Non-existence of non-topological solitons in some types of gauge field theories in Minkowski space

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Abstract

In this paper the conditions, under which non-topological solitons are absent in Yang-Mills theory coupled to a non-linear scalar field in Minkowski space, are obtained. It is also shown that non-topological solitons are absent in a theory describing massive complex vector field coupled to electromagnetic field in Minkowski space.

1 Introduction

In this paper we present some non-existence results for gauge field theories, which can be obtained with the help of the so-called "scaling arguments". These scaling arguments are based on the use of appropriate form of variations of fields. For the first time this technique was used in [1, 2] to show the absence of solitons in a non-linear scalar field theory (see also discussions in [3, 4]). Later such methods were used in more complicated cases, for example, in the case of Yang-Mills field coupled to a scalar field (see [4]) and for skyrmions, monopoles and instantons (see [5]), but only for static configurations of fields.

In this paper we will use the scaling arguments to show the absence of non-topological solitons (when all the fields vanish at spatial infinity) in some types of gauge field theories in Minkowski space even if configurations of fields are time-dependent. We restrict ourselves to the case when solutions to equations of motion are periodical in time with period $T<\infty$ (but not necessary of the simplest form $\sim e^{i\omega t}$). This restriction is necessary to make the proof mathematically rigorous, below we will discuss it in more detail. We will obtain the conditions for the scalar field potential which ensure the absence of non-topological solitons in a theory describing Yang-Mills field coupled to a non-linear scalar field (analogous scaling arguments were applied to the case of Klein-Gordon-Maxwell system with some simplifications, see [6]). For the non-negative scalar field potential we get a restriction, which is in agreement with that obtained in [7] using a different proof. Another consequence of our analysis is the absence of non-topological solitons in pure Yang-Mills theory, which of course is the well-known result and was obtained in [8]-[14] also using different proofs. Finally we will apply our methods to the system of U(1)-charged massive vector field to show the absence of non-topological solitons in this case.

2 Yang-Mills field coupled to a non-linear scalar field

Let us consider the following form of the four-dimensional action:

$$S=\int d^4x\left[\eta^{\mu\nu}(D_\mu\phi)^\dagger D_\nu\phi-V(\phi^\dagger\phi)-\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu}\right], \tag{1}$$

where $\eta_{\mu\nu} = diag(1,-1,-1,-1)$ is the flat Minkowski metric,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gC^{abc}A^b_\mu A^c_\nu, \tag{2}$$

$$D_\mu\phi = \partial_\mu\phi - igT^a A^a_\mu\phi, \tag{3}$$

where C_{abc} are the structure constants of a compact gauge group and T^a are generators of the group in the representation space of field ϕ. For simplicity action (1) describes scalar field in the fundamental representation, but all the results presented below are valid for any representation. We also suppose that

$$V(\phi^\dagger\phi)|_{\phi^\dagger\phi=0} = 0, \quad \frac{dV(\phi^\dagger\phi)}{d(\phi^\dagger\phi)}|_{\phi^\dagger\phi=0} < \infty. \tag{4}$$

The latter condition ensures that the trivial solution is $\phi \equiv 0$, $A_\mu = gT^a A^a_\mu \equiv 0$. Condition $V(\phi^\dagger\phi)|_{\phi^\dagger\phi=0} = 0$ is imposed for simplicity, in order not to deal with additive constants. We also suppose that:

1. there are no sources which are external to the system described by action (1);
2. solutions to equations of motion are periodical in time with period $T < \infty$ up to the coordinate shift $\vec{x} \to \vec{x} - \vec{l}$ with a constant vector $\vec{l}$ such that $|\vec{l}| < \infty$, i.e. for all fields the relation $\Psi(t,\vec{x}) \equiv \Psi(t+T, \vec{x}-\vec{l})$ must hold for any t, where $\Psi(t,\vec{x})$ schematically represents the fields under consideration (this means that we also consider solitons which may not be at rest, solitons moving with a speed smaller than the speed of light can be brought to the "at rest" form with the help of Lorentz transformations, but obviously this can not be done for solitons moving with the speed of light).

