

The Work of Ngô Bao Châu

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In August 2010, Ngô Bao Châu was awarded a Fields Medal for his deep work relating the Hitchin fibration to the Arthur-Selberg trace formula, and in particular for his proof of the Fundamental Lemma for Lie algebras [27], [28].

1. the Trace Formula

A function $h : G \rightarrow \mathbb{C}$ on a finite group G is a *class function* if $h(g^{-1}xg) = h(x)$ for all $x, g \in G$. A class function is constant on each conjugacy class. A basis of the vector space of class functions is the set of characteristic functions of conjugacy classes.

A *representation* of G is a homomorphism $\pi : G \rightarrow GL(V)$ from G to a group of invertible linear transformations on a complex vector space V . It follows from the matrix identity

$$\text{trace}(B^{-1}AB) = \text{trace}(A)$$

that the function $g \mapsto \text{trace}(\pi(g))$ is a class function. This function is called an *irreducible character* if V has no proper G -stable subspace. A basic theorem in finite group theory asserts that the set of irreducible characters forms a second basis of the vector space of class functions on G .

A *trace formula* is an equation that gives the expansion of a class function h on one side of the equation in the basis of characteristic functions of conjugacy classes C and on the other side in the basis of irreducible characters

$$\sum_C b_C \text{char}_C = h = \sum_\pi a_\pi \text{trace } \pi,$$

for some complex coefficients b_C and a_π depending on h . The side of the equation with conjugacy classes is called the *geometric side* of the trace formula and the side with irreducible characters is called the *spectral side*.

When G is no longer assumed to be finite, some analysis is required. We allow G to be a Lie group or more generally a locally compact topological group. The vector space V may be infinite-dimensional so that a trace of a linear transformation of V need not converge. To improve convergence, the irreducible character is no longer viewed as a function, but rather as a distribution

$$f \mapsto \text{trace} \int_G \pi(g) f(g) dg,$$

where f runs over smooth compactly supported test functions on the group, and dg is a G -invariant measure. Similarly, the characteristic function of conjugacy class is replaced with a distribution that integrates a test function

f over the conjugacy class C with respect to an invariant measure:

$$(1) \quad f \mapsto \int_C f(g^{-1}xg) dg.$$

The integral (1) is called an *orbital integral*. A trace formula in this setting becomes an identity that expresses a class distribution (called an *invariant distribution*) on the geometric side of the equation as a sum of orbital integrals and on the spectral side of the equation as a sum of distribution characters.

The celebrated Selberg trace formula is an identity of this general form for the invariant distribution associated with the representation of $SL_2(\mathbb{R})$ on $L^2(SL_2(\mathbb{R})/\Gamma)$, for a discrete subgroup Γ . Arthur generalized the Selberg trace formula to reductive groups of higher rank.

2. History

The Fundamental Lemma (FL) is a collection of identities of orbital integrals that arise in connection with a trace formula. It takes several pages to write all of the definitions that are needed for a precise statement of the lemma [17]. Fortunately, the significance of the lemma and the main ideas of the proof can be appreciated without the precise statement.

Langlands conjectured these identities in lectures on the trace formula in Paris in 1980 and later put them in more precise form with Shelstad [21], [22]. Over time, supplementary conjectures were formulated, including a *twisted* conjecture by Kottwitz and Shelstad, and a *weighted* conjecture by Arthur [20], [1]. Identities of orbital integrals on the group can be reduced to slightly easier identities on the Lie algebra [23]. Papers by Waldspurger rework the conjectures into the form eventually used by Ngô in his solution [35], [33]. Over the years, Chaudouard, Goresky, Kottwitz, Laumon, MacPherson, and Waldspurger among others have made fundamental contributions that led up to the proof of the FL or extended the results afterwards [24], [14], [15], [9], [10], [11]. It is hard to do justice to all those who have contributed to a problem that has been intensively studied for decades, while giving special emphasis to the spectacular breakthroughs by Ngô.

