# THE $GL_n(q)$-MODULE STRUCTURE OF THE SYMMETRIC ALGEBRA AROUND THE STEINBERG MODULE

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ABSTRACT. We determine the graded composition multiplicity in the symmetric algebra $S^\bullet(V)$ of the natural $GL_n(q)$-module $V$, or equivalently in the coinvariant algebra of $V$, for a large class of irreducible modules around the Steinberg module. This was built on a computation, via connections to algebraic groups, of the Steinberg module multiplicity in a tensor product of $S^\bullet(V)$ with other tensor spaces of fundamental weight modules.

## 1. INTRODUCTION

The symmetric algebra $S^\bullet(V)$ is naturally a graded module over the finite general linear group $GL_n(q)$, where $V = \mathbb{F}^n$ is the standard $GL_n(q)$-module over an algebraically closed field $\mathbb{F}$ of characteristic $p$ and $q = p^r$ for $r \geq 1$. Dickson's classical theorem [Di] states that the algebra of $GL_n(q)$-invariants in $S^\bullet(V)$ is a polynomial algebra in $n$ generators, and this work served as the starting point of all the subsequent works on the $GL_n(q)$-module structure of $S^\bullet(V)$ and closely related modules.

The composition multiplicity of the Steinberg module St in $S^\bullet(V) \otimes \wedge^\bullet(V) \otimes \mathrm{Det}^k$, where $\wedge^\bullet(V)$ denotes the exterior algebra of $V$ and Det denotes the determinant module, has been determined in various special cases by Kuhn, Mitchell, and Priddy [KM, Mi, MP] and in full generality by the authors [WW]. The topological approach of [Mi, MP] using Steenrod algebra worked only in a prime field and it is not clear how to develop further along this line. On the other hand, the approach of [KM, WW] is based on a modular version of a formula of Curtis in terms of parabolic subgroup invariants (for closely related work see [Mui, MT]). These parabolic subgroup invariants were determined in a constructive manner, and it seems difficult to extend the approach much further.

By a basic observation of Mitchell [Mi], finding the graded multiplicity of a simple module $L$ in the symmetric algebra $S^\bullet(V)$ is equivalent to finding the graded composition multiplicity of $L$ in the coinvariant algebra of $V$ which is a graded regular representation of $GL_n(q)$. A full answer for every simple $GL_n(q)$-module is beyond the reach for now as it would imply the degrees of all principal indecomposable modules (PIMs).

The main goal of the paper is to find an elegant closed formula for the graded composition multiplicity in $S^\bullet(V)$ for a large class of simple $GL_n(q)$-modules around the Steinberg module (i.e., simple modules of highest weights not far from the Steinberg weight $(q-1)\rho$). The twisting by the determinant module plays an important role in this paper.

Our new approach is based on the intimate and deep connections between representations of $GL_n(q)$ and of the algebraic group $GL_n(\mathbb{F})$, and it is a two-step process. First, via connections to algebraic groups, we compute the graded composition multiplicity of St in various tensor modules of the form $S^\bullet(V) \otimes N$ for some natural $GL_n(\mathbb{F})$-modules

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$N$. Secondly, such a composition multiplicity of St when combined with classical results on PIMs of $GL_n(q)$ are used to derive a closed formula for the graded composition multiplicity in $S^\bullet(V)$ for a large class of simple $GL_n(q)$-modules around the Steinberg module (up to twists by Det).

Let us explain in some detail. Bendel, Nakano and Pillen [BNP] has recently developed an amazing link between the Ext-groups of $GL_n(q)$ and of $GL_n(\mathbb{F})$, and used it to find upper bounds for cohomology of finite groups. As explained to us by Pillen (see Section 2), the machinery of [BNP] can be used effectively to transform the problem of computing the Steinberg module multiplicity in a rational $GL_n(\mathbb{F})$-module with a good filtration (viewed as a $GL_n(q)$-module) into a problem of counting multiplicities in infinitely many rational $GL_n(\mathbb{F})$-modules with good filtrations. By a classical result of J.-P. Wang [Wa], the latter becomes essentially a highly nontrivial combinatorial problem of counting multiplicities of irreducible characters in characteristic zero. In our cases of interest, the intricate combinatorial problem can be eventually solved with a key tool being the Pieri's formula. In this way, we are able to determine the graded multiplicity of the Steinberg module in $S^\bullet(V) \otimes \wedge^m(V) \otimes \mathrm{Det}^k$ (see Theorem 3.1) and more generally in $S^\bullet(V) \otimes \wedge^\nu(V) \otimes \mathrm{Det}^k$ for suitable partitions $\nu$ and suitable $k$ (see Theorem 3.4). Theorem 3.1 recovers in a different form one of the main results in [WW, Theorem C].

Note that $\mathrm{Hom}_{GL_n(q)}(\mathrm{St}, S^\bullet(V) \otimes N) \cong \mathrm{Hom}_{GL_n(q)}(\mathrm{St} \otimes N^*, S^\bullet(V))$ for a finite dimensional $GL_n(q)$-module $N$, and that $\mathrm{St} \otimes N^*$ is projective. The results of Ballard on PIMs [Ba] (which was inspired by Humphreys and Verma [HV] and improved by Chastkofsky [Ch] and Jantzen [J1]) allow us to find an explicit decomposition of $\mathrm{St} \otimes N^*$ for suitable $N$ into a direct sum of PIMs. We derive from this and Theorem 3.4 a closed formula for the graded composition multiplicity in $S^\bullet(V)$ for a large class of simple modules around the Steinberg module; see Theorem 4.7. In light of an observation in [Mi], Theorem 4.7 affords an equivalent reformulation in terms of the coinvariant algebra of $V$ in place of $S^\bullet(V)$; see Theorem 4.10. Also, from Theorem 3.1 and results of Tsushima [Ts] on PIMs (a special case of which goes back to Lusztig [Lu]), we recover the main results of Carlisle and Walker [CW], who obtained a multiplicity formula for several simple modules very close to St in $S^\bullet(V)$ using an ingenious combinatorial and semigroup approach.

Our work opens a new and effective way of studying the $GL_n(q)$-module structure of $S^\bullet(V)$ via its connection to algebraic groups. At the end of the paper, we formulate several open problems, and speculate a formula on the composition multiplicity in the socle of $S^\bullet(V)$ for a family of simple modules $L(\mu) \otimes \mathrm{Det}^k$.

The paper is organized as follows. In Section 2 we recall the basics of the algebraic group $GL_n(\mathbb{F})$ and of the finite group $GL_n(q)$ (a basic reference in this direction is the book of Humphreys [Hu]), and formulate a key formula derived from [BNP]. We determine the graded multiplicity of St in the tensor products of $S^\bullet(V)$ with various natural $GL_n(q)$-modules in Section 3. This is then applied in Section 4 to determine an explicit formula for the graded composition multiplicity in $S^\bullet(V)$, or equivalently in the coinvariant algebra of $V$, for a large class of simple $GL_n(q)$-modules.

## 2. THE PRELIMINARIES

2.1. **Finite group $GL_n(q)$ and algebraic group $GL_n(\mathbb{F})$.** Let $GL_n(\mathbb{F})$ be the general linear group over an algebraically closed field $\mathbb{F}$ of prime characteristic $p > 0$. Let $T$ be the maximal torus consisting of diagonal matrices in $GL_n(\mathbb{F})$ and $B$ be the Borel subgroup consisting of upper triangular matrices. Denote by $\Phi^+$ (resp. $\Phi^-$) the corresponding positive (resp. negative) root system. Then we have the Weyl group $W = S_n$, the set of simple roots $\Pi = \{\alpha_1, \dots, \alpha_{n-1}\}$ and the normalized bilinear form satisfying $(\alpha, \alpha) = 2$ for $\alpha \in \Phi$. Let $X = X(T)$ be the integral weight lattice which can be identified with $\mathbb{Z}^n$ and denote the set of dominant integral weights by

$$X^+ = \{\lambda \mid \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n, \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}.$$

For $\lambda \in X^+$, there exists a simple $GL_n(\mathbb{F})$-module $L(\lambda)$ of highest weight $\lambda$. These $GL_n(\mathbb{F})$-modules are pairwise non-isomorphic and exhaust the isomorphism classes of simple $GL_n(\mathbb{F})$-modules. For $\lambda \in X^+$, let $\nabla(\lambda) := \text{ind}_B^{GL_n(\mathbb{F})} \lambda$ be the induced module and $\Delta(\lambda) := \nabla(-w_0 \lambda)^*$ be the Weyl module of highest weight $\lambda$, where $w_0$ is the longest element in $W$. It is known that $\nabla(\lambda)$ has a unique simple submodule isomorphic to $L(\lambda)$ and $\Delta(\lambda)$ has a unique simple quotient isomorphic to $L(\lambda)$.

Let Fr : $GL_n(\mathbb{F}) \to GL_n(\mathbb{F})$ denote the Frobenius map, and let $q = p^r$ for $r \geq 1$. The fixed point subgroup of the $r$th iterate of the Frobenius map can be identified with $GL_n(q)$. Denote the set of $q$-restricted weights in $X^+$ by

$$X_r = \{\lambda \in X^+ \mid (\lambda, \alpha_i) < q, 1 \leq i \leq n-1\}$$

The restrictions to $GL_n(q)$ of the simple $GL_n(\mathbb{F})$-modules $L(\lambda)$ with $\lambda \in X_r$ form a complete set of pairwise non-isomorphic simple $GL_n(q)$-module (cf. [Hu], [J2, II.3]). In particular, the restriction of $L((q-1)\rho)$ to $GL_n(q)$ is called the Steinberg module and denoted by St $=$ St$_r$, where $\rho = (n-1, n-2, \dots, 1, 0)$. We shall also write

(2.1) $$\rho_i = n - i, \qquad i = 1, \dots, n.$$

Recall that a $GL_n(\mathbb{F})$-module $N$ has a good filtration (also called a $\nabla$-filtration) if it admits a filtration with successive quotients of the form $\nabla(\lambda)$, $\lambda \in X^+$ [J2, II 4.16]. Denote by $[N : \nabla(\lambda)]$ the multiplicity of $\nabla(\lambda)$ appearing in a good filtration of $N$. We have the following lemma (cf. [J2, II, Proposition 4.16]).