Let us discuss the second assumption. We can rewrite the initial action as

$$S = \int_{-\infty}^{\infty} dt \int d^3x L[\Psi(t,\vec{x})] = \sum_{n=-\infty}^{\infty} \int_{nT}^{(n+1)T} dt \int d^3x L[\Psi(t,\vec{x})] = \tag{5}$$
$$= \sum_{n=-\infty}^{\infty} \int_0^T dt \int d^3x L[\Psi(t+nT,\vec{x})] =$$
$$= \sum_{n=-\infty}^{\infty} \int_0^T dt \int d^3x L[\Psi(t+nT,\vec{x}-n\vec{l})] = \sum_{n=-\infty}^{\infty} \int_0^T dt \int d^3x L[\Psi(t,\vec{x})],$$

where n is integer. Thus, we can use the effective action

$$S = \int_0^T dt \int d^3x L[\Psi(t,\vec{x})] \tag{6}$$

instead of the initial one. We will discuss the necessity of our restriction to consider periodical solutions at the end of this section.

Now let us proceed to the system described by action (1). The corresponding effective action takes the form

$$S = \int_0^T dt \int d^3x \left[(D_0\phi)^\dagger D_0\phi - (D_i\phi)^\dagger D_i\phi - V(\phi^\dagger\phi) + \right. \tag{7}$$
$$\left. + \frac{1}{2}F^a_{0i}F^a_{0i} - \frac{1}{4}F^a_{ij}F^a_{ij}\right],$$

where $i, j = 1, 2, 3$. Let us denote

$$\int_0^T dt \int d^3x (D_0\phi)^\dagger D_0\phi = \Pi_0 \geq 0, \tag{8}$$

$$\int_0^T dt \int d^3x (D_i\phi)^\dagger D_i\phi = \Pi_1 \geq 0, \tag{9}$$

$$\int_0^T dt \int d^3x \frac{1}{2} F^a_{0i} F^a_{0i} = \Pi_{A0} \geq 0, \tag{10}$$

$$\int_0^T dt \int d^3x \frac{1}{4} F^a_{ij} F^a_{ij} = \Pi_{A1} \geq 0. \tag{11}$$

We suppose that all these integrals are finite.

We will be looking for smooth solutions to equations of motion, following from (1), of the form

$$\lim_{x^i \to \pm\infty} \phi(t, \vec{x}) = 0, \tag{12}$$

$$\lim_{x^i \to \pm\infty} A_\mu(t, \vec{x}) = 0. \tag{13}$$

Theorem 1. *For the potential of form (4) periodical in time up to a coordinate shift non-topological solitons of form (12), (13) with integrals (8)-(11) and integrals* $\int_0^T dt \int d^3x \frac{dV(\phi^\dagger\phi)}{d(\phi^\dagger\phi)} \phi^\dagger \phi$, $\int_0^T dt \int d^3x V(\phi^\dagger\phi)$ *finite are absent in the theory with action (1) if*

$$2\gamma \frac{dV(\phi^\dagger\phi)}{d(\phi^\dagger\phi)} \phi^\dagger\phi - 3V(\phi^\dagger\phi) \geq 0, \qquad \frac{1}{2} < \gamma \leq \frac{3}{2} \tag{14}$$

for any ϕ *(or at least for that range of values of the field* ϕ *which is supposed for a solution).*

Proof:

Let us suppose that there exists a solution $\phi(t, \vec{x})$, $A_\mu(t, \vec{x})$. With the help of (8)-(11) and using the periodicity of the fields we rewrite the effective action as

$$S = \Pi_0 - \Pi_1 - \int_0^T dt \int d^3x V\left(\phi^\dagger(\vec{x})\phi(\vec{x})\right) + \Pi_{A0} - \Pi_{A1}. \tag{15}$$

Now let us consider the following modification of our solution

$$\phi(t, \vec{x}) \to \lambda^\gamma \phi(t, \lambda\vec{x}), \tag{16}$$

$$A^a_0(t, \vec{x}) \to A^a_0(t, \lambda\vec{x}), \tag{17}$$

$$A^a_i(t, \vec{x}) \to \lambda A^a_i(t, \lambda\vec{x}) \tag{18}$$

with a real parameter λ. The action on this modified solution takes the form

$$S_\lambda = \lambda^{2\gamma-3}\Pi_0 - \lambda^{2\gamma-1}\Pi_1 - \tag{19}$$

$$-\lambda^{-3} \int_0^T dt \int d^3x V\left(\lambda^{2\gamma}\phi^\dagger(\vec{x})\phi(\vec{x})\right) + \lambda^{-1}\Pi_{A0} - \lambda\Pi_{A1}.$$

