On Subdivisions of Dihedral Sets

Sho Saito*

December 3, 2010

Abstract

We investigate subdivisions of simplicial, cyclic, and dihedral sets by using Drinfeld's expression of geometric realization as a filtered colimit.

1 Introduction

Drinfeld [[2]] pointed out that the geometric realization $|X|$ of a simplicial set X can be expressed as a filtered colimit. Namely,

$$|X| = \operatorname{colim}_{F\subset[0,1]:\text{finite}} X[\pi_0([0,1]\setminus F)].$$

Analogously, he also showed that if X is a cyclic set then its realization is given by

$$|X| = \operatorname{colim}_{F\subset\mathbb{R}/\mathbb{Z}:\text{finite}} X[\pi_0((\mathbb{R}/\mathbb{Z})\setminus F)].$$

It follows that geometric realization commutes with finite limits, and the group of orientation-preserving homeomorphisms of the interval $[0,1]$ (respectively, the circle S^1) acts on the realization of the simplicial (resp. cyclic) set X.

In this paper we will first survey his method in section 2, introducing a similar expression for dihedral sets and proving that the group of homeomorphisms of the circle acts on their realization . Then we will see how these expressions clarify the description of subdivisions of simplicial and cyclic sets in section 3. If $\operatorname{sd}_r X$ is the r-fold subdivision of the simplicial (resp. cyclic) set X defined by

*Second year graduate student at Graduate School of Mathematics, Nagoya University, email: m09019h@math.nagoya-u.ac.jp

Bökstedt-Hsiang-Madsen [1], where r is a positive integer, it will be shown that its realization is expressed by

$$|\mathrm{sd}_r\, X| = \mathrm{colim}_{F\subset[0,r]:\mathrm{finite}}\, X[\pi_0([0,r]\setminus F)]$$

$$(\text{resp. } |\mathrm{sd}_r\, X| = \mathrm{colim}_{F\subset\mathbb{R}/r\mathbb{Z}:\mathrm{finite}}\, X[\pi_0((\mathbb{R}/r\mathbb{Z})\setminus F)].)$$

Thus it becomes clear that $|\mathrm{sd}_r\, X|$ and $|X|$ are canonically homeomorphic, and in cyclic case this homeomorphism is compatible with the actions by the circle on both sides . We will show that the subdivision $\mathrm{sbd}_r\, Y$ of a dihedral set Y, which was defined by Spaliński [5] to be a simplicial set, has an even richer structure of $\Delta_{r,\mathrm{sq}}$-set , and that its realization is given by

$$|\mathrm{sbd}_r\, Y| = \mathrm{colim}_{F\subset\mathbb{R}/2r\mathbb{Z}:\mathrm{finite}}\, Y[\pi_0((\mathbb{R}/2r\mathbb{Z})\setminus F)].$$

Here $\Delta_{r,\mathrm{sq}}D$ is the category that makes the family $\{D_{2r(n+1)}\}_{n\geq 0}$ of dihedral groups of order $4r(n+1)$ into a crossed simplicial group in the sense of Loday [3]. This makes it easy to verify that the dihedral groups simplicially act on $\mathrm{sbd}_r\, Y$, so that we can understand the action by dihedral groups on $|Y|$ simplicially. We will also consider a simplicial action by cyclic groups on $\mathrm{sbd}_r\, Y$ and show that by taking fixed points we again obtain a dihedral set.

2 Drinfeld's method

In this section we will summarize Drinfeld's results ([[2]]) on realization of simplicial sets and cyclic sets. We will also derive a similar formula for realization of dihedral sets.

2.1 Simplicial sets

Let Δ be the category of the finite linearly ordered sets $[n] = \{0 < \cdots < n\}$ ($n \geq 0$) and order-preserving maps. By definition a **simplicial set** is a functor $X : \Delta^{\mathrm{op}} \to \mathrm{Sets}$. Let Δ_{big} be the category of all non-empty finite linearly ordered sets and order-preserving maps. Note that for every object $\mathcal{A} = \{a_0 < \cdots < a_n\}$ of Δ_{big} there is a unique isomorphism $i_{\mathcal{A}} : \mathcal{A} \to [n]$ that sends a_i to i. A simplicial set $X : \Delta^{\mathrm{op}} \to \mathrm{Sets}$ can be extended by using the isomorphisms $i_{\mathcal{A}}$ to a functor $\widetilde{X} : \Delta^{\mathrm{op}}_{\mathrm{big}} \to \mathrm{Sets}$. More precisely, $\widetilde{X}$ sends an object $\mathcal{A}$ to $X[m]$ if $|\mathcal{A}| = m+1$ and a morphism $f : \mathcal{A} \to \mathcal{B}$ to $X(\widetilde{f}) : X[n] \to X[m]$, where $|\mathcal{A}| = m+1$, $|\mathcal{B}| = n+1$ and $\widetilde{f}$ is the unique map in Δ that makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{i_{\mathcal{A}}} & [m] \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \widetilde{f}} \\ \mathcal{B} & \xrightarrow{i_{\mathcal{B}}} & [n] \end{array}$$

If $\widetilde{X}'$ is another extension, i.e. a functor $\Delta^{\mathrm{op}}_{\mathrm{big}} \to \mathrm{Sets}$ that is identical to X on Δ^{op}, then there exists a unique natural isomorphism $\kappa : \widetilde{X} \to \widetilde{X}'$ such that $\kappa\,|_{\Delta^{\mathrm{op}}} = \mathrm{id}_X$.

For each finite subset F of the interval $[0,1]$, order the set of connected components $\pi_0([0,1] \setminus F)$ by declaring $[x] \leq [y]$ if $x \leq y \in [0,1] \setminus F$. If $F \subset G$ there is an order-preserving map $\pi_0([0,1] \setminus G) \to \pi_0([0,1] \setminus F)$. Denote by $\mathcal{F}$ the set of all finite subsets of $[0,1]$. The set $\mathcal{F}$ is ordered by inclusion and thus viewed as a category. Note that this category $\mathcal{F}$ is a filtering.

The **geometric realization** of the simplicial set X is by definition the k-space given by the colimit $\mathrm{colim}_{\Delta[n][-] \to X[-]} \Delta[n]$, where $\Delta[n] = \{(z_0, \ldots, z_n) \in [0,1]^{n+1} \mid \sum_{i=0}^{n} z_i = 1\}$ and where the indices are the all simplicial maps from the standard simplicial simplicies to X. Drinfeld [[2]] pointed out that we may use $\mathrm{Sim}^n = \{(x_1, \ldots, x_n) \mid 0 \leq x_1 \leq \cdots \leq x_n \leq 1\}$ instead of $\Delta[n]$, and Sim^n can be written as the filtered colimit $\mathrm{colim}_{F \in \mathcal{F}} \widetilde{\Delta[n]}[\pi_0([0,1] \setminus F)]$ by identifying $(x_1, \ldots, x_n) \in \mathrm{Sim}^n$ with the piecewise constant function $f : [0,1] \to [n]$ defined by $f(x) = i$ for $x_i < x < x_{i+1}$. Since $X = \mathrm{colim}_{\Delta[n][-] \to X[-]} \Delta[n][-]$, we have:

Theorem 2.1 *The geometric realization $|X|$ of the simplicial set X is given by the filtered colimit $\mathrm{colim}_{F \in \mathcal{F}} \widetilde{X}[\pi_0([0,1] \setminus F)]$.*

From this we immediately know that geometric realization, at least as a functor to the category Sets of sets, commutes with finite limits, and that the group $\mathrm{Aut}[0,1]$ of orientation-preserving homeomorphism of $[0,1]$ acts on $|X|$. Even stronger assertions hold:

Corollary 2.2

1. *Geometric realization, considered as a functor to the category $\mathcal{K}$ of k-spaces, commutes with finite limits.*

2. *Topologize $\mathrm{Aut}[0,1]$ by the subspace topology of the standard k-space topology of $\mathrm{Hom}_{\mathcal{K}}([0,1],[0,1])$. Then $\mathrm{Aut}[0,1]$ acts continuously on the realization $|X|$ of a simplicial set X.*

(Proof) 1. Since we are working in $\mathcal{K}$, it suffices to show that the canonical map

$$|\Delta[m][-] \times \Delta[n][-]| \to |\Delta[m][-]| \times |\Delta[n][-]|$$

is a homeomorphism. By the theorem this map is a continuous bijection, with target a Hausdorff space. Moreover, the simplicial set $\Delta[m][-] \times \Delta[n][-]$ has only finitely many non-degenerate simplices. From this we know that the domain space $|\Delta[m][-] \times \Delta[n][-]|$ is a quotient of a disjoint union of finitely many compact spaces $\Delta[m][k] \times \Delta[n][k] \times \Delta[k]$, so that $|\Delta[m][-] \times \Delta[n][-]|$ itself is compact. Hence the canonical map, which is a continuous bijection from a compact space to a Hausdorff space, is a homeomorphism.

2. An orientation preserving homeomorphism $\alpha : [0,1] \to [0,1]$ gives rise to an isomorphism of linearly ordered sets $\alpha_F : \pi_0([0,1] \setminus F) \to \pi_0([0,1] \setminus \alpha(F))$ for every finite $F \subset [0,1]$. The action by α

$$\rho_\alpha : |X| = \mathrm{colim}_{F \in \mathcal{F}} \widetilde{X}[\pi_0([0,1] \setminus F)] \to \mathrm{colim}_{F \in \mathcal{F}} \widetilde{X}[\pi_0([0,1] \setminus F)]$$

is given by $\rho_\alpha \circ \mathrm{in}_F = \mathrm{in}_{\alpha(F)} \circ \widetilde{X}[\alpha_F^{-1}]$. If we express $|X|$ as $\mathrm{colim}_{\Delta[n][-] \to X[-]} \mathrm{Sim}^n$, then ρ_α is the map obtained by taking $x = (x_1, \dots, x_n) \in \mathrm{Sim}^n$ to $(\alpha(x_1), \dots, \alpha(x_n)) \in \mathrm{Sim}^n$ for every $\Delta[n][-] \to X[-]$. We wish to show that the map $\mu : \mathrm{Aut}[0,1] \times \mathrm{colim}_{\Delta[n][-] \to X[-]} \mathrm{Sim}^n \to \mathrm{colim}_{\Delta[n][-] \to X[-]} \mathrm{Sim}^n$ is continuous. An adjunction argument in $\mathcal{K}$ reduce this to the problem of verifying that the map $\mathrm{Aut}[0,1] \to \mathrm{Hom}_{\mathcal{K}}(\mathrm{Sim}^n, \mathrm{Sim}^n)$, $\alpha \mapsto \alpha \times \cdots \times \alpha \ |_{\mathrm{Sim}^n}$ is continuous for every $\Delta[n][-] \to X[-]$, which is done in general topology. See the appendix for details. ■

2.2 Cyclic sets

A similar argument applies to cyclic sets. Cyclic sets are defined by using a category consisting of certain $\mathbb{Z}_+$-categories.

A $\mathbb{Z}_+$-**category** is a category C together with endomorphisms $1_c : c \to c$, $c \in \mathrm{ob}\, C$, satisfying $1_c \neq \mathrm{id}_c$ for every object c and $f \circ 1_{c_1} = 1_{c_2} \circ f$ for every morphism $f : c_1 \to c_2$. A $\mathbb{Z}_+$-**functor** (resp. **isomorphism**) is a functor (resp. isomorphism) between $\mathbb{Z}_+$-categories which preserves the structural endomorphisms. The most basic example of a $\mathbb{Z}_+$-category is the circle $S^1 = \mathbb{R}/\mathbb{Z}$. Morphisms from x to y are homotopy classes of continuous maps $f : [0,1] \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} = S^1$ with $[0,1] \to \mathbb{R}$ non-decreasing, such that $f(0) = x$ and $f(1) = y$. The $\mathbb{Z}_+$-category structure is given by 1_x =(class of degree 1 loop based at x).

If $F \subset S^1$ is a finite subset, the set of connected components $\pi_0(S^1 \setminus F)$ can be considered as a $\mathbb{Z}_+$-category. The set of morphisms from c to d is defined by choosing representatives $x_c \in c$ and $x_d \in d$:

$$\mathrm{Hom}_{\pi_0(S^1 \setminus F)}(c, d) = \mathrm{Hom}_{S^1}(x_c, x_d).$$

If $F \subset G$ there is a $\mathbb{Z}_+$-functor $\pi_0(S^1 \setminus G) \to \pi_0(S^1 \setminus F)$.

Write $[n]_{\mathrm{cyc}}$ for the full subcategory

$$\{[0], [1/(n+1)], \ldots, [n/(n+1)]\} \subset S^1,$$

where $[-]$ denotes the class in $S^1 = \mathbb{R}/\mathbb{Z}$. (For notational simplicity we will frequently omit such brackets.) The category $[n]_{\mathrm{cyc}}$ inherits a $\mathbb{Z}_+$-category structure from S^1. Denote by Λ (resp. by Λ_{big}) the category of the $\mathbb{Z}_+$-categories $[n]_{\mathrm{cyc}}$ (resp. small $\mathbb{Z}_+$-categories isomorphic to some $[n]_{\mathrm{cyc}}$) and $\mathbb{Z}_+$-functors. By definition a **cyclic set** is a functor $X : \Lambda^{\mathrm{op}} \to \mathrm{Sets}$. Choose one isomorphism $i_\lambda : \lambda \to [m_\lambda]_{\mathrm{cyc}}$ for each $\lambda \in \mathrm{ob}\, \Lambda_{\mathrm{big}} \setminus \mathrm{ob}\, \Lambda$. We extend the cyclic set X to a functor $\widetilde{X} : \Lambda_{\mathrm{big}}^{\mathrm{op}} \to \mathrm{Sets}$, by defining on objects $\widetilde{X}[\lambda] = X[m_\lambda]_{\mathrm{cyc}}$ and on morphisms $\widetilde{X}[f] = X[\widetilde{f}] : X[m_\mu]_{\mathrm{cyc}} \to X[m_\lambda]_{\mathrm{cyc}}$, where $f : \lambda \to \mu$ is a map in Λ_{big} and $\widetilde{f}$ is the unique map in Λ that makes the following diagram commute:

$$\begin{array}{ccc} \lambda & \xrightarrow{i_\lambda} & [m_\lambda]_{\mathrm{cyc}} \\ {\scriptstyle f}\downarrow & & {\scriptstyle \widetilde{f}}\downarrow \\ \mu & \xrightarrow{i_\mu} & [m_\mu]_{\mathrm{cyc}} \end{array}$$