**Lemma 2.1.** *Let $N$ be a $GL_n(\mathbb{F})$-module admitting a good filtration. Then, for each $\lambda \in X^+$,*

$$[N : \nabla(\lambda)] = \dim \text{Hom}_{GL_n(\mathbb{F})}(\Delta(\lambda), N),$$
$$\text{Ext}^i(\Delta(\lambda), N) = 0, \quad \forall i \geq 1.$$

The following fundamental result is due to J.-P. Wang [Wa] (cf. [J2, II, Proposition 4.19]).

**Lemma 2.2.** [Wa] *If $N$ and $N'$ are $GL_n(\mathbb{F})$-modules admitting good filtrations, then the tensor product $N \otimes N'$ also has a good filtration.*

2.2. **Relating $GL_n(q)$ to $GL_n(\mathbb{F})$.** Define the induced $GL_n(\mathbb{F})$-module

$$\mathcal{G}_r(\mathbb{F}) = \text{ind}_{GL_n(q)}^{GL_n(\mathbb{F})}(\mathbb{F}).$$

The basic properties of $\mathcal{G}_r(\mathbb{F})$ (for more general reductive groups) were described by Bendel, Nakano and Pillen, and then used to give upper bounds on dimensions of cohomology of finite groups of Lie type.

**Lemma 2.3.** [BNP, Proposition 2.3] *Let $M, N$ be rational $GL_n(\mathbb{F})$-modules. Then,*

$$\text{Ext}^i_{GL_n(q)}(M, N) \cong \text{Ext}^i_{GL_n(\mathbb{F})}(M, N \otimes \mathcal{G}_r(\mathbb{F})), \quad \forall i \geq 0.$$

**Lemma 2.4.** [BNP, Proposition 2.4] *As a $GL_n(\mathbb{F})$-module, $\mathcal{G}_r(\mathbb{F})$ has a filtration with factors $\nabla(\lambda) \otimes \nabla(-w_0\lambda)^{(r)}$ of multiplicity one for each $\lambda \in X^+$.*

The proof of the following useful proposition, which follows from the above results of [BNP], was communicated to us by Cornelius Pillen.

**Proposition 2.5.** *Let $N$ be a finite dimensional rational $GL_n(\mathbb{F})$-module admitting a good filtration. Then*

$$(2.2) \qquad \dim \text{Hom}_{GL_n(q)}(\text{St}, N) = \sum_{\lambda \in X^+} [N \otimes \nabla(\lambda) : \nabla((q-1)\rho + q\lambda)].$$

*Proof.* We first observe by Lemma 2.1 and [J2, II, 3.19] that

$$\text{Ext}^i_{GL_n(\mathbb{F})}(\text{St}, N \otimes \nabla(\lambda) \otimes \nabla(-w_0\lambda)^{(r)})$$

$$\cong \text{Ext}^i_{GL_n(\mathbb{F})}(\text{St} \otimes \Delta(\lambda)^{(r)}, N \otimes \nabla(\lambda))$$

$$\cong \text{Ext}^i_{GL_n(\mathbb{F})}(\Delta((q-1)\rho + q\lambda), N \otimes \nabla(\lambda))$$

$$(2.3) \qquad = \begin{cases} \text{Hom}_{GL_n(\mathbb{F})}(\Delta((q-1)\rho + q\lambda), N \otimes \nabla(\lambda)), & \text{if } i = 0 \\ 0, & \text{if } i > 0. \end{cases}$$

It follows that $\text{Hom}_{GL_n(\mathbb{F})}(\text{St}, N \otimes \nabla(\lambda) \otimes \nabla(-w_0\lambda)^{(r)})$ is nonzero only for finitely many $\lambda \in X^+$.

Let $\mathcal{G}$ be a $GL_n(q)$-module which has a $\nabla(\lambda) \otimes \nabla(-w_0\lambda)^{(r)}$-filtration. For a $GL_n(\mathbb{F})$-submodule $S$ of $\mathcal{G}$ having a $\nabla(\lambda) \otimes \nabla(-w_0\lambda)^{(r)}$-filtration, we define a $GL_n(\mathbb{F})$-module $Q$ by the short exact sequence

$$0 \longrightarrow S \longrightarrow \mathcal{G} \longrightarrow Q \longrightarrow 0.$$

Then $Q \cong \mathcal{G}/M$ also has a $\nabla(\lambda) \otimes \nabla(-w_0\lambda)^{(r)}$-filtration. The short exact sequence induces a long exact sequence with initial terms

$$0 \longrightarrow \text{Hom}_{GL_n(\mathbb{F})}(\text{St}, N \otimes S) \longrightarrow \text{Hom}_{GL_n(\mathbb{F})}(\text{St}, N \otimes \mathcal{G})$$

$$\longrightarrow \text{Hom}_{GL_n(\mathbb{F})}(\text{St}, N \otimes Q) \longrightarrow \text{Ext}^1_{GL_n(\mathbb{F})}(St, N \otimes S)$$

where the last term $\text{Ext}^1$ vanishes by (2.3). Hence we obtain the following identity:

$$\dim \text{Hom}_{GL_n(\mathbb{F})}(\text{St}, N \otimes \mathcal{G}) = \dim \text{Hom}_{GL_n(\mathbb{F})}(\text{St}, N \otimes S) + \dim \text{Hom}_{GL_n(\mathbb{F})}(\text{St}, N \otimes Q).$$

By repeatedly applying this identity, we obtain that

$$(2.4) \quad \dim \mathrm{Hom}_{GL_n(\mathbb{F})}(\mathrm{St}, N \otimes \mathcal{G}_r(\mathbb{F})) = \sum_{\lambda \in X^+} \dim \mathrm{Hom}(\mathrm{St}, N \otimes \nabla(\lambda) \otimes \nabla(-w_0\lambda)^{(r)})$$

where all but finitely many summands on the right-hand side are zero.

By (2.3), (2.4), Lemma 2.1, Lemma 2.2, and Lemma 2.3 (for $M = \mathrm{St}$), we obtain that

$$\dim \mathrm{Hom}_{GL_n(q)}(\mathrm{St}, N) = \sum_{\lambda \in X^+} \dim \mathrm{Hom}(\mathrm{St}, N \otimes \nabla(\lambda) \otimes \nabla(-w_0\lambda)^{(r)})$$

$$= \sum_{\lambda \in X^+} \dim \mathrm{Hom}_{GL_n(\mathbb{F})}(\Delta((q-1)\rho + q\lambda), N \otimes \nabla(\lambda))$$

$$= \sum_{\lambda \in X^+} [N \otimes \nabla(\lambda) : \nabla((q-1)\rho + q\lambda)].$$

The proposition is proved. □

## 3. THE STEINBERG MODULE MULTIPLICITY IN A TENSOR PRODUCT

In this section, we will compute the multiplicity of the Steinberg module St of $GL_n(q)$ in $S^\bullet(V) \otimes \wedge^{\mu'}(V) \otimes \mathrm{Det}^k$ and $S^\bullet(V) \otimes L(\mu) \otimes \mathrm{Det}^k$, for $0 \leq k \leq q-2$ and certain partitions $\mu = (\mu_1, \ldots, \mu_n)$ of length $\leq n$.

### 3.1. **Some notations.** We introduce the following notations:

$$\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in X,$$
$$\omega_m = (1^m) = (1, \ldots, 1, 0, \ldots, 0) \in X^+, \quad 1 \leq m \leq n.$$

For $\lambda \in X^+$, set $|\lambda| = \sum_i \lambda_i$. We denote by $L(\lambda)_\mathbb{C}$ the irreducible representation of highest weight $\lambda$ of the general linear Lie algebra $\mathfrak{gl}_n(\mathbb{C})$, and $[M : L(\lambda)_\mathbb{C}]$ the multiplicity of $L(\lambda)_\mathbb{C}$ in a semisimple $\mathfrak{gl}_n(\mathbb{C})$-module $M$.

Let $\mathbb{Z}_+$ denote the set of nonnegative integers. We denote by $e_k(x_1, \ldots, x_n)$ the $k$-th elementary symmetric polynomial for $k \in \mathbb{Z}_+$, and denote by $e_\nu(x_1, \ldots, x_n)$ the elementary symmetric polynomial associated to a partition $\nu$ whose first part $\nu_1 \leq n$. We denote by $m_\mu(x_1, \ldots, x_n)$ the monomial symmetric polynomial associated to a partition $\mu$ of length $\leq n$. Denote by $\mu'$ the conjugate partition of $\mu$.

For a formal series $f(t) \in \mathbb{Z}[[t]]$, denote by $[t^a]f(t)$ the coefficient of $t^a$ in $f(t)$ for $a \in \mathbb{Z}_+$.

Let Det denote the one-dimensional determinant $GL_n(q)$-module. Note that $\mathrm{Det} \cong \wedge^n(V) \cong \nabla(\omega_n)$, and that $\mathrm{Det}^{q-1}$ is the trivial module.

Recall the Steinberg module St of $GL_n(q)$ is absolutely irreducible and projective. For a graded $GL_n(q)$-module $N^\bullet = \oplus_i N^i$, we denote by

$$H_{\mathrm{St}}(N^\bullet; t) = \sum_i t^i \dim \mathrm{Hom}_{GL_n(q)}(\mathrm{St}, N^i)$$

the graded multiplicity of St in $N^\bullet$. Similarly, we denote by $H_{\mathrm{St}}(N^\bullet; t, s)$ (in two variables $t$ and $s$) the graded multiplicity for St in a bi-graded $GL_n(q)$-module $N^\bullet$.

3.2. **The multiplicity of** St **in** $S^\bullet(V) \otimes \wedge^m(V) \otimes \mathrm{Det}^k$**.** We shall use the connection to the algebraic group $GL_n(\mathbb{F})$ to determine the graded multiplicity of the Steinberg module St in $S^\bullet(V) \otimes \wedge^m(V) \otimes \mathrm{Det}^k$.