Since we suppose that $\phi(t,\vec{x})$, $A_\mu(t,\vec{x})$ is a solution to the equations of motion, variation of the action on this solution should vanish for any variations of the fields. For the case of the modifications described by (16)-(18) it means that

$$\frac{dS_\lambda}{d\lambda}|_{\lambda=1} = (2\gamma-3)\Pi_0 - (2\gamma-1)\Pi_1 - \\ - \int_0^T dt \int d^3x \left(2\gamma\frac{dV(\phi^\dagger\phi)}{d(\phi^\dagger\phi)}\phi^\dagger\phi - 3V(\phi^\dagger\phi)\right) - \Pi_{A0} - \Pi_{A1} = 0. \tag{20}$$

Indeed, $\lambda=1$ in (16)-(18) corresponds to a solution.

Now we are ready to consider the consequences following from equation (20).

1. $\frac{1}{2}<\gamma<\frac{3}{2}$. If
$$2\gamma\frac{dV(\phi^\dagger\phi)}{d(\phi^\dagger\phi)}\phi^\dagger\phi - 3V(\phi^\dagger\phi) \geq 0 \tag{21}$$
for any ϕ, then $\Pi_0=\Pi_1=\Pi_{A0}=\Pi_{A1}$ ($2\gamma\frac{dV(\phi^\dagger\phi)}{d(\phi^\dagger\phi)} - 3V(\phi^\dagger\phi)=0$ also), in this case $\phi\equiv 0$, $F^a_{\mu\nu}\equiv 0$ (the latter equality means that A_μ is a pure gauge and we can set $A_\mu\equiv 0$) and solitons of form (12), (13) are absent in the theory.

2. $\gamma=\frac{3}{2}$. If
$$\frac{dV(\phi^\dagger\phi)}{d(\phi^\dagger\phi)}\phi^\dagger\phi - V(\phi^\dagger\phi) \geq 0 \tag{22}$$
for any ϕ, then $\Pi_1=\Pi_{A0}=\Pi_{A1}$, in this case $A_\mu\equiv 0$, $\phi=\phi(t)\equiv 0$ (see (12)) and solitons of form (12), (13) are also absent in the theory.

3. $\gamma=\frac{1}{2}$. If
$$\frac{dV(\phi^\dagger\phi)}{d(\phi^\dagger\phi)}\phi^\dagger\phi - 3V(\phi^\dagger\phi) \geq 0 \tag{23}$$
for any ϕ, then $\Pi_0=\Pi_{A0}=\Pi_{A1}$, in this case $A_\mu\equiv 0$, $\phi=\phi(\vec{x})$. To obtain restrictions for this case, we can use the results of the well-known Derrick theorem [2], which states that in scalar field theories described by the standard action solitons are absent if $V(\phi^\dagger\phi)\geq 0$. Thus, solitons of form (12), (13) are absent in the theory if
$$\frac{dV(\phi^\dagger\phi)}{d(\phi^\dagger\phi)}\phi^\dagger\phi \geq 3V(\phi^\dagger\phi) \geq 0 \tag{24}$$
for any $\phi\neq 0$. But the latter inequality is less severe then that following from (22) for $V(\phi^\dagger\phi)\geq 0$.

End of the proof.
Remark: When considering static configuration of fields we can also take the following modifications of the fields:

$$\phi(t,\vec{x}) = \phi(\vec{x}) \to \lambda^\gamma\phi(\vec{x}), \tag{25}$$
$$A^a_0(t,\vec{x}) = A^a_0(\vec{x}) \to \lambda^\beta A^a_0(\vec{x}), \tag{26}$$
$$A^a_i(t,\vec{x}) = A^a_i(\vec{x}) \to A^a_i(\vec{x}) \tag{27}$$

with $\gamma > 0$, $\beta < -\gamma$. Then we get

$$\frac{dS_\lambda^\phi}{d\lambda}|_{\lambda=1} = 2(\gamma+\beta)\Pi_0 - \tag{28}$$

$$-2\gamma\left[\Pi_1 + \int_0^T dt \int d^3x \frac{dV(\phi^\dagger\phi)}{d(\phi^\dagger\phi)}\phi^\dagger\phi\right] + 2\beta\Pi_{A0} = 0.$$

Thus if

$$\frac{dV(\phi^\dagger\phi)}{d(\phi^\dagger\phi)}\phi^\dagger\phi \geq 0, \tag{29}$$

then using (8)-(10) we get $\phi \equiv 0$ and $A_0 \equiv 0$ (up to a gauge transformations). Then it is very easy to show that $A_i \equiv 0$ (again up to a gauge transformations), for example, applying (18) to the action containing A_i only. Thus, we get the absence of static solitons if condition (29) fulfills. This restriction was previously obtained in [15] (see also [16]).