With the exception of the FL for the special linear group $SL(n)$, which can be solved with representation theory; starting in the early 1980s, all plausible lines of attack on the general problem have been geometric. Indeed, a geometric approach is suggested by direct computations of these integrals in special cases, which give their values as the number of points on hyperelliptic curves over finite fields [19], [16].

To motivate the FL, we must recall the bare outlines of the ambitious program launched by Langlands in the late 1960s to use representation theory to understand vast tracts of number theory. Let F be a finite field extension of the field of rational numbers \mathbb{Q} . The ring of adeles \mathbb{A} of F is a locally compact topological ring that contains F and has the property that F embeds discretely in \mathbb{A} with a compact quotient $F \backslash \mathbb{A}$. The ring of adeles is a convenient starting point for the analytic treatment of the number field F . If G is a reductive group defined over F with center Z , then $G(F)$ is a discrete subgroup of $G(\mathbb{A})$ and the quotient $G(F)Z(\mathbb{A}) \backslash G(\mathbb{A})$ has finite volume. A representation π of $G(\mathbb{A})$ that appears in the spectral decomposition of

$$L^2(G(F)Z(\mathbb{A}) \backslash G(\mathbb{A}))$$

is said to be an *automorphic representation*. The automorphic representations (by descending to the quotient by $G(F)$) are those that encode the number-theoretic properties of the field F . The theory of automorphic representations just for the two linear groups $G = GL(2)$ and $GL(1)$ already encompasses the classical theory of modular forms and global class field theory.

There is a complex-valued function $L(\pi, s)$, $s \in \mathbb{C}$, called an automorphic L -function, attached to each automorphic representation π . (The L -function also depends on a representation of a dual group, but we skip these details.) Langlands's philosophy can be summarized as two objectives:

- (1) Show that many L -functions that routinely arise in number theory are automorphic.
- (2) Show that automorphic L -functions have wonderful analytic properties.

There are two famous examples of this philosophy. In Riemann's paper on the zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

he proved that it has a functional equation and meromorphic continuation by relating it to a θ -series (an automorphic entity) and then using the analytic properties of the θ -series. Wiles proved Fermat's Last Theorem by showing that the L -function $L(E, s)$ of every semi-stable elliptic curve over \mathbb{Q} is automorphic. From automorphicity follows the analytic continuation and functional equation of $L(E, s)$.

The Arthur-Selberg trace formula has emerged as a general tool to reach the first objective (1) of Langlands's philosophy. To relate one L -function to another, two trace formulas are used in tandem (Figure 1). An automorphic L -function can be encoded on the spectral side of the Arthur-Selberg trace

formula. A second L -function is encoded on the spectral side of a second trace formula of a possibly different kind, such as a topological trace formula. By equating the geometric sides of the two trace formulas, identities of orbital integrals yield identities of L -functions.

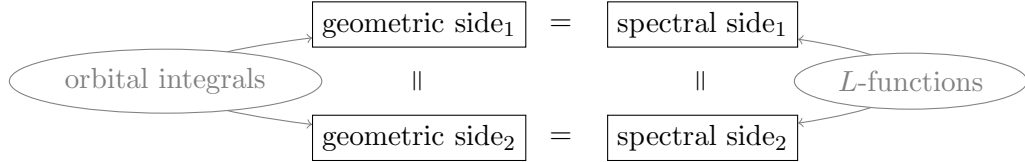


FIGURE 1. A pair of trace formulas can transform identities of orbital integrals into identities of L -functions.

The value of the FL lies in its utility. The FL can be characterized as the minimal set of identities that must be proved in order to put the trace formula in a useable form for applications to number theory, such as those mentioned at the end of this report.

3. the Hitchin Fibration

Ngô's proof of the FL is based on the Hitchin fibration [18].

Every endomorphism A of a finite dimensional vector space V has a characteristic polynomial

$$(2) \quad \det(t - A) = t^n + a_1 t^{n-1} + \dots + a_n.$$

Its coefficients a_i are symmetric polynomials of the eigenvalues of A . This determines a *characteristic map* $\chi : \text{end}(V) \rightarrow \mathfrak{c}$, from the Lie algebra of endomorphisms of V to the vector space \mathfrak{c} of coefficients (a_1, \dots, a_n) . This construction generalizes to a characteristic map $\chi : \mathfrak{g} \rightarrow \mathfrak{c}$ for every reductive Lie algebra \mathfrak{g} , by evaluating a set of symmetric polynomials on \mathfrak{g} .