**Theorem 3.1.** *Suppose $0 \le m \le n$.*
*(1) If $1 \le k \le q-2$, then the graded multiplicity of St in $S^\bullet(V) \otimes \wedge^m(V) \otimes \mathrm{Det}^k$ is*

$$H_{\mathrm{St}}\big(S^\bullet(V) \otimes \wedge^m(V) \otimes \mathrm{Det}^k; t\big) = \frac{t^{-n+(q-k)\frac{q^n-1}{q-1}}}{\prod_{i=1}^n(1-t^{q^i-1})} \, e_m(t^{-1}, t^{-q}, \ldots, t^{-q^{n-1}}).$$

*(2) The graded multiplicity of St in $S^\bullet(V) \otimes \wedge^m(V)$ is given by*

$$H_{\mathrm{St}}\big(S^\bullet(V) \otimes \wedge^m(V); t\big)$$
$$= \frac{t^{-n+\frac{q^n-1}{q-1}}}{\prod_{i=1}^n(1-t^{q^i-1})}\Big((1-t^{q^n-1}) \, e_m(t^{-1}, t^{-q}, \ldots, t^{-q^{n-2}}) + t^{q^n-1} e_m(t^{-1}, t^{-q}, \ldots, t^{-q^{n-1}})\Big).$$

*Proof.* Let us fix $a \in \mathbb{Z}_+$. Observe that $S^a(V) \cong \nabla(a\omega_1)$, and $\wedge^m(V) \cong \nabla(\omega_m)$. It follows by Lemma 2.2 that $S^a(V) \otimes \wedge^m(V) \otimes \mathrm{Det}^k$ has a good filtration. By Proposition 2.5, we turn the problem into a multiplicity problem in characteristic zero:

$$\dim \mathrm{Hom}_{GL_n(q)}(\mathrm{St}, S^a(V) \otimes \wedge^m(V) \otimes \mathrm{Det}^k)$$
$$= \sum_{\lambda \in X^+} \Big[\nabla(a\omega_1) \otimes \nabla(\omega_m) \otimes \nabla(\omega_n)^{\otimes k} \otimes \nabla(\lambda) : \nabla((q-1)\rho + q\lambda)\Big]$$
$$(3.1) \qquad = \sum_{\lambda \in X^+} \Big[L(a\omega_1)_\mathbb{C} \otimes L(\omega_m)_\mathbb{C} \otimes L(\omega_n)_\mathbb{C}^{\otimes k} \otimes L(\lambda)_\mathbb{C} : L((q-1)\rho + q\lambda)_\mathbb{C}\Big].$$

By applying Pieri's formula twice (cf. [FH, Proposition 15.25]), we deduce that

$$L(a\omega_1)_\mathbb{C} \otimes L(\omega_m)_\mathbb{C} \otimes L(\omega_n)_\mathbb{C}^{\otimes k} \otimes L(\lambda)_\mathbb{C}$$
$$\cong \oplus_{a_i \in \mathbb{Z}_+, a_1 + \cdots + a_n = a, \lambda_i + k + a_i \le \lambda_{i-1}+k} L(\omega_m)_\mathbb{C} \otimes L(\lambda_1 + k + a_1, \ldots, \lambda_n + k + a_n)_\mathbb{C}$$
$$\cong \oplus L((\lambda_1 + k + a_1, \ldots, \lambda_n + k + a_n) + \varepsilon_{i_1} + \cdots + \varepsilon_{i_m})_\mathbb{C},$$

where the summation is over the tuples $(a_1, \ldots, a_n)$ and $(i_1, \ldots, i_m)$ satisfying (3.2)-(3.6) below:

$$(3.2) \qquad\qquad 1 \le i_1 < \cdots < i_m \le n,$$
$$(3.3) \qquad\qquad a_1 + \ldots + a_n = a,$$
$$(3.4) \qquad\qquad a_1, \ldots, a_n \in \mathbb{Z}_+,$$
$$(3.5) \qquad\qquad \lambda_i + k + a_i \le \lambda_{i-1} + k,$$
$$(3.6) \qquad\qquad \lambda + k\omega_n + (a_1, \ldots, a_n) + \varepsilon_{i_1} + \cdots + \varepsilon_{i_m} \in X^+.$$

Hence, the multiplicity $[L(a\omega_1)_\mathbb{C} \otimes L(\omega_m)_\mathbb{C} \otimes L(\omega_n)_\mathbb{C}^{\otimes k} \otimes L(\lambda)_\mathbb{C} : L((q-1)\rho + q\lambda)_\mathbb{C}]$ is the same as number of the tuples $(a_1, \ldots, a_n)$ and $(i_1, \ldots, i_m)$ which satisfy (3.2)–(3.5) and the following additional equation

$$(3.7) \qquad\qquad \lambda + k\omega_n + (a_1, \ldots, a_n) + \varepsilon_{i_1} + \ldots + \varepsilon_{i_m} = (q-1)\rho + q\lambda.$$

Note that (3.6) is implied by (3.7). Regarding (3.7) as a defining equation for $(a_1, \ldots, a_n)$, we are reduced to counting the cardinality of the set $\Gamma_\lambda^m$ which consists of the tuples $(i_1, \ldots, i_m)$ satisfying (3.2) and the following additional conditions (3.8)-(3.12) (recall from (2.1) the notation $\rho_i$):

(3.8) $\qquad (q-1)|\rho| + (q-1)|\lambda| - nk - m = a,$

(3.9) $\qquad (q-1)\rho_i + (q-1)\lambda_i - k \geq 0, \quad$ for $i \neq i_1, \ldots, i_m,$

(3.10) $\qquad (q-1)\rho_i + (q-1)\lambda_i - k - 1 \geq 0, \quad$ for $i = i_1, \ldots, i_m,$

(3.11) $\qquad (q-1)\rho_i + q\lambda_i \leq \lambda_{i-1} + k, \quad$ for $i \neq i_1, \ldots, i_m,$

(3.12) $\qquad (q-1)\rho_i + q\lambda_i - 1 \leq \lambda_{i-1} + k, \quad$ for $i = i_1, \ldots, i_m, 2 \leq i \leq n.$

Note here that (3.3) gives rise to (3.8), (3.4) gives rise to (3.9) and (3.10), while (3.5) gives rise to (3.11) and (3.12). Hence, we have

$$\sum_{\lambda \in X^+} \left[ L(a\omega_1)_\mathbb{C} \otimes L(\omega_m)_\mathbb{C} \otimes L(\lambda)_\mathbb{C} : L((q-1)\rho + q\lambda)_\mathbb{C} \right]$$

$$= \sum_{\lambda \in X^+} \#\Gamma_\lambda^m$$

$$= \#\{(\lambda, (i_1, \ldots, i_m)) \mid \lambda \in X^+, (i_1, \ldots, i_m) \in \Gamma_\lambda^m\}$$

(3.13) $$= \sum_{1 \leq i_1 < \cdots < i_m \leq n} \#\{\lambda \mid \lambda \in X^+ \text{ satisfies (3.8)-(3.12)}\}.$$

(1) Suppose $1 \leq k \leq q - 2$. For fixed $1 \leq i_1 < i_2 < \ldots < i_m \leq n$, let us examine the conditions (3.8)-(3.12) closely. It follows by (3.9)-(3.10) for $i = n$ that $\lambda_n \geq 1$ and hence $\lambda_i \geq 1$ for all $i = 1, \ldots, n$ for $\lambda \in X^+$; Moreover, the inequalities $\lambda_i \geq 1$ for all $i$ guarantee the validity of (3.9)-(3.10) in general. Set

$$\bar{\lambda}_n = \lambda_n - 1$$

and set, for $2 \leq i \leq n$,

$$\bar{\lambda}_{i-1} = \begin{cases} \lambda_{i-1} + k - ((q-1)\rho_i + q\lambda_i - 1), & \text{if } i = i_1, \ldots, i_m \\ \lambda_{i-1} + k - ((q-1)\rho_i + q\lambda_i), & \text{otherwise.} \end{cases}$$

Then the conditions (3.9)-(3.12) hold for $\lambda \in X^+$ if and only if $(\bar{\lambda}_1, \ldots, \bar{\lambda}_n) \in \mathbb{Z}_+^n$. By a direct computation, one further checks that the condition (3.8) is reformulated in terms of the $\bar{\lambda}_i$'s as

(3.14) $$a = -n + (q-k)\sum_{i=1}^n q^{i-1} - \sum_{j=1}^m q^{i_j - 1} + \sum_{i=1}^n (q^i - 1)\bar{\lambda}_i.$$

Hence the equation (3.13) can be rewritten as

$$\sum_{\lambda \in X^+} \left[ L(a\omega_1)_{\mathbb{C}} \otimes L(\omega_m)_{\mathbb{C}} \otimes L(\omega_n)_{\mathbb{C}}^{\otimes k} \otimes L(\lambda)_{\mathbb{C}} : L((q-1)\rho + q\lambda)_{\mathbb{C}} \right]$$

$$= \sum_{1 \le i_1 < \cdots < i_m \le n} \# \left\{ (\bar{\lambda}_1, \ldots, \bar{\lambda}_n) \in \mathbb{Z}_+^n \text{ that satisfy } (3.14) \right\}$$

$$= [t^a] \frac{t^{-n+(q-k)\frac{q^n-1}{q-1}}}{\prod_{i=1}^n (1 - t^{q^i-1})} e_m(t^{-1}, t^{-q}, \ldots, t^{-q^{n-1}}).$$

This together with (3.1) implies Part (1) of the theorem.