Corollary 1. *1. For $V(\phi^\dagger\phi) \geq 0$, solitons of form (12), (13) are absent if*

$$\frac{dV(\phi^\dagger\phi)}{d(\phi^\dagger\phi)}\phi^\dagger\phi - V(\phi^\dagger\phi) \geq 0. \tag{30}$$

2. For $V(\phi^\dagger\phi) \leq 0$, solitons of form (12), (13) are absent if

$$\frac{dV(\phi^\dagger\phi)}{d(\phi^\dagger\phi)}\phi^\dagger\phi - 3V(\phi^\dagger\phi) > 0. \tag{31}$$

3. The restrictions presented above are valid in a system with scalar field only.

4. Non-topological solitons satisfying the conditions presented above are absent in pure Yang-Mills theory.

The proof of this corollary follows directly from Theorem 1.
Remark: for the first time the no-go condition (30) for $V(\phi^\dagger\phi) \geq 0$ and the potential $V(\phi^\dagger\phi)$ including a positive mass term was obtained in a different way in [7]. The absence of solitons ("classical lumps") in pure Yang-Mills theory was shown long ago for the static [8], periodic [9] and general cases [10, 11, 12, 13, 14].

It should be noted that the results presented above can be obtained by taking equations of motion following from action (1), multiplying them by variations of fields following from (16)-(18) (or from (25)-(27)), integrating over the four-volume and combining the results coming from different equations of motion. Indeed, variational principle gives us equations of motion and thus all the results obtained above can be also obtained with the help of equations of motion. In this sense the proof presented in formulas (25)-(29) does not differ from that of [15, 16]. Moreover, as it was noted in [5], in principle one can get an infinite number of integral identities using equations of motion (or local identities such as conservation of the energy-momentum tensor which follow from equations of motion). But it seems that considering direct transformations of the action under rescalings of the fields is technically simpler at least for obtaining no-go results.

In the end of this section let us discuss why we restrict ourselves to considering solutions periodical in time. The motivation is the following. When we obtain equations of motions we suppose that variations of fields vanish at spatial and time infinity. But the variations of the fields coming from (16)-(18) clearly do not tend to zero at time infinity in general case and we have a contradiction between the forms of variations used to obtain equations of motion and the forms of variations used to show the absence of some solutions to these equations of motion. In principle, in the case of non-vanishing variations there can arise additional terms in the action coming from surface terms at time infinity (analogous problem, but in the case of the Derrick theorem applied to a finite space-time domain, was discussed in [17]). Considering solutions periodical in time up to a coordinate shift makes the surface terms at $t = 0$ and $t = T$, which arise when obtaining equations of motion from effective action (7), be modulo equal and vanish. The latter ensures that equations of motions obtained from the initial action and those obtained from the effective action coincide, as well as possible solutions. In this case the procedure of obtaining the no-go results described in this section appears to be consistent with equations of motions coming from the original action.

3 Charged massive vector field

The scaling arguments, used above for a theory describing Yang-Mills field coupled to a non-linear scalar field, can be used to show the absence of solitons in other gauge field theories. As an example, let us consider the massive complex vector field coupled to the electromagnetic field. The action of this theory has the form

$$S = \int d^4x \left[-\frac{1}{2}\eta^{\mu\rho}\eta^{\nu\sigma}W^-_{\mu\nu}W^+_{\rho\sigma} + m^2\eta^{\mu\nu}W^-_\mu W^+_\nu - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \right] \tag{32}$$

with $m \neq 0$, where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{33}$$

$$D_\mu W^\pm_\nu = \partial_\mu W^\pm_\nu \mp ieA_\mu W^\pm_\nu, \tag{34}$$

$$W^\pm_{\mu\nu} = D_\mu W^\pm_\nu - D_\nu W^\pm_\mu. \tag{35}$$

Again we suppose that:

1. there are no sources which are external to the system described by action (32);
2. all fields are smooth and vanish at spatial infinity;
3. solutions to equations of motion are periodical in time with period T up to a coordinate shift.