If $\widetilde{X}'$ is another extension, i.e. a functor $\Lambda_{\mathrm{big}}^{\mathrm{op}} \to \mathrm{Sets}$ that is identical to X on Λ^{op}, then there exists a unique natural isomorphism $\kappa : \widetilde{X} \to \widetilde{X}'$ such that $\kappa \mid_{\Lambda^{\mathrm{op}}} = \mathrm{id}_X$. Indeed, if κ is such a natural isomorphism then there is a commutative diagram

$$\begin{array}{ccc} \widetilde{X}(\lambda) & \xrightarrow{\kappa(\lambda)} & \widetilde{X}'(\lambda) \\ {\scriptstyle \widetilde{X}(i_\lambda)}\uparrow & & {\scriptstyle \widetilde{X}'(i_\lambda)}\uparrow \\ X[m_\lambda]_{\mathrm{cyc}} & = & X[m_\lambda]_{\mathrm{cyc}} \end{array}$$

for each $\lambda \in \mathrm{ob}\, \Lambda_{\mathrm{big}}$. This forces $\kappa(\lambda) = \widetilde{X}'[i_\lambda] \circ [\widetilde{X}(i_\lambda)]^{-1}$.

The **geometric realization** of the cyclic set X is the geometric realization of the simplicial set $X \mid_{\Delta^{\mathrm{op}}} : \Delta^{\mathrm{op}} \to \mathrm{Sets}$, where Δ is considered as a subcategory of Λ via the functor $\Delta \to \Lambda$, $[n] \mapsto [n]_{\mathrm{cyc}}$. The image of $f : [m] \to [n]$ by this functor takes $k/(m+1)$ to $f(k)/(n+1)$. The following cyclic set version of Theorem 2.1 is also a result of Drinfeld [[2]].

Theorem 2.3 *Denote by $\mathcal{F}'$ the ordered set of all finite subsets of S^1, which is again a filtered category. Then the geometric realization $|X|$ of the cyclic set X is given by the filtered colimit $\operatorname{colim}_{F\in\mathcal{F}'} \widetilde{X}[\pi_0(S^1 \setminus F)]$.*

The proof of the following corollary is similar to that of the corollary 2.2. The difference is that we use instead of Sim^n the set $\mathrm{Sim}^n_{\mathrm{cyc}}$ of points $(x_0, \ldots, x_n)$ of $(S^1)^{n+1}$ such that $x_1, \ldots, x_n$ are in the correct cyclic order (and in the dihedral case treated in the next subsection, the set $\mathrm{Sim}^n_{\mathrm{dih}}$ of points $(x_0, \ldots, x_n)$ of $(S^1)^{n+1}$ such taht $x_0, \ldots, x_n$ or $x_n, \ldots, x_0$ are in the correct cyclic order). See Drinfeld [[2]] for details.

Corollary 2.4 *1. Geometric realization of cyclic sets, considered as a functor to $\mathcal{K}$, commutes with finite limits.*

2. The group $\mathrm{Aut}\, S^1$ of orientation-preserving homeomorphisms of S^1 (in particular, the special orthogonal group $SO(2) \subset \mathrm{Aut}\, S^1$) acts continuously on the realization of a cyclic set.

2.3 Dihedral sets

When defining cyclic sets we used the category Λ, which contains Δ as a subcategory and the group of automorphisms on each object $[n]_{\mathrm{cyc}}$ is isomorphic to the cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$. There is a more general notion due to Loday [3] (Definition 6.3.0).

Definition 2.1 *A* **crossed simplicial group** *is a family of groups $\{G_n\}_{n\geq 0}$ together with a category ΔG that has one object $[n]$ for each $n \geq 0$, containing Δ as a subcategory, and satisfies the following conditions:*

1. *The group of automorphisms $\mathrm{Aut}_{\Delta G}[n]$ on each $[n]$ is isomorphic to the group G_n^{op}.*
2. *Every morphism $[m] \to [n]$ can be uniquely written as the composite $\phi \circ g$ with $\phi \in \mathrm{Hom}_\Delta([m], [n])$ and $g \in \mathrm{Aut}_{\Delta G}[m]$.*

An important example of a crossed simplicial group is given by the family $\{D_{n+1}\}_{n\geq 0}$, where D_{n+1} is the dihedral group of order $2(n+1)$. We define the category ΔD to have the same set of objects as Λ. The set of morphisms from $[m]_{\mathrm{cyc}}$ to $[n]_{\mathrm{cyc}}$ is the disjoint union of the sets of covariant and contravariant $\mathbb{Z}_+$-functors from $[m]_{\mathrm{cyc}}$ to $[n]_{\mathrm{cyc}}$. (**Remarks**: 1. If C is a $\mathbb{Z}_+$-category, the $\mathbb{Z}_+$-category structure on the opposite category C^{op} is given by $1_c = (1_c)^{\mathrm{op}}$ for $c \in \mathrm{ob}\, C^{\mathrm{op}}$.

2. Considering the *disjoint union* means that *a functor cannot be both covariant and contravariant*. As a consequence, e.g. $\mathrm{Hom}_{\Delta D}([0]_{\mathrm{cyc}}, [0]_{\mathrm{cyc}})$ has two elements: the identity $\mathrm{id}_{[0]_{\mathrm{cyc}}} : [0]_{\mathrm{cyc}} \to [0]_{\mathrm{cyc}}$ as a covariant functor, and $\mathrm{id}_{[0]_{\mathrm{cyc}}} : [0]_{\mathrm{cyc}} \to [0]_{\mathrm{cyc}}^{\mathrm{op}}$ as a contravariant functor.)

A functor $X : (\Delta D)^{\mathrm{op}} \to \mathrm{Sets}$ is called a **dihedral set**. Let $\Delta_{\mathrm{big}} D$ be the category of small $\mathbb{Z}_+$-categories isomorphic to some $[n]_{\mathrm{cyc}}$ and covariant and contravariant $\mathbb{Z}_+$-functors. The dihedral set X can be extended uniquely up to unique isomorphism to a functor $\widetilde{X} : (\Delta_{\mathrm{big}} D)^{\mathrm{op}} \to \mathrm{Sets}$. The **geometric realization** of the dihedral set X is the realization of the simplicial set $X\mid_{\Delta^{\mathrm{op}}}$, or equivalently, the realization of the cyclic set $X\mid_{\Lambda^{\mathrm{op}}}$. I.e.

$$|X| = |X\mid_{\Lambda^{\mathrm{op}}}| = \mathrm{colim}_{F \in \mathcal{F}'}\, \widetilde{(X\mid_{\Lambda^{\mathrm{op}}})}[\pi_0(S^1 \setminus F)].$$