(2) Suppose $k = q - 1$. (Note that $\text{Det}^0 \cong \text{Det}^{q-1}$, and we could also prove Part (2) of the theorem by arguing using $k = 0$). The argument here is similar to (1) above, and so it will be sketchy. For fixed $1 \le i_1 < i_2 < \ldots < i_m \le n$, we set

$$\bar{\lambda}_n = \begin{cases} \lambda_n - 1, & \text{if } i_m \le n - 1 \\ \lambda_n - 2, & \text{if } i_m = n. \end{cases}$$

and set, for $2 \le i \le n$,

$$\bar{\lambda}_{i-1} = \begin{cases} \lambda_{i-1} + (q-1) - ((q-1)\rho_i + q\lambda_i - 1), & \text{if } i = i_1, \ldots, i_m \\ \lambda_{i-1} + (q-1) - ((q-1)\rho_i + q\lambda_i), & \text{otherwise.} \end{cases}$$

Again it can be verified as before that the conditions (3.9)-(3.12) hold if and only if $(\bar{\lambda}_1, \ldots, \bar{\lambda}_n) \in \mathbb{Z}_+^n$. Then (3.13) can be rewritten as

$$\sum_{\lambda \in X^+} [L(a\omega_1)_{\mathbb{C}} \otimes L(\omega_m)_{\mathbb{C}} \otimes L(\lambda)_{\mathbb{C}} : L((q-1)\rho + q\lambda)_{\mathbb{C}}]$$

$$= \sum_{1 \le i_1 < \cdots < i_m \le n-1} \# \left\{ (\bar{\lambda}_1, \ldots, \bar{\lambda}_n) \in \mathbb{Z}_+^n \ \middle| \ a = -n + \sum_{i=1}^n q^{i-1} - \sum_{j=1}^m q^{i_j-1} + \sum_{i=1}^n (q^i - 1)\bar{\lambda}_i \right\}$$

$$+ \sum_{1 \le i_1 < \cdots < i_m = n} \# \left\{ (\bar{\lambda}_1, \ldots, \bar{\lambda}_n) \in \mathbb{Z}_+^n \ \middle| \ a = -n + \sum_{i=1}^n q^{i-1} + (q^n - 1) \right.$$

$$\left. - \sum_{j=1}^m q^{i_j-1} + \sum_{i=1}^n (q^i - 1)\bar{\lambda}_i \right\}$$

$$= [t^a] \frac{t^{-n+\frac{q^n-1}{q-1}}}{\prod_{i=1}^n (1 - t^{q^i-1})} e_m(t^{-1}, t^{-q}, \ldots, t^{-q^{n-2}})$$

$$+ [t^a] \frac{t^{-n+\frac{q^n-1}{q-1}}}{\prod_{i=1}^n (1 - t^{q^i-1})} t^{q^n-1} (e_m(t^{-1}, t^{-q}, \ldots, t^{-q^{n-2}}, t^{-q^{n-1}}) - e_m(t^{-1}, t^{-q}, \ldots, t^{-q^{n-2}}))$$

$$= [t^a] \frac{t^{-n+\frac{q^n-1}{q-1}}}{\prod_{i=1}^n (1 - t^{q^i-1})} \left( (1 - t^{q^n-1}) e_m(t^{-1}, t^{-q}, \ldots, t^{-q^{n-2}}) \right.$$

$$\left. + t^{q^n-1} e_m(t^{-1}, t^{-q}, \ldots, t^{-q^{n-1}}) \right).$$

Therefore together with (3.1) we have proved Part (2) of the theorem.                    □

*Remark* 3.2. Observe that $S^\bullet(V) \otimes \wedge^\bullet(V) \otimes \mathrm{Det}^k$ is naturally a bi-graded $GL_n(q)$-module. Theorem 3.1(2) can be converted into a formula for the bi-graded multiplicity of St in $S^\bullet(V) \otimes \wedge^\bullet(V)$ as follows:

$$H_{\mathrm{St}}\big(S^\bullet(V) \otimes \wedge^\bullet(V); t, s\big)$$
$$= \frac{t^{-n+\frac{q^n-1}{q-1}}}{\prod_{i=1}^n (1-t^{q^i-1})} \left( (1-t^{q^n-1}) \prod_{i=0}^{n-2}(1+st^{q^{-i}}) + t^{q^n-1} \prod_{i=0}^{n-1}(1+st^{q^{-i}}) \right)$$
$$= \frac{t^{-n}(st^{q^n-1}+t^{q^n-1})\prod_{i=0}^{n-2}(s+t^{q^i})}{\prod_{i=1}^n(1-t^{q^i-1})}.$$

Similarly (and more easily), Theorem 3.1(1) is converted into the following formula for $1 \leq k \leq q-2$:

$$H_{\mathrm{St}}(S^\bullet(V) \otimes \wedge^\bullet(V) \otimes \mathrm{Det}^k; t, s) = t^{-n+(q-1-k)\frac{q^n-1}{q-1}} \cdot \frac{\prod_{i=0}^{n-1}(s+t^{q^i})}{\prod_{i=1}^n(1-t^{q^i-1})}.$$

In this way, we obtain a new proof of [WW, Theorem C], which was in turn a generalization of the earlier work [Mi, MP] (where $q$ is assumed to be a prime).

There is an isomorphism of $GL_n(q)$-modules $\wedge^{n-m}(V) \cong \wedge^m(V)^* \otimes \wedge^n(V)$ and $\wedge^n(V) \cong \mathrm{Det}$, where $W^*$ denotes the dual module of a $GL_n(q)$-module $W$. Hence, we have an isomorphism of $GL_n(q)$-modules

$$(3.15) \qquad \wedge^m(V)^* \cong \wedge^{n-m}(V) \otimes \mathrm{Det}^{-1} \cong \wedge^{n-m}(V) \otimes \mathrm{Det}^{q-2}, \qquad 0 \leq m \leq n.$$

Theorem 3.1 can now be converted into the following form using (3.15).

**Corollary 3.3.** *The graded multiplicity of* St *in* $S^\bullet(V) \otimes \wedge^m(V)^* \otimes \mathrm{Det}^k$ *is given by*

$$H_{\mathrm{St}}\big(S^\bullet(V) \otimes \wedge^m(V)^* \otimes \mathrm{Det}^k; t\big)$$
$$= \begin{cases} \dfrac{t^{-n+\frac{q^n-1}{q-1}}}{\prod_{i=1}^n(1-t^{q^i-1})} \Big((1-t^{q^n-1}) \, e_{n-m}(t^{-1},t^{-q},\ldots,t^{-q^{n-2}}) \\ \qquad\qquad\qquad +t^{q^n-1} \, e_{n-m}(t^{-1},t^{-q},\ldots,t^{-q^{n-1}})\Big), & \text{if } k=1 \\[2ex] \dfrac{t^{-n+(q+1-k)\frac{q^n-1}{q-1}}}{\prod_{i=1}^n(1-t^{q^i-1})} \, e_{n-m}(t^{-1},t^{-q},\ldots,t^{-q^{n-2}},t^{-q^{n-1}}), & \text{if } 2 \leq k \leq q-1. \end{cases}$$

### 3.3. The multiplicity of St in $S^\bullet(V) \otimes \wedge^\nu(V) \otimes \mathrm{Det}^k$.

For a partition $\nu = (\nu_1, \ldots, \nu_\ell)$ with $\nu_1 \leq n$, denote by $\wedge^\nu(V)$ the $GL_n(q)$-module

$$\wedge^\nu(V) = \wedge^{\nu_1}(V) \otimes \cdots \otimes \wedge^{\nu_\ell}(V).$$

Recall $e_\nu$ denotes the elementary symmetric polynomial associated to $\nu$. The following is a generalization of Theorem 3.1(1) (which corresponds to the case $\ell = 1$ below).

**Theorem 3.4.** *Let* $\nu = (\nu_1, \ldots, \nu_\ell)$ *be a partition with length* $\ell(\nu) = \ell$ *and* $\nu_1 \leq n$. *Let* $k$ *be a positive integer such that* $k + \ell \leq q-1$. *Then the graded multiplicity of* St *in*

$S^\bullet(V) \otimes \wedge^\nu(V) \otimes \mathrm{Det}^k$ *is given by*

$$H_{\mathrm{St}}\big(S^\bullet(V) \otimes \wedge^\nu(V) \otimes \mathrm{Det}^k; t\big) = \frac{t^{-n+(q-k)\frac{q^n-1}{q-1}}}{\prod_{i=1}^n (1-t^{q^i-1})} \; e_\nu(t^{-1}, t^{-q}, \ldots, t^{-q^{n-1}}).$$

*Proof.* Fix $a \in \mathbb{Z}_+$. By arguments similar to Theorem 3.1, one can show using Proposition 2.5 and Pieri's formula that, for $a \geq 0$,

$$\dim \mathrm{Hom}_{GL_n(q)}(St, \; S^a(V) \otimes \wedge^\nu(V) \otimes \mathrm{Det}^k)$$
$$= \sum_{\lambda \in X^+} \big[ L(a\omega_1)_\mathbb{C} \otimes L(\omega_{\nu_1})_\mathbb{C} \otimes \cdots \otimes L(\omega_{\nu_\ell})_\mathbb{C} \otimes L(\omega_n)_\mathbb{C}^{\otimes k} \otimes L(\lambda)_\mathbb{C} : L((q-1)\rho + q\lambda)_\mathbb{C} \big]$$

(3.16)
$$= \sum_{\lambda \in X^+} |\Gamma_\lambda^\nu|,$$

where $\Gamma_\lambda^\nu$ is the set consisting of the sequences $\big((a_1, \ldots, a_n), (i_u^b | 1 \leq u \leq \nu_b, 1 \leq b \leq \ell)\big)$ satisfying (3.3)-(3.5) and the following additional conditions (3.17)-(3.19):

$$(3.17) \qquad\qquad 1 \leq i_1^j < \cdots < i_{\nu_j}^j \leq n, \quad 1 \leq j \leq \ell,$$

$$(3.18) \qquad\qquad \lambda + (a_1, \ldots, a_n) + k\omega_n + \sum_{b=1}^j \sum_{u=1}^{\nu_b} \varepsilon_{i_u^b} \in X^+, \quad 1 \leq j \leq \ell,$$

$$(3.19) \qquad\qquad \lambda + (a_1, \ldots, a_n) + k\omega_n + \sum_{b=1}^\ell \sum_{u=1}^{\nu_b} \varepsilon_{i_u^b} = (q-1)\rho + q\lambda.$$

Note that (3.18) will automatically hold if $\big((a_1, \ldots, a_n), (i_u^b | 1 \leq u \leq \nu_b, 1 \leq b \leq \ell)\big)$ satisfies (3.17) and (3.19). This is clear once we visualize the weight $(q-1)\rho + q\lambda$ as a Young diagram whose consecutive rows differ by at least $(q-1)$, and removing at most $\ell$ boxes in each row gives rise to new Young diagrams corresponding to the weights in (3.18) (recall here $\ell \leq q-1$ by assumption).

