{A}_+)$ has a coprime factorization over $\mathcal{A}_+$, it also has a *normalized* coprime factorization over $\mathcal{A}_+$.

Putting these remarks together, we see that $\mathbb{S}(\mathcal{A}_+)$ is exactly the set of all plants in $\mathbb{F}(\mathcal{A}_+)$ that are stabilizable over $\mathcal{A}_+$.

Definition 3.2 (ν-metric d_ν on $\mathbb{S}(\mathcal{A}_+)$)**.** For $p_1, p_2 \in \mathbb{S}(\mathcal{A}_+)$, with the normalized coprime factorizations $p_1 = n_1 d_1^{-1}$ and $p_2 = n_2 d_2^{-1}$, we define

$$d_\nu(p_1, p_2) := \begin{cases} \|\widetilde{G}_2 G_1\|_\infty & \text{if } G_1^* G_2 \in \text{inv } \mathcal{A} \text{ and } W(G_1^* G_2) = (0, 0), \\ 1 & \text{otherwise.} \end{cases} \tag{3.3}$$

where the notation is as in Subsections 3.1-3.5.

We have the following; see [1]:

Theorem 3.3. *d_ν given by* (3.3) *is a metric on* $\mathbb{S}(\mathcal{A}_+)$.

Moreover, stabilizability is a robust property of the plant in this new ν-metric. In order to see this, we first introduce the notion of stability margin for a pair comprising a plant and its controller.

Definition 3.4. Given $p, c \in \mathbb{F}(\mathcal{A}_+)$, the *stability margin* of the pair (p, c) is defined by

$$\mu_{p,c} = \begin{cases} \|H(p,c)\|_\infty^{-1} & \text{if } p \text{ is stabilized by } c, \\ 0 & \text{otherwise.} \end{cases}$$

The number $\mu_{p,c}$ can be interpreted as a measure of the performance of the closed loop system comprising p and c: larger values of $\mu_{p,c}$ correspond to better performance, with $\mu_{p,c} > 0$ if c stabilizes p.

The following was proved in [1]:

Theorem 3.5. *If $p, p' \in \mathbb{S}(\mathcal{A}_+)$ and $c \in \mathbb{S}(\mathcal{A}_+)$, then $\mu_{p',c} \geq \mu_{p,c} - d_\nu(p, p')$.*

The above result says that stabilizability is a robust property of the plant, since if c stabilizes p with a stability margin $\mu_{p,c} > m$, and p' is another plant which is close to p in the sense that $d_\nu(p', p) \leq m$, then c is also guaranteed to stabilize p'.

We will now derive an alternative expression for the ν-metric, which is reminiscent of Georgiou's formula for the gap-metric from [6].

Proposition 3.6. *If $p_1, p_2 \in \mathbb{S}(\mathcal{A}_+)$, then*

$$d_\nu(p_1, p_2) = \inf_{\substack{q \in \text{inv } \mathcal{A}, \\ W(q)=(0,0)}} \|G_1 - G_2 q\|_\infty.$$

Proof. Let $q \in \text{inv } \mathcal{A}$ and $W(q) = (0, 0)$. We have

$$\begin{aligned} \|G_1 - G_2 q\|_\infty &= \left\| \begin{bmatrix} G_2^* \\ \widetilde{G}_2 \end{bmatrix} (G_1 - G_2 q) \right\|_\infty \quad \left(\text{as } \begin{bmatrix} G_2 & \widetilde{G}_2^* \end{bmatrix} \begin{bmatrix} G_2^* \\ \widetilde{G}_2 \end{bmatrix} = I\right) \\ &= \left\| \begin{bmatrix} G_2^* G_1 - q \\ \widetilde{G}_2 G_1 \end{bmatrix} \right\|_\infty \quad (\text{since } \widetilde{G}_2 G_2 = 0 \text{ and } G_2^* G_2 = I) \\ &\geq \|\widetilde{G}_2 G_1\|_\infty. \end{aligned}$$

So if $G_2^* G_1 \in \text{inv } \mathcal{A}$ and $W(G_2^* G_1) = (0, 0)$, then from the above it follows that $\|G_1 - G_2 q\|_\infty \geq \|\widetilde{G}_2 G_1\|_\infty = d_\nu(p_1, p_2)$. As the choice of q above was arbitrary, we obtain

$$\inf_{\substack{q \in \text{inv } \mathcal{A}, \\ W(q)=(0,0)}} \|G_1 - G_2 q\|_\infty \geq d_\nu(p_1, p_2). \tag{3.4}$$

If we define $q_0 := G_2^* G_1 \in \mathcal{A}$, then $q_0 \in \text{inv } \mathcal{A}$ and $W(q_0) = (0, 0)$, and so

$$\begin{aligned} \inf_{\substack{q \in \text{inv } \mathcal{A}, \\ W(q)=(0,0)}} \|G_1 - G_2 q\|_\infty &\leq \|G_1 - G_2 q_0\|_\infty = \left\| \begin{bmatrix} G_2^* G_1 - q_0 \\ \widetilde{G}_2 G_1 \end{bmatrix} \right\|_\infty \\ &= \left\| \begin{bmatrix} 0 \\ \widetilde{G}_2 G_1 \end{bmatrix} \right\|_\infty = \|\widetilde{G}_2 G_1\|_\infty = d_\nu(p_1, p_2). \end{aligned}$$

From this and (3.4), the claim in the proposition follows for the case when $G_2^* G_1 \in \text{inv } \mathcal{A}$ and $W(G_2^* G_1) = (0,0)$.