Note that the extension $\widetilde{(X\mid_{\Lambda^{\mathrm{op}}})}$ of $X\mid_{\Lambda^{\mathrm{op}}}$ to $\Lambda^{\mathrm{op}}_{\mathrm{big}}$ is nothing but the restriction $\widetilde{X}\mid_{\Lambda^{\mathrm{op}}_{\mathrm{big}}}$ of $\widetilde{X}$ to $\Lambda^{\mathrm{op}}_{\mathrm{big}}$, because of the following commutative diagram of categories:

$$\begin{array}{ccccc}
\Lambda^{\mathrm{op}} & \longrightarrow & (\Delta D)^{\mathrm{op}} & \xrightarrow{X} & \mathrm{Sets} \\
\downarrow & & \downarrow & & \| \\
\Lambda^{\mathrm{op}}_{\mathrm{big}} & \longrightarrow & (\Delta_{\mathrm{big}} D)^{\mathrm{op}} & \xrightarrow{\widetilde{X}} & \mathrm{Sets}\,.
\end{array}$$

Therefore we see

$$|X| = \mathrm{colim}_{F \in \mathcal{F}'}\, \widetilde{X}\mid_{\Lambda^{\mathrm{op}}_{\mathrm{big}}} [\pi_0(S^1 \setminus F)] = \mathrm{colim}_{F \in \mathcal{F}'}\, \widetilde{X}[\pi_0(S^1 \setminus F)].$$

Theorem 2.5 *The geometric realization $|X|$ of the dihedral set X is given by the filtered colimit* $\mathrm{colim}_{F \in \mathcal{F}'}\, \widetilde{X}[\pi_0(S^1 \setminus F)]$. *In particular, geometric realization of dihedral sets, considered as a functor to $\mathcal{K}$, commutes with finite limits.*

Any homeomorphism of S^1, even an orientation-reversing one, gives rise to an isomorphism $\rho_\alpha : |X| \to |X|$ defined by

$$\rho_\alpha \circ \mathrm{in}_F = \mathrm{in}_{\alpha(F)} \circ \widetilde{X}[\alpha_F^{-1}],$$

where $\alpha_F : \pi_0(S^1 \setminus F) \to \pi_0(S^1 \setminus \alpha(F))$ is the covariant or contravariant $\mathbb{Z}_+$-isomorphim induced by α. Therefore,

Corollary 2.6 *The group* $\mathrm{Homeo}\, S^1$ *of homeomorphisms of S^1 (in particular, the orthogonal group $O(2) \subset$* $\mathrm{Homeo}\, S^1$*) acts continuously on the realization of a dihedral set.*

3 Subdivisions

3.1 Simplicial sets

The following definition of subdivisions of simplicial sets is due to Bökstedt-Hsiang-Madsen [1]. For every positive integer r, let $\operatorname{sd}_r : \Delta \to \Delta$ be the functor defined on objects by $\operatorname{sd}_r[n] = [r(n+1)-1]$ and on morphisms by $\operatorname{sd}_r[f](a(m+1)+b) = a(n+1)+f(b)$, where $f : [m] \to [n]$, $0 \le a < r$, and $0 \le b \le m$. The composite $\operatorname{sd}_r X = X \circ \operatorname{sd}_r$, which is again a simplicial set, is called the **r-fold subdivision** of X.

By Drinfeld's formula, its realization is given by

$$|\operatorname{sd}_r X| = \operatorname{colim}_{F \in \mathcal{F}} \widetilde{\operatorname{sd}_r X}[\pi_0([0,1] \setminus F)].$$

For each $F \in \mathcal{F}$, let F_r denote the finite set $\{n + x \mid 0 \le n < r, x \in F \cup \{0,1\}\} \subset [0,r]$. If n is the cardinarity of $\pi_0([0,1] \setminus F)$, then that of $\pi_0([0,r] \setminus F_r)$ is rn. In this case we have $\widetilde{\operatorname{sd}_r X}[\pi_0([0,1] \setminus F)] = \operatorname{sd}_r X[n-1] = X[rn-1] = \widetilde{X}[\pi_0([0,r] \setminus F_r)]$. (Note that we may proceed by choosing a particular extension $\widetilde{\operatorname{sd}_r X}$ of $\operatorname{sd}_r X$, since other extensions are all isomorphic to the chosen one up to unique isomorphism. The same thing can be said for extension of X.) Therefore the realization of the subdivision can be rewritten as

$$|\operatorname{sd}_r X| = \operatorname{colim}_{F \in \mathcal{F}} \widetilde{X}[\pi_0([0,r] \setminus F_r)].$$

Since the subsets in $[0,r]$ of the form F_r with $F \in \mathcal{F}$ forms a subcategory of the category $\mathcal{F}_r$ of all finite subsets in $[0,r]$, $|\operatorname{sd}_r X| = \operatorname{colim}_{F \in \mathcal{F}} \widetilde{X}[\pi_0([0,r] \setminus F_r)]$ is a subspace of $\operatorname{colim}_{F \in \mathcal{F}_r} \widetilde{X}[\pi_0([0,r] \setminus F)]$. Now, for every $F \in \mathcal{F}_r$ there exists a set $F' \in \mathcal{F}$ such that $F \subset (F')_r$. Indeed, $\{x \in [0,1] \mid n + x \in F \text{ for some } 0 \le n < r\} \subset [0,1]$ is such a set. There results a map $\widetilde{X}[\pi_0([0,r] \setminus F)] \to \widetilde{X}[\pi_0([0,r] \setminus (F')_r)]$. This implies that every element in $\operatorname{colim}_{F \in \mathcal{F}_r} \widetilde{X}[\pi_0([0,r] \setminus F)]$ can be represented by an element of $\widetilde{X}[\pi_0([0,r] \setminus F_r)]$ with some $F \in \mathcal{F}$. Hence,

Theorem 3.1 *The realization of* $\operatorname{sd}_r X$ *is given by*

$$|\operatorname{sd}_r X| = \operatorname{colim}_{F \in \mathcal{F}_r} \widetilde{X}[\pi_0([0,r] \setminus F)].$$

In particular, the bijection $[0,r] \to [0,1]$, $x \mapsto x/r$ defines a homeomorphism

$$D_r : |\operatorname{sd}_r X| = \operatorname{colim}_{F \in \mathcal{F}_r} \widetilde{X}[\pi_0([0,r] \setminus F)] \to \operatorname{colim}_{F \in \mathcal{F}} \widetilde{X}[\pi_0([0,1] \setminus F)] = |X|.$$