Fix once and for all a smooth projective curve X of genus g over a finite field k .

In its simplest form, a *Higgs pair* (E, ϕ) is what we obtain when we allow an element Z of the Lie algebra $\text{end}(V)$ to vary continuously along the curve X . As we vary along the curve, the vector space V sweeps out a vector bundle E on X , and the element $Z \in \text{end}(V)$ sweeps out a section ϕ of the bundle $\text{end}(E)$ or of the bundle $\text{end}(E) \otimes \mathcal{O}_X(D)$ when the section acquires finitely many poles prescribed by a divisor D of X . Extending this construction to a general reductive Lie group G with Lie algebra \mathfrak{g} , a Higgs pair (E, ϕ) consists of a principal G -bundle E and a section ϕ of the bundle $\text{ad}(E) \otimes \mathcal{O}_X(D)$

associated with E and the adjoint representation of G on \mathfrak{g} . For each X , G , and D , there is a moduli space \mathcal{M} (or more correctly, moduli stack) of all Higgs pairs (E, ϕ) .

The *Hitchin fibration* is the morphism obtained when we vary the characteristic map $\chi : \mathfrak{g} \rightarrow \mathfrak{c}$ along a curve X . For each Higgs pair (E, ϕ) , we evaluate the characteristic map $p \mapsto \chi(\phi_p)$ of the endomorphism ϕ at each point $p \in X$. This function belongs to the set \mathcal{A} of a global sections of the bundle $\mathfrak{c} \otimes \mathcal{O}_X(D)$ over X . The Hitchin fibration is this morphism $\mathcal{M} \rightarrow \mathcal{A}$.

Abelian varieties occur naturally in the Hitchin fibration. To illustrate, we return to the Lie algebra $\mathfrak{g} = \text{end}(V)$. For each section $a = (a_1, \dots, a_n) \in \mathcal{A}$, the characteristic polynomial

$$(3) \quad t^n + a_1(p)t^{n-1} + \dots + a_n(p) = 0,$$

defines an n -fold cover Y_a of X (called the *spectral curve*). By construction, each point of the spectral curve is a root of the characteristic polynomial at some $p \in X$. We consider the simple setting when Y_a is smooth and the discriminant of the characteristic polynomial is sufficiently generic. A Higgs pair (E, ϕ) over the section a determines a line (a one-dimensional eigenspace of ϕ with eigenvalue that root) at each point of the spectral curve, and hence a line bundle on Y_a . This establishes a map from Higgs pairs over a to $\text{Pic}(Y_a)$, the group of line bundles on the spectral curve Y_a . Conversely, just as linear maps can be constructed from eigenvalues and eigenspaces, Higgs pairs can be constructed from line bundles on the spectral curve Y_a . The connected component $\text{Pic}^0(Y_a)$ is an abelian variety. Even outside this simple setting, the group of symmetries of the Hitchin fiber over $a \in \mathcal{A}$ has an abelian variety as a factor.

4. The Proof of the FL

Shifting notation (as justified in [34], [11]), we let F be the field of rational functions on a curve X over a finite field k . One of the novelties of Ngô's work is to treat the FL as identities over the global field F , rather than as local identities at a given place of X . By viewing each global section of $\mathcal{O}_X(D)$ as a rational function on X , each point $a \in \mathcal{A}$ is identified with an F -valued point $a \in \mathfrak{c}(F)$. The preimage of a under the characteristic map χ is a union of conjugacy classes in $\mathfrak{g}(F)$, and therefore corresponds to terms of the Arthur-Selberg trace formula for the Lie algebra. The starting point of Ngô's work is the following geometric interpretation of the trace formula.