Now let $q \in \text{inv } \mathcal{A}$ be such that $W(q) = (0,0)$ and $\|G_1 - G_2 q\|_\infty < 1$. Using $G_1^* G_1 = 1$, we see that

$$\|1 - G_1^* G_2 q\|_\infty = \|G_1^*(G_1 - G_2 q)\|_\infty \le \|G_1^*\|_\infty \|G_1 - G_2 q\|_\infty < 1 \cdot 1 = 1.$$

So $G_1^* G_2 q = 1 - (1 - G_1^* G_2 q)$ is invertible as an element of $\mathcal{A}$. Consider the map $H : [0,1] \to \text{inv } \mathcal{A}$ given by $H(t) = 1 - t(1 - G_1^* G_2 q)$, $t \in [0,1]$. By the homotopic invariance of the index,

$$(0,0) = W(1) = W(H(0)) = W(H(1)) = W(G_1^* G_2 q).$$

As $W(q) = (0,0)$, we obtain that $W(G_1^* G_2) = (0,0)$. So we have shown that if there is a $q \in \mathcal{A}$ such that $q \in \text{inv } \mathcal{A}$, $W(q) = (0,0)$ and $\|G_1 - G_2 q\|_\infty < 1$, then $G_1^* G_2 \in \text{inv } \mathcal{A}$ and $W(G_1^* G_2) = (0,0)$. Thus if either $G_1^* G_2 \notin \text{inv } \mathcal{A}$ or $G_1^* G_2 \in \text{inv } \mathcal{A}$ but $W(G_1^* G_2) \neq (0,0)$, then for all $q \in \mathcal{A}$ such that $q \in \text{inv } \mathcal{A}$, $W(q) = (0,0)$, we have that $\|G_1 - G_2 q\|_\infty \ge 1$, and so

$$\inf_{\substack{q \in \text{inv } \mathcal{A}, \\ W(q)=(0,0)}} \|G_1 - G_2 q\|_\infty \ge 1 = d_\nu(p_1, p_2).$$

Also, with $q_n := \frac{1}{n} I$, $q_n \in \text{inv } \mathcal{A}$ and $W(q_n) = (0,0)$. We have

$$\|G_1 - G_2 q_n\|_\infty \le \|G_1\|_\infty + \|G_2\|_\infty \|q_n\|_\infty \le 1 + 1 \cdot \frac{1}{n}.$$

Hence

$$\inf_{\substack{q \in \text{inv } \mathcal{A}, \\ W(q)=(0,0)}} \|G_1 - G_2 q\|_\infty \le \inf_n \|G_1 - G_2 q_n\|_\infty \le \inf_n \left(1 + \frac{1}{n}\right) = 1 = d_\nu(p_1, p_2).$$

Consequently, $\displaystyle\inf_{\substack{q \in \text{inv } \mathcal{A}, \\ W(q)=(0,0)}} \|G_1 - G_2 q\|_\infty = 1 = d_\nu(p_1, p_2).$ □

4. The gap-metric

In this section we will recall the gap-metric topology for unstable plants over the ring $\mathcal{A}_+$. We will also prove a few technical lemmas which will be used in the next section in order to prove our main result.

Definition 4.1 (Graph of a system)**.** For $p \in \mathbb{S}(\mathcal{A}_+)$, with the normalized coprime factorization $p = nd^{-1}$, we define the *graph of* p, denoted by $\mathcal{G}$, to be the following subspace of the Hardy space $H^2(\mathbb{C}^2)$:

$$\mathcal{G} = GH^2 = \left\{ \begin{bmatrix} n\varphi \\ d\varphi \end{bmatrix} : \varphi \in H^2 \right\}.$$

Using the fact that there exist $x, y \in \mathcal{A}_+$ such that $xn + yd = 1$, it is easy to see that the graph $\mathcal{G}$ is a *closed* subspace of $H^2 \times H^2$. We denote the orthogonal projection from $H^2 \times H^2$ onto $\mathcal{G}$ by $P_{\mathcal{G}}$.

Definition 4.2 (Gap-metric d_g)**.** For $p_1, p_2 \in \mathbb{S}(\mathcal{A}_+)$, with the normalized coprime factorizations $p_1 = n_1 d_1^{-1}$ and $p_2 = n_2 d_2^{-1}$, we define

$$d_g(p_1, p_2) := \|P_{\mathcal{G}_1} - P_{\mathcal{G}_2}\|_{\mathcal{L}(H^2 \times H^2)}. \tag{4.1}$$

We will need a few technical results on the gap-metric d_g. For a self-contained account of these results, we refer the reader to [12]. It can be checked that d_g given by (4.1) is well-defined. Since the gap-metric is a metric on the set of closed subspaces of a Hilbert space, it follows that d_g given by (4.1) is a metric on $\mathbb{S}(\mathcal{A}_+)$.