3.2 Cyclic sets

We recall subdivisions of cyclic sets from Bökstedt-Hsiang-Madsen [1], in a simpler manner not mentioning face, degeneracy and cyclic operators. The definition is analogous, but subdivisions of cyclic sets are not cyclic sets but Λ_r-sets. The category Λ_r is defined to make the family $\{\mathbb{Z}/r(n+1)\mathbb{Z}\}_{n\geq 0}$ into a crossed simplicial group, by using the $\mathbb{Z}_+$-category $S^1_r = \mathbb{R}/r\mathbb{Z}$ instead of $S^1 = \mathbb{R}/\mathbb{Z}$. Denote by $[n]_r$ the subset (considered as a $\mathbb{Z}_+$-subcategory) $\{[k + l/(n+1)] \mid 0 \leq k < r, 0 \leq l \leq n\}$ of S^1_r. We define the category Λ_r (resp. $\Lambda_{r,\mathrm{big}}$) to have as objects the $\mathbb{Z}_+$-categories $[n]_r \subset S^1_r$, $n \geq 0$, (resp. $\mathbb{Z}_+$-categories isomorphic to some $[n]_r$) and to have as morphisms from $[m]_r$ to $[n]_r$, $\mathbb{Z}_+$-functors satisfying $f(x+1) = f(x)+1$. (resp. as morphisms from $\mathcal{A}$ to $\mathcal{B}$, $\mathbb{Z}_+$-functors f such that there exists a map f' in Λ_r such that the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\iota_{\mathcal{A}}} & [m]_r \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ \mathcal{B} & \xrightarrow{\iota_{\mathcal{B}}} & [n]_r \end{array}$$

commutes, where $\iota_{\mathcal{A}}$ and $\iota_{\mathcal{B}}$ are chosen isomorphisms. This definition is independent of the choices of $\iota_{\mathcal{A}}$ and $\iota_{\mathcal{B}}$.) A Λ_r**-set** is a functor $X : \Lambda_r^{\mathrm{op}} \to \mathrm{Sets}$, and there is a unique (up to unique isomorphism) extension $\widetilde{X}$ to $\Lambda_{r,\mathrm{big}}$.

The **geometric realization** $|X|$ of the Λ_r-set X is defined to be the geometric realization of the simplicial set $X\ |_{\Delta^{\mathrm{op}}}$. Let $\mathcal{F}'_r$ denote the category of all finite subset of S^1_r such that the cardinarity of $\pi_0(S^1_r \setminus F)$ equals $r(n+1)$ with some $n \geq 0$. Then we have the following formula:

$$|X| = \mathrm{colim}_{F\in\mathcal{F}'_r} \widetilde{X}[\pi_0(S^1_r \setminus F)].$$

If α is an orientation-preserving homeomorphism on S^1_r, it induces an isomorphism $\alpha_F : \pi_0(S^1_r \setminus F) \to \pi_0(S^1_r \setminus \alpha(F))$ for each $F \in \mathcal{F}'_r$, and in turn a homeomorphims $\rho_\alpha : |X| \to |X|$ defined by

$$\rho_\alpha \circ \mathrm{in}_F = \mathrm{in}_{\alpha(F)} \circ \widetilde{X}[\alpha_F^{-1}].$$

In the particular case where α is given by the translation $S^1_r \ni x \mapsto x + a \in S^1_r$ by $a \in S^1_r = \mathbb{R}/r\mathbb{Z}$, the action $\rho_a : |X| \to |X|$ is given by

$$\rho_a \circ \mathrm{in}_F = \mathrm{in}_{F+a} \circ \widetilde{X}[\pi_0(S^1_r \setminus (F+a)) \overset{\cong}{\to} \pi_0(S^1_r \setminus F)].$$

Hence,

Theorem 3.2 *1. Geometric realization of Λ_r-sets as a functor to $\mathcal{K}$ commutes with finite limits.*

2. The group $\operatorname{Aut} S^1_r$ of all orientation-preserving homeomorphisms of S^1_r (in particular, S^1_r itself) acts continuously on the realization of a Λ_r-set.

Let $\mathrm{sd}_r : \Lambda_r \to \Lambda$ be the functor that is defined on objects by $\mathrm{sd}_r[n] = [r(n+1)-1]$ and on morphisms by $\mathrm{sd}_r[f] = \rho_n^{-1} \circ f \circ \rho_m : [r(m+1)-1] \to [r(n+1)-1]$, where $f : [m] \to [n]$ is a map in Λ_r and ρ_m and ρ_n are the set bijections from $[m]_r$ to $[r(m+1)-1]_{\mathrm{cyc}}$, and from $[n]_r$ to $[r(n+1)-1]_{\mathrm{cyc}}$, respectively, induced by the isomorphism $\rho : S^1_r \to S^1$, $x \mapsto x/r$. This functor is an extension of the subdivision functor for simplicial sets, in the sense that the diagram

$$\begin{array}{ccc} \Lambda_r & \xrightarrow{\mathrm{sd}_r} & \Lambda \\ \uparrow & & \uparrow \\ \Delta & \xrightarrow{\mathrm{sd}_r} & \Delta \end{array}$$

commutes. The ***r*-fold subdivision** $\mathrm{sd}_r Y$ of a cyclic set Y is defined to be the composite $Y \circ \mathrm{sd}_r$. By an argument similar to that in the previous subsection, we have:

Theorem 3.3 *The realization of $\mathrm{sd}_r Y$ is given by*

$$|\mathrm{sd}_r Y| = \mathrm{colim}_F \widetilde{Y}[\pi_0(S^1_r \setminus F)],$$

where F runs through all finite subsets of S^1_r. The bijection $\rho : S^1_r \to S^1$ defines a homeomorphism

$$D_r : |\mathrm{sd}_r Y| = \mathrm{colim}_F \widetilde{Y}[\pi_0(S^1_r \setminus F)] \to \mathrm{colim}_{F \in \mathcal{F}'} \widetilde{Y}[\pi_0(S^1 \setminus F)] = |Y|.$$

More precisely, $D_r \circ \mathrm{in}_F = \mathrm{in}_{F/r} \circ \widetilde{Y}[\pi_0(S^1 \setminus (F/r)) \xrightarrow{\cong} \pi_0(S^1_r \setminus F)]$.

The following proposition follows:

Proposition 3.4 *Let a be an element of S^1_r and ρ_a the action on $|\mathrm{sd}_r Y|$ induced by the translation $S^1 \ni x \mapsto x + a \in S^1_r$, and similar for $a/r \in S^1$ and $\rho_{a/r} : |Y| \to |Y|$. Then the diagram*

$$\begin{array}{ccc} |\mathrm{sd}_r Y| & \xrightarrow{D_r} & |Y| \\ {\scriptstyle \rho_a}\downarrow & & \downarrow{\scriptstyle \rho_{a/r}} \\ |\mathrm{sd}_r Y| & \xrightarrow{D_r} & |Y| \end{array}$$

is commutative.