For $1 \leq i \leq n$, denote

$$c_i = \#\{(b, u) \mid i_u^b = i, 1 \leq u \leq \nu_b, 1 \leq b \leq \ell\}.$$

Hence, regarding (3.19) as a defining relation for $(a_1, \ldots, a_n)$, we see that the set $\Gamma_\lambda^\nu$ has the same cardinality as the set consisting of the sequences $(i_u^b | 1 \leq u \leq \nu_b, 1 \leq b \leq \ell)$ satisfying (3.17) and (3.20)-(3.22) below:

$$(3.20) \qquad\qquad (q-1)\rho_i + (q-1)\lambda_i - k - c_i \geq 0, \quad 1 \leq i \leq n,$$
$$(3.21) \qquad\qquad (q-1)\rho_i + q\lambda_i - c_i \leq \lambda_{i-1} + k, \quad 2 \leq i \leq n,$$
$$(3.22) \qquad\qquad (q-1)|\rho| + (q-1)|\lambda| - nk - |\nu| = a.$$

Therefore, the identity (3.16) can be rewritten as

$$\dim \operatorname{Hom}_{GL_n(q)}(\mathrm{St},\ S^a(V) \otimes \wedge^\nu(V) \otimes \mathrm{Det}^k)$$

$$= \sum_{\lambda \in X^+} \#\left\{\text{the tuples } \left(i_u^b | 1 \leq u \leq \nu_b, 1 \leq b \leq \ell\right) \text{ that satisfy } (3.17),(3.20)\text{--}(3.22)\right\}$$

(3.23)

$$= \sum_{1 \leq i_1^j < \cdots < i_{\nu_j}^j \leq n,\ 1 \leq j \leq \ell} \#\left\{\lambda \in X^+ \text{ that satisfy } (3.20)\text{--}(3.22)\right\}.$$

Given $1 \leq i_1^j < \cdots < i_{\nu_j}^j \leq n$ for $1 \leq j \leq \ell$, we set $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_n)$ for $\lambda \in X^+$ with

$$\bar{\lambda}_n = \lambda_n - 1,$$

$$\bar{\lambda}_{i-1} = \lambda_{i-1} + k - ((q-1)\rho_i + q\lambda_i - c_i), \quad 2 \leq i \leq n.$$

We claim that (3.20) and (3.21) hold for $\lambda \in X^+$ if and only if $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_n) \in \mathbb{Z}_+^n$. In fact, (3.21) is clearly equivalent to that $\bar{\lambda}_i \geq 0$ for $1 \leq i \leq n-1$. Also, (3.20) for $i = n$ (which reads that $(q-1)\lambda_n - k - c_n \geq 0$) holds if and only if $\bar{\lambda}_n = \lambda_n - 1 \geq 0$, since $1 \leq k + c_n \leq k + \ell \leq q - 1$ by assumption. It is further readily checked that $\lambda \in X^+$ follows from $\bar{\lambda} \in \mathbb{Z}_+^n$. So it remains to see that if $(\bar{\lambda}_1, \ldots, \bar{\lambda}_n) \in \mathbb{Z}_+^n$ then $(q-1)\rho_i + (q-1)\lambda_i - k - c_i \geq 0$ for $1 \leq i \leq n-1$; This follows from the facts that $k + c_i \leq k + \ell \leq q - 1$ and $\lambda_i \geq \lambda_n \geq 1$.

On the other hand, a direct calculation shows that

$$\sum_{i=1}^n (q^i - 1)\bar{\lambda}_i = \sum_{i=1}^n (q-1)\lambda_i - (q^n - 1) + \sum_{i=1}^n k(q^{i-1} - 1)$$

$$- \sum_{i=1}^n (q^{i-1} - 1)(q-1)\rho_i + \sum_{i=1}^n (q^{i-1} - 1)c_i.$$

It follows that

$$(q-1)|\rho| + (q-1)|\lambda| - nk - |\nu|$$

$$= \sum_{i=1}^n (q^i - 1)\bar{\lambda}_i + (q^n - 1) + \sum_{i=1}^n q^{i-1}(q-1)\rho_i$$

$$- \sum_{i=1}^n k(q^{i-1} - 1) - \sum_{i=1}^n (q^{i-1} - 1)c_i - nk - |\nu|$$

$$= \sum_{i=1}^n (q^i - 1)\bar{\lambda}_i + (q^n - 1) + (-n + 1 + q + \cdots + q^{n-1})$$

$$- \sum_{i=1}^n kq^{i-1} - \sum_{i=1}^n q^{i-1}c_i + \sum_{i=1}^n c_i - |\nu|$$

$$= \sum_{i=1}^n (q^i - 1)\bar{\lambda}_i + (-n) + (q - k)\frac{q^n - 1}{q - 1} - \sum_{i=1}^n q^{i-1}c_i,$$

since $\sum_{i=1}^{n} c_i = \nu_1 + \ldots + \nu_\ell = |\nu|$. This together with the identity

$$\sum_{i=1}^{n} q^{i-1} c_i = \sum_{1 \leq b \leq \ell, 1 \leq u \leq \nu_b} q^{i_u^b - 1}$$

implies that (3.22) is equivalent to

$$(3.24) \qquad a = \sum_{i=1}^{n} (q^i - 1)\bar{\lambda}_i + (-n) + (q-k)\frac{q^n - 1}{q - 1} - \sum_{1 \leq b \leq \ell, 1 \leq u \leq \nu_b} q^{i_u^b - 1}.$$

Summarizing, we can rewrite (3.23) as

$$\dim \operatorname{Hom}_{GL_n(q)}(\operatorname{St}, \ S^a(V) \otimes \wedge^\nu(V) \otimes \operatorname{Det}^k)$$

$$= \sum_{1 \leq i_1^j < \cdots < i_{\nu_j}^j \leq n, \ 1 \leq j \leq \ell} \#\{(\bar{\lambda}_1, \ldots, \bar{\lambda}_n) \in \mathbb{Z}_+^n \text{ that satisfy } (3.24)\}$$

$$= [t^a] \frac{t^{-n+(q-k)\frac{q^n-1}{q-1}}}{\prod_{i=1}^{n}(1 - t^{q^i - 1})} \prod_{j=1}^{\ell} e_{\nu_j}(t^{-1}, t^{-q}, \ldots, t^{-q^{n-1}})$$

$$= [t^a] \frac{t^{-n+(q-k)\frac{q^n-1}{q-1}}}{\prod_{i=1}^{n}(1 - t^{q^i - 1})} e_\nu(t^{-1}, t^{-q}, \ldots, t^{-q^{n-1}}).$$

Therefore,

$$H_{\operatorname{St}}\big(S^\bullet(V) \otimes \wedge^\nu(V) \otimes \operatorname{Det}^k; t\big)$$

$$= \sum_{a \geq 0} \big(\dim \operatorname{Hom}_{GL_n(q)}(St, S^a(V) \otimes \wedge^\nu(V) \otimes \operatorname{Det}^k)\big) \cdot t^a$$

$$= \frac{t^{-n+(q-k)\frac{q^n-1}{q-1}}}{\prod_{i=1}^{n}(1 - t^{q^i - 1})} e_\nu(t^{-1}, t^{-q}, \ldots, t^{-q^{n-1}}).$$

The theorem is proved.                                                                        □

*Remark* 3.5. Using a similar argument, we can in principle obtain a (very messy in general) formula for the graded multiplicity of St in $S^\bullet(V) \otimes \wedge^\nu(V) \otimes \operatorname{Det}^k$ without the assumption that $k + \ell \leq q - 1$, generalizing Theorem 3.1(2). For example, the formula in the case $\nu = (1, 1, \ldots, 1)$ is given by

$$H_{\operatorname{St}}\big(S^\bullet(V) \otimes V^{\otimes \ell} \otimes \operatorname{Det}^k; t\big)$$

$$= \frac{t^{-n+(q-k)\frac{q^n-1}{q-1}}}{\prod_{i=1}^{n}(1 - t^{q^i - 1})} \sum_{c=0}^{q-1-k} \binom{\ell}{c}(t^{-1} + t^{-q} + \cdots + t^{-q^{n-2}})^{\ell-c}(t^{-q^{n-1}})^c$$

$$+ \frac{t^{-n+(2q-1-k)\frac{q^n-1}{q-1}}}{\prod_{i=1}^{n}(1 - t^{q^i - 1})} \sum_{c=q-k}^{\ell} \binom{\ell}{c}(t^{-1} + t^{-q} + \cdots + t^{-q^{n-2}})^{\ell-c}(t^{-q^{n-1}})^c$$

for $1 \leq k \leq q - 1$.

By (3.15), one has

(3.25) $$\wedge^\nu(V)^* \cong \wedge^{n-\nu_\ell}(V) \otimes \cdots \otimes \wedge^{n-\nu_1}(V) \otimes \mathrm{Det}^{q-1-\ell}$$

for any partition $\nu$ with $\nu_1 \leq n$. Hence, Theorem 3.4 can be converted into the following form and vice versa.

**Corollary 3.6.** *Let $\nu = (\nu_1, \ldots, \nu_\ell)$ be a (nonempty) partition with length $\ell(\nu) = \ell$ such that $\nu_1 \leq n$. For $\ell < k \leq q - 1$, the graded multiplicity of $\mathrm{St}$ in $S^\bullet(V) \otimes \wedge^\nu(V)^* \otimes \mathrm{Det}^k$ is given by*

$$H_{\mathrm{St}}\big(S^\bullet(V) \otimes \wedge^\nu(V)^* \otimes \mathrm{Det}^k; t\big) = \frac{t^{-n+(q-k)\frac{q^n-1}{q-1}}}{\prod_{i=1}^n(1-t^{q^i-1})} e_\nu(t, t^q, \ldots, t^{q^{n-1}}).$$

3.4. **The multiplicity of $\mathrm{St}$ in $S^\bullet(V) \otimes L(\mu) \otimes \mathrm{Det}^k$.** We now compute the multiplicity of $\mathrm{St}$ in $S^\bullet(V) \otimes L(\mu) \otimes \mathrm{Det}^k$. Our more restrictive condition on $\mu$ ensures that $L(\mu)$ appears as a direct summand in the Schur duality.