For $p_1, p_2 \in \mathbb{S}(\mathcal{A}_+)$, $d_g(p_1, p_2) = \max\{\vec{\delta}(p_1, p_2), \vec{\delta}(p_2, p_1)\}$, where $\vec{\delta}(\cdot, \cdot)$ denotes the *directed gap*, defined by

$$\vec{\delta}(p_1, p_2) := \|(I - P_{\mathcal{G}_2}) P_{\mathcal{G}_1}\|_{\mathcal{L}(H^2 \times H^2)}.$$

If $d_g(p_1, p_2) < 1$, then $d_g(p_1, p_2) = \vec{\delta}(p_1, p_2) = \vec{\delta}(p_2, p_1)$ [7, Prop. 3, p.675]. In [6], it was shown that

$$d_g(p_1, p_2) = \max\left\{ \inf_{q \in H^\infty} \|G_1 - G_2 q\|_\infty \ , \ \inf_{q \in H^\infty} \|G_2 - G_1 q\|_\infty \right\}.$$

For $p_1, p_2 \in \mathbb{S}(\mathcal{A}_+)$, the infimums above can be taken over $\mathcal{A}_+$ instead of H^∞, and this follows from [9, Theorem 11.3.3].

Lemma 4.3. *If $p_1, p_2 \in \mathbb{S}(\mathcal{A}_+)$, then*

$$\inf_{q \in H^\infty} \|G_1 - G_2 q\|_\infty = \inf_{q \in \mathcal{A}_+} \|G_1 - G_2 q\|_\infty.$$

Proof. Clearly $m := \inf_{q \in H^\infty} \|G_1 - G_2 q\|_\infty \leq \inf_{q \in \mathcal{A}_+} \|G_1 - G_2 q\|_\infty =: M$. Define

$$V = \begin{bmatrix} G_2 & G_1 \\ 0 & 1 \end{bmatrix}, \quad W := V^\star \begin{bmatrix} I & 0 \\ 0 & -M^2 \end{bmatrix} V.$$

(For $X \in (H^\infty)^{p \times m}$, $X^\star \in (L^\infty)^{m \times p}$ is defined by $X^\star(iy) = (X(iy))^*$, $y \in \mathbb{R}$.) Suppose that $m < M$. Then there exists a $q \in H^\infty$ such that $\|G_1 - G_2 q\|_\infty < M$. Now we apply [9, Theorem 11.3.3, p.654] to conclude that the q can in fact be chosen in $\mathcal{A}_+$. For this, a few technical assumptions have to be verified first, and we give these details in the following paragraph for the interested reader.

(First of all, the Standing Hypothesis [9, 11.0.1, p.611] is satisfied, since $\mathcal{A}_+$ does satisfy [9, Hypothesis 8.4.7., p.384], by [9, Theorem 8.4.9(β), p.385]. Secondly, the Standing Hypothesis [9, 11.3.1, p.654] is satisfied, since $G_2^* G_2 = 1$. Actually, there are two extraneous assumptions in 11.3.1, but neither is used in the part of the proofs required here, and these extraneous assumptions are anyway satisfied in our case. Now as the Assumption (FI1$\frac{1}{2}$s) of [9, Theorem 11.3.3, p.654] holds, also (FI13s) holds. By the last sentence of [9, Theorem 11.3.6, p.659], as W has entries from $\mathcal{A}_+$, there exists a $q \in \mathcal{A}_+$ such that $\|G_1 - G_2 q\|_\infty < M$.)

Consequently, $m = M$. ∎

We use the notation $P_{\mathcal{G}_1}|_{\mathcal{G}_2}$ to mean the restriction of $P_{\mathcal{G}_1}$ to $\mathcal{G}_2$, namely, the operator from $\mathcal{G}_2$ to $\mathcal{G}_1$, given by

$$P_{\mathcal{G}_1}|_{\mathcal{G}_2} g_2 = P_{\mathcal{G}_1} g_2, \quad g_2 \in \mathcal{G}_2.$$

Then $\ker(P_{\mathcal{G}_1}|_{\mathcal{G}_2}) = \{g_2 \in \mathcal{G}_2 : P_{\mathcal{G}_1} g_2 = 0\} = \mathcal{G}_2 \cap (\ker P_{\mathcal{G}_1}) = \mathcal{G}_2 \cap \mathcal{G}_1^\perp$. Also, for $g_1 \in \mathcal{G}_1$ and $g_2 \in \mathcal{G}_2$, we have

$$\begin{aligned} \langle P_{\mathcal{G}_1}|_{\mathcal{G}_2} g_2, g_1\rangle_{\mathcal{G}_1} &= \langle P_{\mathcal{G}_1} g_2, g_1\rangle_{\mathcal{G}_1} = \langle g_2, g_1\rangle_{H^2(\mathbb{C}^2)} \\ &= \langle g_2, P_{\mathcal{G}_2} g_1\rangle_{H^2(\mathbb{C}^2)} = \langle g_2, P_{\mathcal{G}_2} g_1\rangle_{\mathcal{G}_2} \\ &= \langle g_2, P_{\mathcal{G}_2}|_{\mathcal{G}_1} g_1\rangle_{\mathcal{G}_2}, \end{aligned}$$

and so $(P_{\mathcal{G}_1}|_{\mathcal{G}_2})^* = P_{\mathcal{G}_2}|_{\mathcal{G}_1}$. Thus $\ker((P_{\mathcal{G}_1}|_{\mathcal{G}_2})^*) = \ker(P_{\mathcal{G}_2}|_{\mathcal{G}_1}) = \mathcal{G}_1 \cap \mathcal{G}_2^\perp$. So if $P_{\mathcal{G}_1}|_{\mathcal{G}_2}$ is a Fredholm operator [11, §2.5.1,p.218], then its Fredholm index is given by $\dim(\mathcal{G}_2 \cap \mathcal{G}_1^\perp) - \dim(\mathcal{G}_1 \cap \mathcal{G}_2^\perp)$.