(Proof) Clear by definition. Indeed, we have

$$\begin{aligned} D_r \circ \rho_a \circ \mathrm{in}_F &= D_r \circ \mathrm{in}_{F+a} \circ \widetilde{Y}[\pi_0(S^1_r \setminus (F+a)) \overset{\cong}{\to} \pi_0(S^1_r \setminus F)] \\ &= \mathrm{in}_{F/r+a/r} \circ \widetilde{Y}[\pi_0(S^1 \setminus (F/r+a/r)) \overset{\cong}{\to} \pi_0(S^1_r \setminus (F+a))] \\ &\circ \widetilde{Y}[\pi_0(S^1_r \setminus (F+a)) \overset{\cong}{\to} \pi_0(S^1_r \setminus F)] \\ &= \mathrm{in}_{F/r+a/r} \circ \widetilde{Y}[\pi_0(S^1 \setminus (F/r+a/r)) \overset{\cong}{\to} \pi_0(S^1_r \setminus F)], \end{aligned}$$

and similarly $\rho_{a/r} \circ D_r \circ \mathrm{in}_F = \mathrm{in}_{F/r+a/r} \circ \widetilde{Y}[\pi_0(S^1 \setminus (F/r+a/r)) \overset{\cong}{\to} \pi_0(S^1_r \setminus F)]$. ■

3.3 Dihedral sets

We repeat the same argument as in the previous subsection to define and study the dihedral version of subdivisions. Define the category $\Delta_r D$ to have the same sets of objects as Λ_r, and to have as morphisms covariant $\mathbb{Z}_+$-functors f satisfying $f(x+1) = f(x)+1$ and contravariant $\mathbb{Z}_+$-functors g satisfying $g(x+1) = g(x) - 1$. For every $n \geq 0$, the automorphism group $\mathrm{Aut}_{\Delta_r D}[n]_r$ is the dihedral group of order $2r(n+1)$. The extended category $\Delta_{r,\mathrm{big}} D$ is defined analogously. A functor $X : \Delta_r D^{\mathrm{op}} \to \mathrm{Sets}$, a $\Delta_r D$**-set**, is extended to a functor $\widetilde{X} : \Delta_{r,\mathrm{big}} D^{\mathrm{op}} \to \mathrm{Sets}$, and such an extension is unique up to unique isomorphism. The **geometric realization** of the $\Delta_r D$-set X is the realization of the underlying simplicial set, and we have the following formula

$$|X| = \mathrm{colim}_{F \in \mathcal{F}'_r} \widetilde{X}[\pi_0(S^1_r \setminus F)].$$

The group of all homeomorphisms of S^1_r acts on $|X|$.

If Y is a dihedral set, its r**-fold subdivision** $\mathrm{sd}_r\, Y$ is defined to be the composite $Y \circ \mathrm{sd_r}$, where sd_r is the functor $\Delta_r D \to \Delta D$ that extends the cyclic subdivision functor $\mathrm{sd}_r : \Lambda_r \to \Lambda$. The realization $|\mathrm{sd}_r\, Y|$ is equal to $\mathrm{colim}_F\, \widetilde{Y}[\pi_0(S^1_r \setminus F)]$, where F runs over all finite subsets of S^1_r, and canonically homeomorphic to $|Y|$.

Remark. Spaliński [5] defined the r-fold subdivision of the dihedral set Y to be the r-fold subdivision of the underlying simplicial set $Y\ |_{\Delta^{\mathrm{op}}}$. Our definition is compatible with Spaliński's one since we have the following commutative diagram:

$$\begin{array}{ccccc} (\Delta_r D)^{\mathrm{op}} & \xrightarrow{\mathrm{sd}_r} & (\Delta D)^{\mathrm{op}} & \xrightarrow{Y} & \mathrm{Sets} \\ \uparrow & & \uparrow & & \| \\ \Delta^{\mathrm{op}} & \xrightarrow{\mathrm{sd}_r} & \Delta^{\mathrm{op}} & \xrightarrow{Y|_{\Delta^{\mathrm{op}}}} & \mathrm{Sets}\,. \end{array}$$

3.3.1 Combination with Quillen's edgewise subdivision

Spaliński [5] defined another subdivision $\mathrm{sbd}_r Y$ of the dihedral set Y, for each $r \geq 1$, combining the subdivision functor sd_r with Quillen's edgewise subdivision functor sq introduced in Segal [4]. The functor $\mathrm{sq} : \Delta \to \Delta$ is given on objects by $\mathrm{sq}[n] = [2n+1]$, and on morphisms by $\mathrm{sq}[f](k) = f(k)$, $\mathrm{sq}[f](2m+1-k) = 2n+1-f(k)$ where $f : [m] \to [n]$, $0 \leq k \leq m$. Spaliński defines $\mathrm{sbd}_r Y$ to be the composite $Y \mid_{\Delta^{\mathrm{op}}} \circ \mathrm{sd}_r \circ \mathrm{sq}$ of the underlying simplicial set of Y with sd_r and sq. Here we will re-define $\mathrm{sbd}_r Y$ to be the composite of Y with $\mathrm{sd}_r : \Delta_r D \to \Delta$ and a certain functor $\mathrm{sq} : \Delta_{r,\mathrm{sq}} D \to \Delta_r D$. Let $\Delta_{r,\mathrm{sq}} D$ denote the category that has as objects the $\mathbb{Z}_+$-subcategories $[n]_{r,\mathrm{sq}} = \{[k + l/(n+1)] \mid 0 \leq k < 2r, 0 \leq l \leq n\} \subset S^1_{2r}$ of the $\mathbb{Z}_+$-category $S^1_{2r} = \mathbb{R}/2r\mathbb{Z}$ and that has as morphisms from $[m]_{r,\mathrm{sq}}$ to $[n]_{r,\mathrm{sq}}$ the covariant $\mathbb{Z}_+$-functors f satisfying

1. $f(x+2) = f(x) + 2$
2. $f(x) + f(-1/(m+1) - x) - 2f(0) = -1/(n+1)$ for $0 \leq x < 1$

and contravariant $\mathbb{Z}_+$-functors g satisfying

1. $g(x+2) = g(x) - 2$
2. $g(x) + g(-1/(m+1) - x) - 2g(0) = -1/(n+1)$ for $0 \leq x < 1$.

The family $\{D_{2r(n+1)}\}_{n \geq 0}$ of dihedral groups of order $4r(n+1)$ becomes a crossed simplicial group by $\Delta_{r,\mathrm{sq}} D$. We have an inclusion $\Delta \to \Delta_{r,\mathrm{sq}} D$, $[n] \mapsto [n]_{r,\mathrm{sq}}$, that sends a map $f : [m] \to [n]$ to the map $\widetilde{f} : [m]_{r,\mathrm{sq}} \to [n]_{r,\mathrm{sq}}$ given by $\widetilde{f}(l/(m+1)) = f(l)/(n+1)$ and $\widetilde{f}(-1/(m+1) - l/(m+1)) = -1/(n+1) - f(l)/(n+1)$ for $0 \leq k \leq m$. (This determines $\widetilde{f}$. Remember that $\widetilde{f}$ should satisfy $\widetilde{f}(x+2) = \widetilde{f}(x) + 2$.) The functor $\mathrm{sq} : \Delta_{r,\mathrm{sq}} D \to \Delta_r$ is defined on objects by $\mathrm{sq}[n]_{r,\mathrm{sq}} = [2n+1]_r$ and on morphisms by $\mathrm{sq}[f] = \rho_{m,\mathrm{sq}} \circ f(x) \circ \rho_{n,\mathrm{sq}}$, where $\rho_{m,\mathrm{sq}} : [m]_{r,\mathrm{sq}} \to [2m+1]_r$ and $\rho_{n,\mathrm{sq}} : [n]_{r,\mathrm{sq}} \to [2n+1]_r$ are $\mathbb{Z}_+$-isomorphisms induced by the bijection $\rho_{\mathrm{sq}} : S^1_{r,\mathrm{sq}} \to S^1_r$, $x \mapsto x/2$.