Theorem 1 (Ngô). *There is an explicit test function f_D , depending on the divisor D , such that for every anisotropic element $a \in \mathcal{A}^{an}$, the sum of the orbital integrals with characteristic polynomial a in the trace formula for f_D*

equals the number of Higgs pairs in the Hitchin fibration over a , counted with multiplicity.

The proof is based on Weil’s description of vector bundles on a curve in terms of the cosets of a compact open subgroup of $G(\mathbb{A})$. Orbital integrals have a similar coset description.

From this starting point, the past thirty years of research on the trace formula can be translated into geometrical properties of the Hitchin fibration. In particular, Ngô formulates and then solves the FL as a statement about counting points in Hitchin fibrations.

The identities of the FL are between the orbital integrals on two different reductive groups G and H . A root system is associated to each reductive group. There is a duality of every root system that interchanges its long and short roots. The two reductive groups of the FL are related only indirectly: the root system dual to that of H is a subset of the root system dual to that of G (Figure 2). Informally, the set of representations of a group is in duality with the group itself, so by a double duality, when the dual root systems are directly related, we might also expect their representation theories to be directly related. This expectation is supported by an overwhelming amount of evidence.

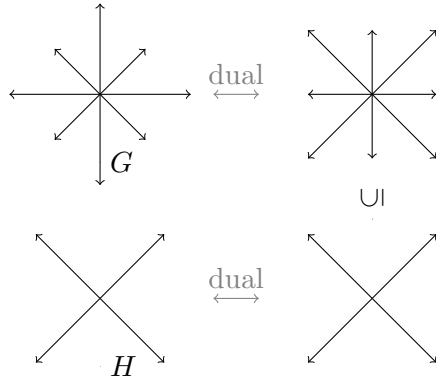


FIGURE 2. The two root systems in each row are in duality. The root system on the bottom right is a subset of the root system on the upper right.

By using the same curve X for both H and G , and by comparing the characteristic maps for the two groups, Ngô produces a map $\nu : \mathcal{A}_H \rightarrow \mathcal{A}_G$ of the bases of the two Hitchin fibrations, but to kill unwanted monodromy he prefers to work with a base-change $\tilde{\nu} : \tilde{\mathcal{A}}_H \rightarrow \tilde{\mathcal{A}}_G$. The particular identities of the FL pick out a subspace $\tilde{\mathcal{A}}_\kappa$ of $\tilde{\mathcal{A}}_G$ containing $\tilde{\nu}(\tilde{\mathcal{A}}_H)$. Restricting the Hitchin fibration to anisotropic elements, to prove the FL, he must compare

fibers of the two (base-changed, anisotropic) Hitchin fibrations $\tilde{\mathcal{M}}_G^{an} \rightarrow \tilde{\mathcal{A}}_\kappa^{an}$ and $\tilde{\mathcal{M}}_H^{an} \rightarrow \tilde{\mathcal{A}}_H^{an}$ over corresponding points of the base spaces.

The base $\tilde{\nu}(\tilde{\mathcal{A}}_H^{an})$ contains a dense open subset of elements that satisfy a transversality condition. For $\mathfrak{g} = \text{end}(V)$ this condition requires the self intersections of the spectral curve (Equation 3) to be transversal (Figure 3). For a particularly nice open subset $\tilde{\mathcal{U}} \subset \tilde{\nu}(\tilde{\mathcal{A}}_H^{an})$ of transversal elements, the number of points in a Hitchin fiber may be computed directly, and the FL can be verified in this case without undue difficulty.

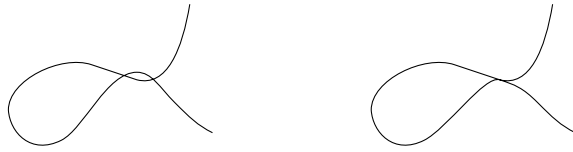


FIGURE 3. After giving a direct proof of the FL under the assumption of transversality (left), Ngô obtains the general case (right) by continuity.