**Theorem 3.7.** *Fix $1 \leq k \leq q - 1$ and $1 \leq d \leq p - 1$. Let $\mu = (\mu_1, \ldots, \mu_n)$ be a partition of $d$ with $\ell(\mu) \leq n$.*

*(1) If $k + \mu_1 \leq q - 1$, then the graded multiplicity of $\mathrm{St}$ in the graded $GL_n(q)$-module $S^\bullet(V) \otimes L(\mu) \otimes \mathrm{Det}^k$ is given by*

$$H_{\mathrm{St}}\big(S^\bullet(V) \otimes L(\mu) \otimes \mathrm{Det}^k; t\big) = \frac{t^{-n+(q-k)\frac{q^n-1}{q-1}}}{\prod_{i=1}^n(1-t^{q^i-1})} s_\mu(t^{-1}, t^{-q}, \ldots, t^{-q^{n-1}}).$$

*(2) If $\mu_1 < k \leq q - 1$, then the graded multiplicity of $\mathrm{St}$ in the graded $GL_n(q)$-module $S^\bullet(V) \otimes L(\mu)^* \otimes \mathrm{Det}^k$ is given by*

$$H_{\mathrm{St}}\big(S^\bullet(V) \otimes L(\mu)^* \otimes \mathrm{Det}^k; t\big) = \frac{t^{-n+(q-k)\frac{q^n-1}{q-1}}}{\prod_{i=1}^n(1-t^{q^i-1})} s_\mu(t, t^q, \ldots, t^{q^{n-1}}).$$

*Proof.* (1) Suppose $k + \mu_1 \leq q - 1$ and let $\mu'$ be the conjugate of $\mu$. Then $\ell(\mu') = \mu_1$ and hence $k + \ell(\mu') \leq q - 1$. By Theorem 3.4, we have

$$H_{\mathrm{St}}\big(S^\bullet(V) \otimes \wedge^{\mu'}(V) \otimes \mathrm{Det}^k; t\big) = \frac{t^{-n+(q-k)\frac{q^n-1}{q-1}}}{\prod_{i=1}^n(1-t^{q^i-1})} e_{\mu'}(t^{-1}, t^{-q}, \ldots, t^{-q^{n-1}}).$$

Since $d \leq p - 1$, the $GL_n(q)$-module $V^{\otimes d}$ is semisimple and so is its submodule $\wedge^{\mu'}(V)$. Hence, we have the following decomposition of $GL_n(q)$-modules:

(3.26) $$\wedge^{\mu'}(V) \cong \sum_{\ell(\gamma) \leq n, \gamma \leq \mu} K_{\gamma'\mu'} L(\gamma) \cong L(\mu) \oplus \sum_{\ell(\gamma) \leq n, \gamma < \mu} K_{\gamma'\mu'} L(\gamma),$$

where $K_{\gamma'\mu'}$ are the Kostka numbers (cf. e.g. [FH]).

We complete the proof of Part (1) by induction on the dominance order of $\mu$. For $\mu$ minimal in the sense that there is no partition $\gamma < \mu$ with $\ell(\gamma) \leq n$, Part (1) reduces to Theorem 3.4.

Now Let $\mu = (\mu_1, \ldots, \mu_n)$ be a general partition of $d$ with $\ell(\mu) \le n$. For $\gamma < \mu$ with $\ell(\gamma) \le n$, we have $\gamma_1 \le \mu_1$ and $k + \gamma_1 \le q - 1$. Thus by induction hyperthesis we have

$$H_{\mathrm{St}}(S^\bullet(V) \otimes L(\gamma) \otimes \mathrm{Det}^k; t) = \frac{t^{-n+(q-k)\frac{q^n-1}{q-1}}}{\prod_{i=1}^n(1-t^{q^i-1})} s_\gamma(t^{-1}, t^{-q}, \ldots, t^{-q^{n-1}}).$$

This together with (3.26) gives us

$$H_{\mathrm{St}}\big(S^\bullet(V) \otimes L(\mu) \otimes \mathrm{Det}^k; t\big)$$

$$= H_{\mathrm{St}}\big(S^\bullet(V) \otimes \wedge^{\mu'}(V) \otimes \mathrm{Det}^k; t\big) - \sum_{\ell(\gamma)\le n, \gamma<\mu} K_{\gamma'\mu'} H_{\mathrm{St}}\big(S^\bullet(V) \otimes L(\gamma) \otimes \mathrm{Det}^k; t\big)$$

$$= \frac{t^{-n+(q-k)\frac{q^n-1}{q-1}}}{\prod_{i=1}^n(1-t^{q^i-1})} \left(e_{\mu'}(t^{-1}, t^{-q}, \ldots, t^{-q^{n-1}}) - \sum_{\ell(\gamma)\le n, \gamma<\mu} K_{\gamma'\mu'} s_\gamma(t^{-1}, t^{-q}, \ldots, t^{-q^{n-1}})\right)$$

$$= \frac{t^{-n+(q-k)\frac{q^n-1}{q-1}}}{\prod_{i=1}^n(1-t^{q^i-1})} s_\mu(t^{-1}, t^{-q}, \ldots, t^{-q^{n-1}}),$$

where the last equality is due to the symmetric function identity

$$e_{\mu'}(x_1, x_2, \ldots, x_n) = \sum_{\gamma \vdash d, \ell(\gamma)\le n, \gamma\le\mu} K_{\gamma'\mu'} s_\gamma(x_1, x_2, \ldots, x_n).$$

(2) Suppose $\mu_1 < k \le q - 1$ and let $\mu'$ be the conjugate of $\mu$. Then $\ell(\mu') = \mu_1$ and hence $\ell(\mu') < k$. By Corollary 3.6 we have

$$H_{\mathrm{St}}(S^\bullet(V) \otimes \wedge^{\mu'}(V)^* \otimes \mathrm{Det}^k; t) = \frac{t^{-n+(q-k)\frac{q^n-1}{q-1}}}{\prod_{i=1}^n(1-t^{q^i-1})} e_{\mu'}(t, t^q, \ldots, t^{q^{n-1}}).$$

On the other hand, by (3.26) one has

$$\wedge^{\mu'}(V)^* \cong \sum_{\gamma \vdash d, \ell(\gamma)\le n, \gamma\le\mu} K_{\gamma'\mu'} L(\gamma)^* \cong L(\mu)^* \oplus \sum_{\gamma \vdash d, \ell(\gamma)\le n, \gamma<\mu} K_{\gamma'\mu'} L(\gamma)^*,$$

The theorem follows by an argument similar to the proof of Part (1).                    $\square$

## 4. The composition multiplicity in $S^\bullet(V)$ around the Steinberg module

In this section, we shall determine the graded multiplicity in the symmetric algebra $S^\bullet(V)$ for a large class of of irreducible modules around the Steinberg module.

Denote by $P(\lambda)$ the projective cover of the irreducible $GL_n(q)$-module $L(\lambda)$ for $\lambda \in X_r$. The graded multiplicity of $L(\lambda)$ in the symmetric algebra $S^\bullet(V)$ is equal to the Hilbert series of the graded space $\mathrm{Hom}_{GL_n(q)}(P(\lambda), S^\bullet(V))$.

4.1. **A case for any prime** $p$. Recall that $q = p^r$. For $\lambda \in X_r$, set

(4.1)                                          $$\lambda^0 = (q-1)\rho + w_0\lambda.$$

According to Tsushima [Ts] (which goes back to Lusztig [Lu] for $m = 1$ and $p > 2$), we have, for $1 \leq m \leq n - 1$, that

$$\mathrm{St} \otimes \wedge^m(V) \cong \begin{cases} P(\omega_m^0), & \text{if } q > 2 \\ P(\omega_m^0) \oplus \mathrm{St}, & \text{if } q = 2, \end{cases}$$

and hence also, for all $1 \leq k \leq q - 1$,

(4.2) $\qquad \mathrm{St} \otimes \wedge^m(V) \otimes \mathrm{Det}^k \cong \begin{cases} P(\omega_m^0 + k\omega_n), & \text{if } q > 2 \\ P(\omega_m^0 + k\omega_n) \oplus \mathrm{St} \otimes \mathrm{Det}^k, & \text{if } q = 2. \end{cases}$

Recall the Kronecker symbol $\delta_{2,q} = 1$ if $q = 2$ and $\delta_{2,q} = 0$ if $q > 2$. For $1 \leq m \leq n - 1$, define $\gamma = (\gamma_1, \dots, \gamma_n) \in X_r$ by $\gamma_i = (q-1)(n-i) - k - 1$ for $1 \leq i \leq m$ and $\gamma_i = (q-1)(n-i) - k$ for $m + 1 \leq i \leq n$.

**Theorem 4.1.** *Suppose $1 \leq m \leq n - 1$ and $1 \leq k \leq q - 1$.*
*(1) For $1 \leq k \leq q - 2$, the graded composition multiplicity of $L(\gamma)$ in $S^\bullet(V)$ is given by*

$$\frac{t^{-n+(q-k)\frac{q^n-1}{q-1}}}{\prod_{i=1}^n (1 - t^{q^i-1})} \, e_m(t^{-1}, t^{-q}, \dots, t^{-q^{n-2}}, t^{-q^{n-1}}).$$

*(2) The graded composition multiplicity of $L(\gamma)$ in $S^\bullet(V)$ is given by*

$$\frac{t^{-n+\frac{q^n-1}{q-1}}}{\prod_{i=1}^n (1 - t^{q^i-1})} \Big( (1 - t^{q^n-1}) \, e_m(t^{-1}, t^{-q}, \dots, t^{-q^{n-2}})$$

$$+ t^{q^n-1} \, e_m(t^{-1}, t^{-q}, \dots, t^{-q^{n-2}}, t^{-q^{n-1}}) \Big) - \delta_{2,q} \frac{t^{-n+\frac{q^n-1}{q-1}}}{\prod_{i=1}^n (1 - t^{q^i-1})}.$$

*Proof.* Suppose $1 \leq k \leq q - 2$; observe that this happens only when $q > 2$.