We will use the following result from [10, p.201].

Lemma 4.4 (Lemma on Closed Subspaces)**.** *Let H be a Hilbert space and let U, V be subspaces of H. Then the following are equivalent:*

(S1) *$U \cap V^\perp = \{0\}$.*
(S2) *Closure of $P_U V$ is U.*

Also, the following are equivalent:

(S3) *$P_U V = U$ and $V \cap U^\perp = \{0\}$.*
(S4) *$\|(I - P_V)P_U\| < 1$ and $\|(I - P_U)P_V\| < 1$.*

Lemma 4.5. *Let $p_1, p_2 \in \mathbb{S}(\mathcal{A}_+)$. Then $d_g(p_1, p_2) < 1$ if and only if the following three conditions hold:*

(1) *$P_{\mathcal{G}_1}|_{\mathcal{G}_2}$ is Fredholm,*
(2) *$\mathcal{G}_1 \cap \mathcal{G}_2^\perp = \{0\}$, and*
(3) *$\mathcal{G}_2 \cap \mathcal{G}_1^\perp = \{0\}$.*

Proof. (Only if) As $P_{\mathcal{G}_1}|\mathcal{G}_2$ is Fredholm, its range is closed, that is, $P_{\mathcal{G}_1}\mathcal{G}_2$ is a closed subspace. Hence from the equivalence of (S1) with (S2) in Lemma 4.4 above, we have that the closure of $P_{\mathcal{G}_1}\mathcal{G}_2$, which is the same as $P_{\mathcal{G}_1}\mathcal{G}_2$, is equal to $\mathcal{G}_1$. Now from the equivalence of (S3) with (S4) in Lemma 4.4, we obtain that $\vec{\delta}(p_1, p_2) = \|(I - P_{\mathcal{G}_2})P_{\mathcal{G}_1}\| < 1$ and $\vec{\delta}(p_2, p_1) = \|(I - P_{\mathcal{G}_1})P_{\mathcal{G}_2}\| < 1$. Hence $d_g(p_1, p_2) < 1$.

(If) As $\vec{\delta}(p_1, p_2) = \|(I - P_{\mathcal{G}_2})P_{\mathcal{G}_1}\| < 1$ and $\vec{\delta}(p_2, p_1) = \|(I - P_{\mathcal{G}_1})P_{\mathcal{G}_2}\| < 1$, by the equivalence of (S3) with (S4) in Lemma 4.4, we obtain $P_{\mathcal{G}_1}\mathcal{G}_2 = \mathcal{G}_1$, and so the range of $P_{\mathcal{G}_1}|_{\mathcal{G}_2}$ is closed. Moreover, $\mathcal{G}_2 \cap \mathcal{G}_1^\perp = \{0\}$. By interchanging the roles of p_1 and p_2, we also get that $\mathcal{G}_1 \cap \mathcal{G}_2^\perp = \{0\}$. □

The following is easy to check.

Lemma 4.6. *Let H_1, H_2 be Hilbert spaces and $T \in \mathcal{L}(H_1, H_2)$, $S \in \mathcal{L}(H_2, H_1)$ be such that $ST = I$. Suppose that U is a subspace of H_1. Then we have that TU is closed if and only if U is closed.*

Proof. (If) Since T is left invertible, $\|x\| = \|STx\| \leq \|S\|\|Tx\|$ $(x \in H_1)$. Suppose $(y_n) = (Tx_n)$ $(x_n \in U)$ is a sequence that converges in H_2. Thus $\|y_n - y_m\| \geq \frac{1}{\|S\|}\|x_n - x_m\|$, showing that (x_n) must converge to some $x \in H_1$. As U is closed, $x \in U$. Thus $y_n = Tx_n \to Tx \in TU$. Hence TU is closed.

(Only if) Now suppose that TU is closed. If (x_n) is a sequence in U that converges to x in H_1, then clearly $Tx_n \to Tx$. But TU is closed, and so $Tx \in TU$. Hence $Tx = Tx'$ for some $x' \in U$. Operating by S, we have $x = STx = STx' = x'$, and so $x = x' \in U$. Thus U is closed. □

For $X \in (L^\infty)^{p\times m}$, T_X denotes the *Toeplitz operator* from $(H^2)^m$ to $(H^2)^p$, given by $T_X\varphi = \Pi_{(H^2)^p}(X\varphi)$ $(\varphi \in (H^2)^m)$, where $X\varphi$ is considered as an element of $(L^2)^p$ and $\Pi_{(H^2)^p}$ denotes the canonical orthogonal projection from $(L^2)^p$ onto $(H^2)^p$.