To complete the proof, Ngô argues by continuity, that because the identities of the FL hold on a dense open subset of $\tilde{\nu}(\tilde{\mathcal{A}}_H^{an})$, the identities are also forced to hold on the closure of the subset, even without transversality. The justification of this continuity principle is the deepest part of his work.

Through the legacy of Weil and Grothendieck, we know the number of points on a variety (or even on a stack if you are brave enough) over a finite field to be determined by the action of the Frobenius operator on cohomology. To cohomology we turn. After translation into this language, the FL takes the form of a desired equality of (the semisimplifications of) two perverse sheaves over a common base space $\tilde{\nu}(\tilde{\mathcal{A}}_H^{an})$. By the BBDG decomposition theorem, over the algebraic closure of k , the perverse sheaves break into direct sums of simple terms, each given as the intermediate extension of a local system on an open subset Z^0 of its support Z [5]. The decomposition theorem already implies a weak continuity principle; each simple factor is uniquely determined by its restriction to a dense open subset of its support. This weak continuity is not sufficient, because it does not rule out the existence of supports Z that are disjoint from the open set of transverse elements.

To justify the continuity principle, Ngô shows that the support Z of each of these sheaves lies in $\tilde{\nu}(\tilde{\mathcal{A}}_H^{an})$ and intersects the open set $\tilde{\mathcal{U}}$ of transverse elements. In rough terms, the continuity principle consists in showing that every cohomology class can be pushed out into the open. There are two parts to the argument: the cohomology class first is pushed into the top degree cohomology, and then from there into the open. In the first part,

the abelian varieties mentioned above enter in a crucial way. By taking cap product operations coming from the abelian varieties, and using Poincaré duality, a nonzero cohomology class produces nonzero class in the top degree cohomology of a Hitchin fiber. This part of his proof uses a stratification of the base of the Hitchin fibration and a delicate inequality relating the dimension of the abelian varieties to the codimension of the strata.

In the second part of the argument, a set of generators of the top degree cohomology of the fiber is provided by the component group π_0 of a Picard group that acts as symmetries on the fibers. Recall that the two groups G and H are related only indirectly through a duality of root systems. At this step of the proof, a duality is called for, and Ngô describes π_0 explicitly, generalizing classical dualities of Kottwitz, Tate, and Nakayama in class field theory. With this dual description of the top cohomology, he is able to transfer information about the support Z on the Hitchin fibration for G to the Hitchin fibration on H and deduce the desired support and continuity theorems. With continuity in hand, the FL follows as described above.

Further accounts of Ngô's work and the proof of the FL appear in [26], [12], [2], [13], [8], [7], [29].

5. Applications

Only in the land of giants does the profound work of a Fields medalist get called a lemma. Its name reminds us nonetheless that the FL was never intended as an end in itself. A lemma it is. Although proved only recently, it has already been put to use as a step in the proofs of the following major theorems in number theory.

- (1) Arthur's forthcoming classification of automorphic representations of classical groups [3].
- (2) The calculation of the cohomology of Shimura varieties and their Galois representations [25], [30].
- (3) The Sato-Tate Conjecture for elliptic curves over a totally real number field [4].
- (4) Iwasawa's Main Conjecture for $GL(2)$ [32], [31].
- (5) The Birch and Swinnerton-Dyer Conjecture for a positive fraction of all elliptic curves over \mathbb{Q} [6].

The proof of the following recent theorem invokes the FL. It is striking that this result in pure arithmetic ultimately relies on the Hitchin fibration, which was originally introduced in the context of completely integrable systems!

Theorem 2 ([4]). *Let n_p be the number of ways a prime p can be expressed as a sum of twelve integers:*

$$n_p = \text{card} \{(a_1, \dots, a_{12}) \in \mathbb{Z}^{12} \mid p = a_1^2 + \dots + a_{12}^2\}.$$

Then the real number

$$t_p = \frac{n_p - 8(p^5 + 1)}{32p^{5/2}}$$

belongs to the interval $[-1, 1]$, and as p runs over all primes, the numbers t_p are distributed within that interval according to the probability measure

$$\frac{2}{\pi} \sqrt{1 - t^2} dt.$$

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