<u>Definition</u> 3.1 *We define* $\mathrm{sbd}_r Y$ *to be the* $\Delta_{r,\mathrm{sq}} D$*-set* $Y \circ \mathrm{sd}_r \circ \mathrm{sq} : \Delta_{r,\mathrm{sq}} D^{\mathrm{op}} \to \mathrm{Sets}$.

The diagram

$$\begin{array}{ccc} (\Delta_{r,\mathrm{sq}} D) & \xrightarrow{\mathrm{sq}} & \Delta D_r \\ \uparrow & & \uparrow \\ \Delta & \xrightarrow{\mathrm{sq}} & \Delta. \end{array}$$

commutes, and in view of this our definition of $\mathrm{sbd}_r Y$ is compatible with that of Spaliński.

We define $\Delta_{r,\mathrm{sq},\mathrm{big}}D$ to be the category of $\mathbb{Z}_+$-categories isomorphic to some $[n]_{r,\mathrm{sq}}$. A morphism from $\mathcal{A}$ to $\mathcal{B}$ in $\Delta_{r,\mathrm{sq},\mathrm{big}}D$ is a covariant or contravariant $\mathbb{Z}_+$-functor f such that there are some isomorphisms $\iota_{\mathcal{A}}$ from $\mathcal{A}$ to $[m]_{r,\mathrm{sq}}$ and $\iota_{\mathcal{B}}$ from $\mathcal{B}$ to $[n]_{r,\mathrm{sq}}$, and some map $g : [m]_{r,\mathrm{sq}} \to [n]_{r,\mathrm{sq}}$ in $\Delta_{r,\mathrm{sq}}D$, that make the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\iota_{\mathcal{A}}} & [m]_{r,\mathrm{sq}} \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle g}\big\downarrow \\ \mathcal{A} & \xrightarrow{\iota_{\mathcal{B}}} & [n]_{r,\mathrm{sq}} \end{array}$$

commute. A $\Delta_{r,\mathrm{sq}}D$-set X is extended to a functor $\widetilde{X}$ from $\Delta_{r,\mathrm{sq},\mathrm{big}}D$ to Sets, such an extension being unique up to unique isomorphism. The geometric realization of X, defined to be the realization of the underlying simplicial set, is given by

$$|X| = \mathrm{colim}_F\, \widetilde{X}[\pi_0(S^1_{2r} \setminus F)],$$

where F runs over finite subsets of S^1_{2r} such that the cardinarity of $\pi_0(S^1_{2r} \setminus F)$ is $2r(n+1)$ with some $n \geq 0$.

<u>Theorem</u> 3.5 *If the $\Delta_{r,\mathrm{sq}}D$-set X is the subdivision $\mathrm{sbd}_r Y$ of the dihedral set Y, then the geometric realizations of $X = \mathrm{sbd}_r Y$, $\mathrm{sd}_r Y$, and Y are given by $\mathrm{colim}_F\, \widetilde{Y}[\pi_0(S \setminus F)]$, where F runs over all finite subsets of $S = S^1_{2r}$, $S = S^1_r$, and $S = S^1$, respectively, and homeomorphic to each other via the isomorphisms induced by $\rho_{\mathrm{sq}} : S^1_{2r} \to S^1_r$ and $\rho : S^1_r \to S^1$. Moreover, these homeomorphisms preserve the actions of $\mathrm{Homeo}\, S^1_{2r}$ on $|\mathrm{sbd}_r Y|$, $\mathrm{Homeo}\, S^1_r$ on $|\mathrm{sd}_r Y|$, and $\mathrm{Homeo}\, S^1$ on $|Y|$.*

3.3.2 Simplicial actions on subdivisions

Let ω_n and $\tau_n : [n]_{r,\mathrm{sq}} \to [n]_{r,\mathrm{sq}}$ be isomorphisms in $\Delta_{r,\mathrm{sq}}D$ given by $\omega_n(x) = -1/(n+1) - x$ and $\widetilde{\tau}_n(x) = 1/(n+1) + x$, respectively. The subgroup of $\mathrm{Aut}[n]_{r,\mathrm{sq}}$ generated by ω_n and $\tau_n = (\widetilde{\tau}_n)^{2(n+1)}$ can be considered as the dihedral group D_r. If $f : [m] \to [n]$ is a map in Δ, the diagram

$$\begin{array}{ccc} [m]_{r,\mathrm{sq}} & \xrightarrow{\omega_n \text{ or } \tau_n} & [m]_{r,\mathrm{sq}} \\ {\scriptstyle \widetilde{f}}\big\downarrow & & {\scriptstyle \widetilde{f}}\big\downarrow \\ [n]_{r,\mathrm{sq}} & \xrightarrow{\omega_n \text{ or } \tau_n} & [n]_{r,\mathrm{sq}} \end{array}$$

commutes by the construction of $\widetilde{f}$. Thus D_r acts on the underlying simplicial set of $\mathrm{sbd}_r Y$, and hence on $|\mathrm{sbd}_r Y|$. This action is nothing but the action obtained by using the action of Homeo S^1_{2r} and by identifying D_r with the subgroup of Homeo S^1_{2r} generated by $\omega : x \mapsto -x$ and $\tau : x \mapsto x+2$.

Consider a simplicial set $[n] \mapsto (\mathrm{sbd}_r Y[n])^{D_r}$. We give a new proof to the following result of Spaliński:

Proposition 3.6 *There is a canonical homeomorphism between* $|(\mathrm{sbd}_r Y[-])^{D_r}|$ *and* $(|Y|)^{D_r}$.

(Proof) The left-hand-side is given by the colimit $\mathrm{colim}_{F\in\mathcal{F}}(\widetilde{\mathrm{sbd}_r Y}[\pi_0([0,1]\setminus F)])^{D_r}$. It is clear that if $x \in |\mathrm{sbd}_r Y| = \mathrm{colim}_{F\in\mathcal{F}} \widetilde{\mathrm{sbd}_r} Y[\pi_0([0,1]\setminus F)]$ is represented by an element of $(\widetilde{\mathrm{sbd}_r} Y[\pi_0([0,1]\setminus F)])^{D_r}$ with some $F \in \mathcal{F}$, then x is fixed by the D_r-action on $|\mathrm{sbd}_r Y|$. The converse also holds. Indeed, suppose $x \in |\mathrm{sbd}_r Y|$ to be represented by $y \in \widetilde{\mathrm{sbd}_r} Y[\pi_0([0,1]\setminus F)]$ and to be fixed by the D_r-action. Then for any $\delta \in D_r$, there is a larger subset $G \subset [0,1]$ containing F such that the images of y and $\delta \cdot y$ in $\widetilde{\mathrm{sbd}_r} Y[\pi_0([0,1]\setminus G)]$ coincides. Then x is represented by this common element $z \in \widetilde{\mathrm{sbd}_r} Y[\pi_0([0,1]\setminus G)]$, and z is fixed by the action of D_r, i.e. $z \in (\widetilde{\mathrm{sbd}_r} Y[\pi_0([0,1]\setminus G)])^{D_r}$. Therefore $|(\mathrm{sbd}_r Y[-])^{D_r}| = \mathrm{colim}_{F\in\mathcal{F}}(\widetilde{\mathrm{sbd}_r} Y[\pi_0([0,1]\setminus F)])^{D_r} = (\mathrm{colim}_{F\in\mathcal{F}} \widetilde{\mathrm{sbd}_r} Y[\pi_0([0,1]\setminus F)])^{D_r} = (|\mathrm{sbd}_r Y|)^{D_r}$. Finally, the canonical homeomorphism from $|\mathrm{sbd}_r Y|$ to $|Y|$, which preserves the appropreate actions on both sides, concludes the proof. ■