Lemma 4.7. *Let $p_1, p_2 \in \mathbb{S}(\mathcal{A}_+)$. Then $P_{\mathcal{G}_1}|_{\mathcal{G}_2}$ is Fredholm if and only if $T_{G_1^*G_2}$ is Fredholm. Moreover, their Fredholm indices coincide.*

Proof. First of all, we note that $T_{G_1^*G_2} = T_{G_1^*}T_{G_2}$ (since G_2 has H^∞ entries). Also, it can be checked that for a matrix X with L^∞ entries $(T_X)^* = T_{X^*}$. Thus $(T_{G_1^*G_2})^* = T_{G_2^*G_1}$.

As T_{G_1} is an isometry, it follows that the orthogonal projection onto the range of T_{G_1}, namely the subspace $\mathcal{G}_1$, is given by $T_{G_1}(T_{G_1})^* = T_{G_1}T_{G_1^*}$. Indeed, with $P := T_{G_1}T_{G_1^*}$, and using $G_1^*G_1 = 1$, we can check that $P^2 = P$, that $P^* = P$ and that P maps onto the range of T_{G_1}:

$$\text{ran } (T_{G_1}T_{G_1^*}) \subset \text{ran } T_{G_1} = \text{ran } (T_{G_1}T_{G_1^*}T_{G_1}) \subset \text{ran } (T_{G_1}T_{G_1^*}).$$

We have that

$$\begin{aligned}
\ker(T_{G_1^*}T_{G_2}) &= \{\varphi \in H^2 : T_{G_1^*}T_{G_2}\varphi = 0\} \\
&= \{\varphi \in H^2 : T_{G_1}T_{G_1^*}T_{G_2}\varphi = 0\} \quad (\text{since } \begin{bmatrix} x_1 & y_1 \end{bmatrix} G_1 = 1) \\
&= \{\varphi \in H^2 : P_{\mathcal{G}_1}T_{G_2}\varphi = 0\} = \{\varphi \in H^2 : T_{G_2}\varphi \in \mathcal{G}_1^\perp\}.
\end{aligned}$$

Consider the map $\iota : \ker(T_{G_1^*}T_{G_2}) \to \mathcal{G}_1^\perp \cap \mathcal{G}_2$ defined by $\iota(\varphi) = T_{G_2}\varphi$ for $\varphi \in \ker(T_{G_1^*}T_{G_2})$. From the above calculation, we see that ι is onto. Also, since $\begin{bmatrix} x_2 & y_2 \end{bmatrix} G_2 = 1$ it follows that ι is one-to-one. So ι is invertible.

The above shows that in case that $P_{\mathcal{G}_1}|_{\mathcal{G}_2}$ and $T_{G_1^*G_2}$ are both Fredholm operators, their Fredholm indices will coincide.

In light of the above, we just need to show that the range of $P_{\mathcal{G}_1}|_{\mathcal{G}_2}$ is closed if and only if the range of $T_{G_1^*G_2}$ is closed. The range of $P_{\mathcal{G}_1}|_{\mathcal{G}_2}$ is

$$P_{\mathcal{G}_1}\mathcal{G}_2 = P_{\mathcal{G}_1}\text{ran } T_{G_2} = T_{G_1}T_{G_1^*}\text{ran } T_{G_2} = T_{G_1}\text{ran } T_{G_1^*G_2}.$$

Since G_1 has a left inverse $\begin{bmatrix} x_1 & y_1 \end{bmatrix} \in \mathcal{A}_+^2$, it follows that T_{G_1} is left-invertible. By Lemma 4.6, the range of $P_{\mathcal{G}_1}|_{\mathcal{G}_2}$ is closed if and only if the range of ran $T_{G_1^*G_2}$ is closed. □

We will need the following result, which follows from [3, Thm. 3, p.150]. Here C_0 denotes the set of continuous functions on $\mathbb{R}$ that vanish at $\pm\infty$.

Proposition 4.8. *Let $F = f + g$, where $f \in AP$ and $g \in C_0$ be such that T_F is Fredholm. Then the following hold:*

(1) *T_f is invertible.*
(2) *$F \in \text{inv}\,(AP + C_0)$.*
(3) *The Fredholm index of T_F is the winding number of $1 + f^{-1}g$.*

Proof. Since T_F is invertible modulo the compacts, it is invertible modulo any bigger ideal which we can take to be the kernel of the symbol map from the Toeplitz C^*-algebra $\mathcal{T}(AP + C_0)$ (generated by T_φ for $\varphi \in AP + C_0$) to $AP + C_0$. Consequently, there must exist $\epsilon > 0$ such that $|f + g| > \epsilon$ on all of $\mathbb{R}$.

Since g is in C_0, it follows that by choosing a large enough we can assume that $|g(x)| < \epsilon/2$ for $x > a$ and hence $|f(x)| > \epsilon/2$ for $x > a$. Since $f \in AP$, it follows that $f(x) \neq 0$ for all $x \in \mathbb{R}$. Therefore f is invertible in AP. Moreover, using [3, Theorem 3, p.150], one knows that its generalized index is $(0, n)$ for some integer n and hence the average winding number of f is zero. Thus T_f is invertible [4, Theorem 11, p.25].