Let us denote

$$\int_0^T dt \int d^3x W^-_{0i}W^+_{0i} = \Pi_{W0} \geq 0, \tag{36}$$

$$\int_0^T dt \int d^3x \frac{1}{2} W^-_{ij}W^+_{ij} = \Pi_{W1} \geq 0, \tag{37}$$

$$m^2 \int_0^T dt \int d^3x W^-_0 W^+_0 = V_0 \geq 0, \tag{38}$$

$$m^2 \int_0^T dt \int d^3x W^-_i W^+_i = V_1 \geq 0, \tag{39}$$

$$\int_0^T dt \int d^3x \frac{1}{2} F_{0i} F_{0i} = \Pi_{A0} \geq 0, \tag{40}$$

$$\int_0^T dt \int d^3x \frac{1}{4} F_{ij} F_{ij} = \Pi_{A1} \geq 0. \tag{41}$$

We also suppose that all these integrals are finite.

Analogously to what was made above, we also suppose that there exists a solution to the corresponding equations of motion and consider the following modification of this solution:

$$W_0^{\pm}(t, \vec{x}) \to \lambda^{\beta-1} W_0^{\pm}(t, \lambda\vec{x}), \tag{42}$$

$$W_i^{\pm}(t, \vec{x}) \to \lambda^{\beta} W_i^{\pm}(t, \lambda\vec{x}), \tag{43}$$

$$A_0^a(t, \vec{x}) \to A_0^a(t, \lambda\vec{x}), \tag{44}$$

$$A_i^a(t, \vec{x}) \to \lambda A_i^a(t, \lambda\vec{x}) \tag{45}$$

with a real parameter λ. For the "modified" action we get

$$S_\lambda = \lambda^{2\beta-3}\Pi_{W0} - \lambda^{2\beta-1}\Pi_{W1} + \lambda^{2\beta-5}V_0 - \\ -\lambda^{2\beta-3}V_1 + \lambda^{-1}\Pi_{A0} - \lambda\Pi_{A1}. \tag{46}$$

Now let us take

$$\beta = \frac{3}{2}. \tag{47}$$

We get

$$\frac{dS_\lambda}{d\lambda}|_{\lambda=1} = -2\Pi_{W1} - 2V_0 - \Pi_{A0} - \Pi_{A1} = 0. \tag{48}$$

From this equation it follows that

$$\Pi_{W1} = V_0 = \Pi_{A0} = \Pi_{A1} \equiv 0, \tag{49}$$

which means that $F_{\mu\nu} \equiv 0$ and we can set $A_\mu \equiv 0$, we also get $W_0^{\pm} \equiv 0$ and $W_{ij}^{\pm} \equiv 0$. With $A_\mu \equiv 0$ we can rewrite $W_{ij}^{\pm} \equiv 0$ as

$$\partial_i W_j^{\pm} - \partial_j W_i^{\pm} \equiv 0. \tag{50}$$

Now let us take equations of motion for the field $W_\mu^{\pm}$ with $A_\mu \equiv 0$. It follows from these equations that

$$\partial^\mu W_\mu^{\pm} = 0. \tag{51}$$

Using the fact that $W_0^{\pm} \equiv 0$ we get

$$\partial^i W_i^{\pm} = 0. \tag{52}$$

Equations (50) and (52) can be rewritten as

$$\operatorname{div} \vec{W}^{\pm} = 0, \qquad \operatorname{rot} \vec{W}^{\pm} = 0, \tag{53}$$

where $\vec{W}^{\pm} = (W_1^{\pm}, W_2^{\pm}, W_3^{\pm})$. Equations (53) imply that

$$\vec{W}^{\pm} = \operatorname{grad} \varphi^{\pm}, \tag{54}$$

$$\Delta\varphi^{\pm} = 0, \tag{55}$$

where $\varphi^{\pm} = \varphi^{\pm}(t, \vec{x})$, $(\varphi^{+})^{*} = \varphi^{-}$. The condition $\int d^3x W_i^- W_i^+ = \int d^3x \partial_i \varphi^- \partial_i \varphi^+ < \infty$ clearly leads to $\varphi^{\pm} = \varphi^{\pm}(t)$ (one can simply use the results of [1, 2] to show it) and therefore

$$\vec{W}^{\pm} \equiv 0. \tag{56}$$

Thus

$$W_{\mu}^{\pm} \equiv 0. \tag{57}$$

Finally, we can make the following statement:

Proposition 1. *Periodical in time up to a coordinate shift non-topological solitons with finite integrals (36)-(41) are absent in the theory described by action (32).*

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