We can also consider an action by the cyclic group C_{2r} on $\mathrm{sbd}_r Y$ by identifying C_{2r} with the subgroup $\langle \tau'_n \rangle$ of $\mathrm{Aut}[n]_{r,\mathrm{sq}}$, where $\tau'_n = \widetilde{\tau}_n^{\,n+1}$, for each $n \geq 0$. It is proved likewise that the realization of $(\mathrm{sbd}_r Y[-])^{C_{2r}}$ is canonically homeomorphic to that of Y. Moreover, in this case the simplicial set $(\mathrm{sbd}_r Y[-])^{C_{2r}}$ has an extra structure. Indeed, ω_n and $\widetilde{\tau}_n$ satisfy $\omega_n^2 = (\widetilde{\tau}_n)^{n+1} = \mathrm{id}_{[n]_{r,\mathrm{sq}}}$ and $\omega_n \widetilde{\tau}_n \omega_n = (\widetilde{\tau}_n)^{-1}$ on $(\mathrm{sbd}_r Y[n])^{C_{2r}}$. Hence, $(\mathrm{sbd}_r Y[-])^{C_{2r}}$ is again a dihedral set.

Appendix

We will complete the proof of Corollary 2.2-(2), by verifying the following lemma:

Lemma 3.7 *The map* $\mu_n : \mathrm{Aut}[0,1] \to \mathrm{Hom}_{\mathcal{K}}(\mathrm{Sim}^n, \mathrm{Sim}^n)$, $\alpha \mapsto \alpha \times \cdots \times \alpha\, |_{\mathrm{Sim}^n}$, *is continuous with respect to the standard k-space topologies on both sides.*

(Proof) Remember that the subbasis of the target space is given by the subsets $N(h,U) = \{f : \mathrm{Sim}^n \to \mathrm{Sim}^n \mid f(h(K)) \subset U\}$ where $h : K \to \mathrm{Sim}^n$ is a continuous map from a compact Hausdorff space K and U is an open set of Sim^n. Hence it suffices to show that $\mu_n^{-1}N(h,U)$ is open in $\mathrm{Aut}[0,1]$. To this end, we fix an arbitrary $\alpha \in \mu_n^{-1}N(h,U)$ and will show that there is an open neighbourhood $N(\alpha)$ of α in $\mathrm{Aut}[0,1]$ such taht $N(\alpha) \subset \mu_n^{-1}N(h,U)$. Since $U \subset \mathrm{Sim}^n$ is open, for every $x \in h(K)$, there exists a positive real number ε_x such that $B''_x = \{y \in \mathrm{Sim}^n \mid |y - \mu_n(\alpha)(x)| < \varepsilon_x\}$ is contained in U. We take a smaller ball $B'_x = \{y \in \mathrm{Sim}^n \mid |y - \mu_n(\alpha)(x)| < \varepsilon_x/n\}$ in B''_x, and put $B_x = \mu_n(\alpha)^{-1}(B'_x) \cap h(K)$. Then $\{B_x\}_{x \in h(K)}$ forms an open cover for $h(K)$. A compactness argument tells us that we can choose finite $x^{(1)}, \ldots, x^{(l)} \in h(K)$ such that $h(K) = \bigcup_{j+1}^{l} B_{x^{(j)}}$. If $\overline{B_x}$ denotes the closure of B_x in $h(K)$, we also have $h(K) = \bigcup_{j+1}^{l} \overline{B_{x^{(j)}}}$. Note that $\overline{B_x}$ is compact (because it is a closed set in a compact set). We let $\iota_x : \overline{B_x} \to h(K) \to \mathrm{Sim}^n$ be the inclusion, and consider for every $i = 1, \ldots, n$ and $j = 1, \ldots, l$, the set $N'(p_i \circ \iota_{x^{(j)}}, p_i(B'_{x^{(j)}})) = \{\beta \in \mathrm{Aut}[0,1] \mid \beta(p_i(\overline{B_{x^{(j)}}})) \subset p_i(B'_{x^{(j)}})\}$, where $p_i : \mathrm{Sim}^n \to [0,1]$ is the projection onto the i-th component. Then $N'(p_i \circ \iota_{x^{(j)}}, p_i(B'_{x^{(j)}}))$ is an open set in $\mathrm{Aut}[0,1]$ containing α. We also have $\bigcap_{1 \le i \le n, 1 \le j \le l} N'(p_i \circ \iota_{x^{(j)}}, p_i(B'_{x^{(j)}})) \subset \mu_n^{-1}(N(h,U))$. Indeed, let β be in the left-hand side and take $x \in h(K)$ arbitrarily. Then there is some j such that $x = (x_1, \ldots, x_n) \in \overline{B_{x^{(j)}}}$. For every $1 \le i \le n$, $\beta(p_i(x)) = \beta(x_i) \in p_i(B'_{x^{(j)}}) \subset \{y_i \in [0,1] \mid |y_i - p_i(\mu_n(\alpha)(x^{(j)}))| < \varepsilon_{x^{(j)}}/n\}$. Hence, $|\mu_n(\beta)(x) - \mu_n(\alpha)(x^{(j)})|^2 = \sum_{i+1}^{n}(\beta(x_i) - \alpha(x_i^{(j)}))^2 \le \sum_{i=1}^{n} \varepsilon_{x^{(j)}}^2/n^2 = \varepsilon_{x^{(j)}}^2/n < \varepsilon_{x^{(j)}}^2$. Therefore $\mu_n(\beta)(x) \in B''_{x^{(j)}} \subset U$. This implies $\beta \in \mu_n^{-1}(N(h,U))$. Thus we take $N(\alpha) = \bigcap_{1 \le i \le n, 1 \le j \le l} N'(p_i \circ \iota_{x^{(j)}}, p_i(B'_{x^{(j)}}))$, obtaining the desired conclusion. ■

References

[1] M. Bökstedt, W. C. Hsiang, and I. Madsen: *The cyclotomic trace and algebraic K-theory of spaces*, Invent. math. 111 (1993), 465-540.

[2] V. Drinfeld: *On the notion of geometric realization*, arXiv:math/0304064v3 [math.CT].

[3] J.-L. Loday: *Cyclic homology*, Grundlehren der mathematischen Wissenschaften, vol. 301, Springer-Verlag, New York, 1971.

[4] G. Segal: *Configuration-spaces and iterated loop-spaces*, Invent. math. 21 (1973) 213-221.

[5] J. Spaliński: *Homotopy theory of dihedral and quaternionic sets*, Topology 39 (2000) 557-572.