Again using [3, Theorem 3, p.150], one can see that the generalized index of T_F equals the sum of the generalized indices of T_f and $T_{1+f^{-1}g}$. But the generalized index of T_f is $(0, 0)$ which completes the proof. □

Proposition 4.9. *If $p_1, p_2 \in \mathbb{S}(\mathcal{A}_+)$, then*

$$d_g(p_1, p_2) = \inf_{q \in \text{inv}\,\mathcal{A}_+} \|G_1 - G_2 q\|_\infty$$

Proof. $\underline{1}^\circ$ Consider first the case when $d_g(p_1, p_2) < 1$. From Lemma 4.5, it follows that $P_{\mathcal{G}_1}|_{\mathcal{G}_2}$ is Fredholm, $\mathcal{G}_1 \cap \mathcal{G}_2^\perp = \{0\}$ and $\mathcal{G}_2 \cap \mathcal{G}_1^\perp = \{0\}$. Furthermore, the Fredholm index of $P_{\mathcal{G}_1}|_{\mathcal{G}_2}$ is 0. By Lemma 4.7, $T_{G_1^* G_2}$ is Fredholm, with Fredholm index 0 too. From Proposition 4.8, it follows that $G_1^* G_2$ is invertible as an element of $AP + C_0$. Thus it is also invertible as an element of $\mathcal{A}$. Also, $W(G_1^* G_2) = (0, 0)$. Now suppose that there is a $q_0 \in \mathcal{A}_+$ such that $\|G_1 - G_2 q_0\|_\infty < 1$. Then $\|I - G_1^* G_2 q_0\|_\infty < 1$ and so $G_1^* G_2 q_0 = 1 - (1 - G_1^* G_2 q_0)$ is invertible in $\mathcal{A}$. Hence $G_1^* G_2 q_0 \in \text{inv}\,\mathcal{A}$. In particular, $q_0 \in \text{inv}\,\mathcal{A}$. Consider the map $H : [0, 1] \to \text{inv}\,\mathcal{A}$ given by $H(t) = 1 - t(1 - G_1^* G_2 q_0)$, $t \in [0, 1]$. By the homotopic invariance,

$$(0, 0) = W(1) = W(H(0)) = W(H(1)) = W(G_1^* G_2 q_0).$$

Since $W(G_1^* G_2) = (0, 0)$, it follows that $W(q_0) = (0, 0)$. Thus by (I4), we obtain that $q_0 \in \text{inv}\,\mathcal{A}_+$. Consequently,

$$1 > d_g(p_1, p_2) = \vec{\delta}(p_1, p_2) = \inf_{q \in \mathcal{A}_+} \|G_1 - G_2 q\|_\infty = \inf_{q \in \text{inv}\,\mathcal{A}_+} \|G_1 - G_2 q\|_\infty.$$

$\underline{2}^\circ$ Now suppose that $d_g(p_1, p_2) = 1$, but that $\vec{\delta}(p_1, p_2) < 1$. Since we have $\vec{\delta}(p_1, p_2) = \|(I - P_{\mathcal{G}_2}) P_{\mathcal{G}_1}\|$, we obtain $\mathcal{G}_1 \cap \mathcal{G}_2^\perp = \{0\}$. For otherwise, if $0 \neq v \in \mathcal{G}_1 \cap \mathcal{G}_2^\perp$, then we have $(I - P_{\mathcal{G}_2}) P_{\mathcal{G}_1} v = v$, and so we would obtain that $\|(I - P_{\mathcal{G}_2}) P_{\mathcal{G}_1}\| \geq \|(I - P_{\mathcal{G}_2}) P_{\mathcal{G}_1} v\| / \|v\| = 1$, a contradiction.

From Lemma 4.5, it now follows that either $\mathcal{G}_2 \cap \mathcal{G}_1^\perp \neq \{0\}$ or $P_{\mathcal{G}_1}|_{\mathcal{G}_2}$ is not Fredholm.

Suppose first that $P_{\mathcal{G}_1}|_{\mathcal{G}_2}$ is Fredholm. Then we must have $\mathcal{G}_2 \cap \mathcal{G}_1^\perp \neq \{0\}$. This gives that the Fredholm index of $P_{\mathcal{G}_1}|_{\mathcal{G}_2}$, namely

$$\dim(\mathcal{G}_2 \cap \mathcal{G}_1^\perp) - \dim(\mathcal{G}_1 \cap \mathcal{G}_2^\perp) = \dim(\mathcal{G}_2 \cap \mathcal{G}_1^\perp) - 0 = \dim(\mathcal{G}_2 \cap \mathcal{G}_1^\perp),$$

is nonzero. By Lemma 4.7, $T_{G_1^* G_2}$ is Fredholm, with Fredholm index nonzero too. It now follows from Proposition 4.8, that $W(G_1^* G_2) = (*, n)$ with the integer $n \neq 0$. By the definition of d_ν, $d_\nu(p_1, p_2) = 1$.

Next assume that $P_{\mathcal{G}_1}|_{\mathcal{G}_2}$ is not Fredholm. Then Lemma 4.7 gives that $T_{G_1^* G_2}$ is not Fredholm either. Now if $G_1^* G_2$ is not invertible in $\mathcal{A}$, then we have $d_\nu(p_1, p_2) = 1$ by definition. On the other hand, if $G_1^* G_2 \in \text{inv } \mathcal{A}$ and $W(G_1^* G_2) = (0, 0)$, it follows from [4, Proposition 6.3, p.27] that $T_{G_1^* G_2}$ is invertible, a contradiction. Thus $W(G_1^* G_2) = (0, 0)$, and so $d_\nu(p_1, p_2) = 1$ in this case as well.