By (3.15) and (4.2) we have

$$\mathrm{St} \otimes (\wedge^m(V) \otimes \mathrm{Det}^k)^* \cong \mathrm{St} \otimes \wedge^{n-m}(V) \otimes \mathrm{Det}^{-1-k} \cong P(\omega_{n-m}^0 - (k+1)\omega_n) = P(\gamma).$$

This implies that

$$\mathrm{Hom}_{GL_n(q)}(P(\gamma), S^\bullet(V))$$

$$\cong \mathrm{Hom}_{GL_n(q)}(\mathrm{St} \otimes (\wedge^m(V) \otimes \mathrm{Det}^k)^*, S^\bullet(V))$$

$$\cong \mathrm{Hom}_{GL_n(q)}(\mathrm{St}, S^\bullet(V) \otimes \wedge^m(V) \otimes \mathrm{Det}^k).$$

Thus, Part (1) of the theorem follows by Theorem 3.1(1). By a similar argument and Theorem 3.1(2), the second part in the case $q > 2$ follows.

Suppose now $q = 2$. Note that the determinant module Det coincides with the trivial module. Using (4.2) we get

$$\mathrm{St} \otimes \wedge^m(V)^* \cong \mathrm{St} \otimes \wedge^{n-m}(V) \cong P(\gamma) \oplus \mathrm{St}$$

and hence, for $a \geq 0$,

$$\dim \mathrm{Hom}_{GL_n(q)}(P(\gamma), S^a(V))$$

$$= \dim \mathrm{Hom}_{GL_n(q)}(\mathrm{St} \otimes \wedge^m(V)^*, S^a(V)) - \dim \mathrm{Hom}_{GL_n(q)}(\mathrm{St}, S^a(V))$$

$$= \dim \mathrm{Hom}_{GL_n(q)}(\mathrm{St}, S^a(V) \otimes \wedge^m(V)) - \dim \mathrm{Hom}_{GL_n(q)}(\mathrm{St}, S^a(V)).$$

The theorem for $q = 2$ now follows from this identity and Theorem 3.1(2). Note a special case of Theorem 3.1(2) for $m = 0$ says that the graded multiplicity of the Steinberg module St in the symmetric algebra is given by $t^{-n+\frac{q^n-1}{q-1}} \prod_{i=1}^{n}(1 - t^{q^i-1})^{-1}$. $\qquad\square$

*Remark* 4.2. Theorem 4.1 under the additional assumption that $q$ is a prime was first established by Carlisle and Walker [CW, Corollary 1.4] using a completely different method. Their formula is equivalent to ours since

$$\frac{t^{-n+(q-k)\frac{q^n-1}{q-1}}}{\prod_{i=1}^{n}(1 - t^{q^i-1})} \, e_m(t^{-1}, t^{-q}, \ldots, t^{-q^{n-1}})$$

$$= \frac{t^{-k\frac{q^n-1}{q-1}}}{\prod_{i=1}^{n}(1 - t^{q^i-1})} \sum_{0 \le i_1 < i_2 < \ldots < i_m \le n-1} t^{q+q^2+\ldots+q^n-q^{i_1}-\ldots-q^{i_m}-n}.$$

4.2. **A general composition multiplicity formula.** Throughout this subsection, we assume that $p > n$, the Coxeter number of $GL_n(q)$.

We denote by $\mathrm{br}(N)$ the Brauer character of a $GL_n(q)$-module $N$ (cf. [Hu]). For any $W$-invariant element $\gamma \in \mathbb{Z}[X]^W$ with $W = S_n$, one writes it as a linear combination in terms of the formal characters of the simple modules $L(\lambda)$ for $\lambda \in X_r$ and define its Brauer character $\mathrm{Br}(\gamma)$ to be the corresponding linear combination of $\mathrm{Br}(L(\lambda))$'s. Denote by $W^\mu$ a set of coset representatives in $W$ of the stabilizer subgroup $W_\mu$ of $\mu$. Then the orbit sum $\sum_{\sigma \in W^\mu} e^{\sigma\mu} \in \mathbb{Z}[X]^W$ has the monomial symmetric polynomial $m_\mu(e^{\varepsilon_1}, \ldots, e^{\varepsilon_n})$ as its formal character, and its Brauer character will be denoted by $\mathrm{Br}(m_\mu)$. Let $\phi$ denote the Brauer character of St. The following is essentially a result of Ballard [Ba, Proposition 7.2] in the case of $GL_n(q)$.

**Lemma 4.3.** *Assume that $p > n$. If $\mu \in X_r$ satisfies $\mu_1 - \mu_n \le p - 1$, then the Brauer character of the projective $GL_n(q)$-module $P(\mu^0)$ is equal to $\phi\mathrm{Br}(m_\mu)$. In particular,*

$$\dim P(\mu^0) = \dim \mathrm{St} \cdot |W^\mu| = \frac{\prod_{i=0}^{n-1}(q^n - q^i)}{\prod_{i=1}^{n}(q^i - 1)} m_\mu(1, 1, \ldots, 1).$$

*Remark* 4.4. For $q = p^r$ with $r \ge 2$, the bound in [Ba, Proposition 7.2] can be stated as $\mu_1 - \mu_n \le p - 1$ as above, while for $q = p$, Ballard imposed the more restrictive assumption that $\mu_1 - \mu_n < (p - 1)/2$ (because of using [Ba, Lemma 7.3]). Chastkofsky remarked in his math review on [Ba] that the bound in Ballard's paper can always be improved to $\mu_1 - \mu_n \le p - 1$, using his work [Ch] (also see Jantzen [J1] for closely related results).

Recall that (cf. [FH]) elementary symmetric functions can be expressed in terms of monomial symmetric functions as follows:

$$(4.3) \qquad\qquad\qquad e_\nu = \sum_{\mu \le \nu'} a_{\nu\mu} m_\mu,$$

where $a_{\nu\mu}$ are nonnegative integers and $a_{\nu\nu'} = 1$. This can be seen by expressing $e_\nu$ in terms of Schur functions $s_\lambda$ and then $s_\lambda$ in terms of $m_\mu$.

**Proposition 4.5.** *Suppose $p > n$ and $0 \leq k \leq q-2$. Let $\nu$ be a partition such that $\nu_1 \leq n$ and $\nu_1' - \nu_n' \leq p - 1$. Then the $GL_n(q)$-module $\mathrm{St} \otimes \wedge^\nu(V) \otimes \mathrm{Det}^k$ can be decomposed as*

$$\mathrm{St} \otimes \wedge^\nu(V) \otimes \mathrm{Det}^k \cong \bigoplus_{\tau \leq \nu', \ell(\tau) \leq n} P(\tau^0 + k\omega_n)^{\oplus a_{\nu\tau}},$$

*where $a_{\nu\tau}$ is defined in (4.3).*

*Proof.* As the module $\mathrm{St} \otimes \wedge^\nu(V) \otimes \mathrm{Det}^k$ is known to be projective, it suffices to check the isomorphism in the proposition on the Brauer character level. We can further assume $k = 0$, as the general case is then obtained easily by tensoring by $\mathrm{Det}^k$. By (4.3), the Brauer character of $\mathrm{St} \otimes \wedge^\nu(V) \otimes \mathrm{Det}^k$ can be written as

$$\mathrm{Br}(\mathrm{St} \otimes \wedge^\nu(V)) = \phi \mathrm{Br}(\wedge^\nu(V)) = \sum_{\tau \leq \nu', \ell(\tau) \leq n} a_{\nu\tau} \phi \mathrm{Br}(m_\tau).$$

Observe that if $\tau \leq \nu'$ then $\tau_1 - \tau_n \leq \nu_1' - \nu_n' \leq p - 1$. It follows by Lemma 4.3 that $\phi \mathrm{Br}(m_\tau)$ is the Brauer character of the projective indecomposable module $P(\tau^0)$. So we have established the desired identity of Brauer characters. $\square$

The projective $GL_n(q)$-module $P(\lambda)$ for $\lambda \in X_r$ has both head and socle isomorphic to $L(\lambda)$, and therefore $P(\lambda)^* \cong P(-w_0\lambda)$. Then thanks to the fact that $\mathrm{St}^* \cong \mathrm{St}$, Proposition 4.5 can be converted into the following.

**Corollary 4.6.** *Suppose $p > n$ and $0 \leq k \leq q - 2$. Let $\nu$ be a partition such that $\nu_1 \leq n$ and $\nu_1' - \nu_n' \leq p - 1$. Then the $GL_n(q)$-module $\mathrm{St} \otimes \wedge^\nu(V)^* \otimes \mathrm{Det}^k$ can be decomposed as*

$$\mathrm{St} \otimes \wedge^\nu(V)^* \otimes \mathrm{Det}^k \cong \sum_{\tau \leq \nu', \ell(\tau) \leq n} P((q - 1)\rho - \tau + k\omega_n)^{\oplus a_{\nu\tau}}.$$

Below is a main result of this section.