Now that we have obtained $d_\nu(p_1, p_2) = 1$, it follows that there is no $q \in \text{inv } \mathcal{A}_+$ such that $\|G_1 - G_2 q\|_\infty < 1$. In other words, for each $q \in \mathcal{A}_+$, $\|G_1 - G_2 q\|_\infty \geq 1$. Also, $\|G_1 - G_2 q\|_\infty \leq \|G_1\|_\infty + \|G_2\|_\infty \|q\|_\infty \leq 1 + 1 \cdot \|q\|_\infty$, and by taking $q_1 = \frac{1}{n} I \in \text{inv } \mathcal{A}_+$, we obtain

$$\inf_{q \in \text{inv } \mathcal{A}_+} \|G_1 - G_2 q\|_\infty \leq \inf_n \left(1 + \frac{1}{n}\right) = 1.$$

Consequently, $\inf\limits_{q \in \text{inv } \mathcal{A}_+} \|G_1 - G_2 q\|_\infty = 1 = d_g(p_1, p_2)$.

<u>3</u>° Now suppose that $d_g(p_1, p_2) = 1 = \vec{\delta}(p_1, p_2) = \vec{\delta}(p_2, p_1)$. We have

$$\inf_{q \in \text{inv } \mathcal{A}_+} \|G_1 - G_2 q\|_\infty \leq \inf_n \left\|G_1 - G_2 \frac{1}{n} I\right\|_\infty \leq \inf_n \left(1 + \frac{1}{n}\right) = 1.$$

Also, $1 = \vec{\delta}(p_1, p_2) = \inf\limits_{q \in \mathcal{A}_+} \|G_1 - G_2 q\|_\infty \leq \inf\limits_{q \in \text{inv } \mathcal{A}_+} \|G_1 - G_2 q\|_\infty$. Thus

$$\inf_{q \in \text{inv } \mathcal{A}_+} \|G_1 - G_2 q\|_\infty = 1 = d_g(p_1, p_2).$$

This completes the proof. □

5. Equivalence of the ν-metric and the gap-metric

Proof of Theorem 1.1. We will show the following for $p_1, p_2 \in \mathbb{S}(\mathcal{A}_+)$:

$$d_g(p_1, p_2) \mu_{\text{opt}}(p_1) \leq d_\nu(p_1, p_2) \leq d_g(p_1, p_2), \tag{5.1}$$

where $\mu_{\text{opt}}(p_1) := \sup\limits_c \mu_{p_1, c}$.

This will prove the fact that the topologies induced by metrics d_g and d_ν on the set $\mathbb{S}(\mathcal{A}_+)$ are identical.

The second inequality in (5.1) is an immediate consequence of the Propositions 3.6 and 3.6. Indeed, we have

$$\begin{aligned} d_\nu(p_1,p_2) &= \inf_{\substack{q\in \text{inv } \mathcal{A}, \\ W(q)=(0,0)}} \|G_1-G_2q\|_\infty \\ &\leq \inf_{\substack{q\in \mathcal{A}_+\cap(\text{inv } \mathcal{A}), \\ W(q)=(0,0)}} \|G_1-G_2q\|_\infty \\ &= \inf_{q\in \text{inv } \mathcal{A}_+} \|G_1-G_2q\|_\infty \quad \text{(using (I4))} \\ &= d_g(p_1,p_2). \end{aligned}$$

Now we will show the first inequality in (5.1). This inequality is trivially satisfied if $d_\nu(p_1,p_2)\geq \mu_{\text{opt}}(p_1)$, since $d_g(p_1,p_2)\leq 1$. So we will only consider the case when $d_\nu(p_1,p_2)<\mu_{\text{opt}}(p_1)$. Thus we can choose a c that stabilizes both p_1 and p_2. (Since the above inequality shows that there exists a c_0 stabilizing p_1 such that $d_\nu(p_1,p_2)<\mu_{p_1,c_0}$. But by Theorem 3.5, it follows that c_0 also stabilizes p_2.) If we now define $q_0:=(\widetilde{K}_0G_1)^{-1}\widetilde{K}_0G_2$, then we have $G_2-G_1q_0=G_2-G_1(\widetilde{K}_0G_1)^{-1}\widetilde{K}_0G_2=(I-G_1(\widetilde{K}_0G_1)^{-1}\widetilde{K}_0)G_2$. Also

$$I-\left[\begin{array}{c} p_1 \\ 1 \end{array}\right](1-c_0p_1)^{-1}\left[\begin{array}{cc} -c_0 & 1 \end{array}\right]=\left[\begin{array}{c} 1 \\ c_0 \end{array}\right](1-p_1c_0)^{-1}\left[\begin{array}{cc} 1 & -p_1 \end{array}\right].$$

that is, $I-G_1(\widetilde{K}_0G_1)^{-1}\widetilde{K}_0=K_0(\widetilde{G}_1K_0)^{-1}\widetilde{G}_1$. Thus

$$G_2-G_1q_0=K_0(\widetilde{G}_1K_0)^{-1}\widetilde{G}_1G_2.$$

Then we use $\|K_0\|\leq 1$ (since $K_0^*K_0=1$) to obtain

$$\begin{aligned} \|G_2-G_1q_0\|_\infty &= \|K_0(\widetilde{G}_1K_0)^{-1}\widetilde{G}_1G_2\|_\infty \\ &\leq \|K_0\|_\infty\|(\widetilde{G}_1K_0)^{-1}\widetilde{G}_1G_2\|_\infty \\ &\leq 1\cdot\|(\widetilde{G}_1K_0)^{-1}\widetilde{G}_1G_2\|_\infty \\ &\leq \|(\widetilde{G}_1K_0)^{-1}\|_\infty\|\widetilde{G}_1G_2\|_\infty. \end{aligned}$$

As for each c, $\mu_{p_1,c}\leq 1$, we have $\mu_{\text{opt}}(p_1)\leq 1$. So $d_\nu(p_1,p_2)<\mu_{\text{opt}}(p_1)\leq 1$, and we obtain $d_\nu(p_1,p_2)=\|\widetilde{G}_1G_2\|_\infty$.

From [1, Propositions 4.2,4.5], $\|(\widetilde{G}_1K_0)^{-1}\|_\infty=1/\mu_{c_0,p_1}=1/\mu_{p_1,c_0}$. So

$$\|G_2-G_1q_0\|_\infty\leq\|(\widetilde{G}_1K_0)^{-1}\|_\infty\|\widetilde{G}_1G_2\|_\infty\leq\frac{d_\nu(p_1,p_2)}{\mu_{p_1,c_0}}.$$

Thus

$$d_g(p_1,p_2)=\inf_{q\in\text{inv } \mathcal{A}}\|G_1-G_2q\|_\infty\leq\|G_1-G_2q_0\|\leq d_\nu(p_1,p_2)/\mu_{p_1,c_0}.$$

This inequality holds for any c_0 that stabilizes p_1 for which there holds $d_\nu(p_1,p_2)<\mu_{p_1,c_0}$. We can choose a sequence $(c_{0,n})$ such $\mu_{p_1,c_{0,n}}\to\mu_{\text{opt}}(p_1)$ as $n\to\infty$. Thus $d_g(p_1,p_2)\leq d_\nu(p_1,p_2)/\mu_{\text{opt}}(p_1)$. This completes the proof the first inequality in (5.1). $\square$

The question of whether our main result, Theorem 1.1 remains true for systems with multiple inputs and multiple outputs (as opposed to just *scalar* inputs and outputs) is open. A key technical difficulty is the validity of the analogue of Proposition 4.8 for matricial data.

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References

[1] J.A. Ball and A.J. Sasane. Extension of the ν-metric. *Complex Analysis and Operator Theory*, to appear.

[2] A. Brudnyi and A.J. Sasane. Sufficient conditions for the projective freeness of Banach algebras. *Journal of Functional Analysis*, in press.

[3] R.G. Douglas. On the C^*-algebra of a one-parameter semigroup of isometries. *Acta Mathematica*, 128:143-151, no. 3-4, 1972.

[4] R.G. Douglas. *Banach algebra techniques in the theory of Toeplitz operators.* Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 15. American Mathematical Society, Providence, R.I., 1973.

[5] A.K. El-Sakkary. The gap metric: robustness of stabilization of feedback systems. *IEEE Transactions on Automatic Control*, 30:240-247, no. 3, 1985.

[6] T.T. Georgiou. On the computation of the gap metric. *Systems Control Letters*, 11:253-257, no. 4, 1988.

[7] T.T. Georgiou and M.C. Smith. Optimal robustness in the gap metric. *IEEE Transactions on Automatic Control*, 35:673-686, no. 6, 1990.

[8] B. Jessen and H. Tornehave. Mean motions and zeros of almost periodic functions. *Acta Mathematica*, 77:137-279, 1945.

[9] K.M. Mikkola. *Infinite-dimensional linear systems, optimal control and algebraic Riccati equations.* Doctoral dissertation, Technical Report A452, Institute of Mathematics, Helsinki University of Technology, 2002.

[10] N.K. Nikolski. *Treatise on the shift operator. Spectral function theory. With an appendix by S.V. Khrushchëv and V. V. Peller.* Grundlehren der Mathematischen Wissenschaften 273, Springer-Verlag, Berlin, 1986.

[11] N.K. Nikolski. *Operators, functions, and systems: an easy reading. Volume 1.* Mathematical Surveys and Monographs, 92. American Mathematical Society, Providence, RI, 2002.

[12] J.R. Partington. *Linear operators and linear systems. An analytical approach to control theory.* London Mathematical Society Student Texts 60, Cambridge University Press, Cambridge, 2004.

[13] M. Vidyasagar. The graph metric for unstable plants and robustness estimates for feedback stability. *IEEE Transactions on Automatic Control*, 29:403-418, no. 5, 1984.

[14] G. Vinnicombe. Frequency domain uncertainty and the graph topology. *IEEE Transactions on Automatic Control*, no. 9, 38:1371-1383, 1993.

[15] G. Zames and A.K. El-Sakkary. Unstable systems and feedback: The gap metric. In *Proceedings of the Allerton Conference*, 380-385, Oct. 1980.

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