**Theorem 4.7.** *Suppose $p > n$ and $0 \leq k \leq q - 2$. Let $\mu$ be a partition with $\ell(\mu) \leq n$ and $\mu_1 - \mu_n \leq p - 1$.*

*(1) If $\mu_1 \leq k$, then the graded composition multiplicity of $L((q - 1)\rho - \mu + k\omega_n)$ in $S^\bullet(V)$ is*

$$\frac{t^{-n+(k+1)\frac{q^n-1}{q-1}}}{\prod_{i=1}^n (1 - t^{q^i-1})} \, m_\mu(t^{-1}, t^{-q}, \ldots, t^{-q^{n-1}}).$$

*(2) If $\mu_1 + k < q - 1$, then the graded composition multiplicity of $L((q-1)\rho + w_0\mu + k\omega_n)$ in $S^\bullet(V)$ is*

$$\frac{t^{-n+(k+1)\frac{q^n-1}{q-1}}}{\prod_{i=1}^n (1 - t^{q^i-1})} \, m_\mu(t, t^q, \ldots, t^{q^{n-1}}).$$

*Proof.* (1) Suppose $\mu_1 \leq k$. We shall prove by induction on dominance order of $\mu$ the equivalent claim that the Hilbert series of $\mathrm{Hom}_{GL_n(q)}(P((q - 1)\rho - \mu + k\omega_n), S^\bullet(V))$ is given by

$$\frac{t^{-n+(k+1)\frac{q^n-1}{q-1}}}{\prod_{i=1}^n (1 - t^{q^i-1})} \, m_\mu(t^{-1}, t^{-q}, \ldots, t^{-q^{n-1}}).$$

For $\mu$ minimal in the sense that it has no partition $\tau < \mu$ with $\ell(\tau) \leq n$, by (4.3) we have

$$e_{\mu'}(t^{-1}, t^{-q}, \ldots, t^{-q^{n-1}}) = m_\mu(t^{-1}, t^{-q}, \ldots, t^{-q^{n-1}})$$

and moreover by Corollary 4.6 we have

$$P((q-1)\rho - \mu + k\omega_n) \cong \mathrm{St} \otimes \wedge^{\mu'}(V)^* \otimes \mathrm{Det}^k.$$

Hence

$$\mathrm{Hom}_{GL_n(q)}(P((q-1)\rho - \mu + k\omega_n), S^\bullet(V))$$
$$\cong \mathrm{Hom}_{GL_n(q)}(\mathrm{St} \otimes \wedge^{\mu'}(V)^* \otimes \mathrm{Det}^k, S^\bullet(V))$$
$$\cong \mathrm{Hom}_{GL_n(q)}(\mathrm{St}, S^\bullet(V) \otimes \wedge^{\mu'}(V) \otimes \mathrm{Det}^{q-1-k}).$$

The claim for such a minimal weight $\mu$ follows by Theorem 3.7(1) since $1 \leq q-1-k \leq q-1$ and $\mu_1 + (q-1-k) \leq q-1$ by the assumption $\mu_1 \leq k$.

If $\tau$ is a partition satisfying $\tau < \mu$, then $\tau_1 \leq \mu_1 \leq k$ and $\tau_1 - \tau_n \leq \mu_1 - \mu_n \leq p-1$. Hence by inductive assumption the Hilbert series of $\mathrm{Hom}_{GL_n(q)}(P((q-1)\rho - \tau + k\omega_n), S^\bullet(V))$ for $\tau < \mu$ is given by

$$\frac{t^{-n+(k+1)\frac{q^n-1}{q-1}}}{\prod_{i=1}^n (1 - t^{q^i-1})} \, m_\tau(t^{-1}, t^{-q}, \ldots, t^{-q^{n-1}}).$$

By Corollary 4.6 and Theorem 3.7(1) the Hilbert series of $\mathrm{Hom}_{GL_n(q)}(P((q-1)\rho - \mu + k\omega_n), S^\bullet(V))$ has the form

$$\frac{t^{-n+(k+1)\frac{q^n-1}{q-1}}}{\prod_{i=1}^n (1 - t^{q^i-1})}\left( e_{\mu'}(t^{-1}, t^{-q}, \ldots, t^{-q^{n-1}}) - \sum_{\tau < \mu} a_{\mu'\tau} m_\tau(t^{-1}, t^{-q}, \ldots, t^{-q^{n-1}})\right)$$
$$= \frac{t^{-n+(k+1)\frac{q^n-1}{q-1}}}{\prod_{i=1}^n (1 - t^{q^i-1})} \, m_\mu(t^{-1}, t^{-q}, \ldots, t^{-q^{n-1}}).$$

Here we have used the identity that $e_{\mu'} = \sum_{\tau \leq \mu} a_{\mu'\tau} m_\tau$ and $a_{\mu'\mu} = 1$. Therefore the first part of theorem is proved. The second part of the theorem follows from a similar argument together using Proposition 4.5 and Theorem 3.7(2). $\square$

*Remark* 4.8. There is a duality formulated in [CW, Proposition 2.12] between the graded composition multiplicity of $L(\mu)$ and that of $L(\mu)^*$ in general. The two parts of Theorem 4.7 fit well with such a duality.

4.3. **The coinvariant algebra.** According to a classical theorem of Dickson [Di], the algebra of $GL_n(q)$-invariants $S^\bullet(V)^{GL_n(q)}$ is a polynomial algebra in $n$ generators, and its Hilbert series is given by

$$(4.4) \qquad\qquad\qquad \frac{1}{\prod_{i=0}^{n-1}(1 - t^{q^n-q^i})}.$$

Consider the following quotient algebra

$$S^\bullet(V)_{GL_n(q)} := S^\bullet(V)/I_+^\bullet,$$

where $I_+^\bullet$ denotes the ideal of $S^\bullet(V)$ generated by homogeneous elements of positive degree in $S^\bullet(V)^{GL_n(q)}$. The graded algebra $S^\bullet(V)_{GL_n(q)}$ is called the *coinvariant algebra* for $GL_n(q)$. We recall the following basic results of Mitchell.

**Lemma 4.9.** [Mi, Proposition 1.3, Theorem 1.4] *As $GL_n(q)$-modules,*
   *(1) $S^\bullet(V)$ has the same compositions series as $S^\bullet(V)^{GL_n(q)} \otimes S^\bullet(V)_{GL_n(q)}$;*
   *(2) $S^\bullet(V)_{GL_n(q)}$ has the same compositions series as the regular module $\mathbb{F}GL_n(q)$.*

Mitchell further pointed out that $S^\bullet(V)_{GL_n(q)}$ is not isomorphic to $\mathbb{F}GL_n(q)$ since $S^\bullet(V)_{GL_n(q)}$ has the trivial module (in degree zero) as a direct summand.

Thanks to Lemma 4.9, Theorem 4.7 admits the following reformulation in terms of the coinvariant algebra.

**Theorem 4.10.** *Suppose $p > n$ and $0 \leq k \leq q - 2$. Let $\mu$ be a partition with $\ell(\mu) \leq n$ and $\mu_1 - \mu_n \leq p - 1$.*
   *(1) If $\mu_1 \leq k$, then the graded composition multiplicity of $L((q-1)\rho - \mu + k\omega_n)$ in the coinvariant algebra $S^\bullet(V)_{GL_n(q)}$ is*

$$\frac{t^{-n+(k+1)\frac{q^n-1}{q-1}} \prod_{i=0}^{n-1}(1 - t^{q^n-q^i})}{\prod_{i=1}^{n}(1 - t^{q^i-1})} \, m_\mu(t^{-1}, t^{-q}, \ldots, t^{-q^{n-1}}).$$

   *(2) If $\mu_1 + k < q - 1$, then the graded composition multiplicity of $L((q-1)\rho + w_0\mu + k\omega_n)$ in the coinvariant algebra $S^\bullet(V)_{GL_n(q)}$ is*

$$\frac{t^{-n+(k+1)\frac{q^n-1}{q-1}} \prod_{i=0}^{n-1}(1 - t^{q^n-q^i})}{\prod_{i=1}^{n}(1 - t^{q^i-1})} \, m_\mu(t, t^q, \ldots, t^{q^{n-1}}).$$

Observe that the limit as $t \mapsto 1$ of either formula in the above theorem is equal to

$$\frac{\prod_{i=0}^{n-1}(q^n - q^i)}{\prod_{i=1}^{n}(q^i - 1)} m_\mu(1, 1, \ldots, 1)$$
$$= \dim \mathrm{St} \cdot m_\mu(1, 1, \ldots, 1)$$
$$= \dim \mathrm{St} \cdot |W^\mu|,$$

which is the dimension of the corresponding projective cover by Lemma 4.3. This is consistent with Lemma 4.9, since the composition multiplicity of a simple module in a regular module of a finite group is always equal to the dimension of its projective cover.

4.4. **Some open problems.** Theorem 4.1 and Theorem 4.7 have provided partial answers to the problem of finding the graded composition multiplicity of an irreducible $GL_n(q)$-module in the symmetric algebra $S^\bullet(V)$. They are obtained by converting the computations of the Steinberg module multiplicity in $S^\bullet(V) \otimes \wedge^\nu(V)$ in Section 3 and the results of Ballard and Tsushima.

*Question* 4.11. How to decompose $\mathrm{St} \otimes N$ into a direct sum of PIMs for a reasonable $GL_n(q)$-module $N$? Can we relax the restriction on $p$?

Suitable generalizations of results of Ballard and Tsushima in answer to the above question would allow one to expand the range of applicability of the approach developed in this paper. The methods developed in this paper seem likely to apply to the following.

*Question* 4.12. Find the composition multiplicity of the Steinberg module (or the simple modules around it) in the symmetric algebra of the natural module for other classical finite groups of Lie type.

The $GL_n(q)$-module $S^\bullet(V)$ is not semisimple, and it makes sense to ask the following.

*Question* 4.13. What is the graded multiplicity of a simple $GL_n(q)$-module $L(\mu)$ in the socle of $S^\bullet(V)$?

Dickson's classical theorem [Di] (see (4.4)) provides a first beautiful answer in case when $L(\mu)$ is the trivial module. Several generalizations have been obtained in [Mui, Mi, MP, KM], culminating in our previous work [WW] which settled this socle mutiplicity question for the simple modules of the form $\wedge^m(V) \otimes \mathrm{Det}^k$ for arbitrary prime powers $q = p^r$, $1 \le m \le n$ and $0 \le k \le q - 2$. The answer in *loc. cit.* fit into the following form, which we ask if it holds for a wider class of $L(\mu)$:

Let $\mu = (\mu_1, \dots, \mu_n)$ be a partition of $d$ with $1 \le d \le p - 1$ and $0 \le k \le q - 2 - \mu_1$. Is the multiplicity of the simple module $L(\mu) \otimes \mathrm{Det}^k$ in the socle of $S^\bullet(V)$ given by

$$\frac{t^{k \cdot \frac{q^n-1}{q-1}}}{\prod_{i=0}^{n-1}(1 - t^{q^n - q^i})} \cdot s_\mu(t, t^q, \dots, t^{q^{n-1}})?$$

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