Orientability of vector bundles over real flag manifolds

Mauro Patrão * Luiz A. B. San Martin †
Laércio J. dos Santos‡ Lucas Seco §

AMS 2010 subject classification: Primary: 14M15, Secondary: 57R22.
Key words an phrases. Vector bundles; Orientability; Flag manifold.

1 Introduction

The purpose of this paper is to give necessary and sufficient conditions for the orientability of vector bundles over flag manifolds of real semi-simple Lie groups. We focus on two kinds of vector bundles whose orientability can be decided by looking at the root systems and Dynkin diagrams associated to the semi-simple groups.

We look at the tangent bundles of the flag manifolds as well as the stable and unstable vector bundles of gradient flows on flag manifolds (see the construction of these bundles in Section 2.4 below).

In both cases there is a Lie group acting on the vector bundle by linear maps in such a way that the action on the base space is transitive. From this property we derive our main method which consists in reducing the orientability question to a computation of signs of determinants. Namely the vector bundle is orientable if and only if each linear map coming from the representation of the isotropy subgroup on the fiber at the origin has positive determinant (see Proposition 3.1 below).

*Department of Mathematics. Universidade Federal de Brasilia. Brasília - DF, Brazil. Supported by CNPq grant no. 310790/09-3
†Institute of Mathematics. Universidade Estadual de Campinas. Campinas - SP, Brazil. Supported by CNPq grant no. 303755/09-1
‡Universidade Federal de São Carlos - Campus de Sorocaba. Sorocaba - SP, Brazil.
§Department of Mathematics. Universidade Federal de Brasilia. Brasília - DF, Brazil.

Using this criterion we get closed formulas, in terms of roots and their multiplicities to decide when one of our vector bundles is orientable (see theorems 4.1 and 4.5, below). In particular, we prove that any of the maximal flag manifolds is orientable. A result already obtained by Kocherlakota [10] as a consequence of the computation of the homology groups of the real flag manifolds. Also in Section 5 we make a detailed analysis of the orientability of the flag manifolds associated to the split real forms of the classical Lie algebras ($A_l = \mathfrak{sl}(l+1,\mathbb{R})$, $B_l = \mathfrak{so}(l,l+1)$, $C_l = \mathfrak{sp}(l,\mathbb{R})$ and $D_l = \mathfrak{so}(l,l)$).

The orientability of the stable and unstable bundles were our original motivation to write this paper. It comes from the computation of the Conley indices for flows on flag bundles in [12]. In this computation one wishes to apply the Thom isomorphism between homologies of the base space and the disk bundle associated to a vector bundle. The isomorphism holds in $\mathbb{Z}$ homology provided the bundle is orientable, asking for criteria of orientability of such bundles. We develop along this line on Section 6.

2 Preliminaries

We recall some facts of semi-simple Lie groups and their flag manifolds (see Duistermat-Kolk-Varadarajan [3], Helgason [6], Knapp [9] and Warner [15]). To set notation let G be a connected noncompact real semi-simple Lie group with Lie algebra $\mathfrak{g}$. Fix a Cartan involution θ of $\mathfrak{g}$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. The form $\langle X, Y\rangle_\theta = -\langle X, \theta Y\rangle$, where $\langle\cdot,\cdot\rangle$ is the Cartan-Killing form of $\mathfrak{g}$, is an inner product. An element $g \in G$ acts in $X \in \mathfrak{g}$ by the adjoint representation and this is denoted by gX.

Fix a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{s}$ and a Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$. We let Π be the set of roots of $\mathfrak{a}$, Π^+ the positive roots corresponding to $\mathfrak{a}^+$, Σ the set of simple roots in Π^+ and $\Pi^- = -\Pi^+$ the negative roots. The Iwasawa decomposition of the Lie algebra $\mathfrak{g}$ reads $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^\pm$ with $\mathfrak{n}^\pm = \sum_{\alpha\in\Pi^\pm} \mathfrak{g}_\alpha$ where $\mathfrak{g}_\alpha$ is the root space associated to α. As to the global decompositions of the group we write $G = KS$ and $G = KAN^\pm$ with $K = \exp\mathfrak{k}$, $S = \exp\mathfrak{s}$, $A = \exp\mathfrak{a}$ and $N^\pm = \exp\mathfrak{n}^\pm$.

The Weyl group W associated to $\mathfrak{a}$ is the finite group generated by the reflections over the root hyperplanes $\alpha = 0$ in $\mathfrak{a}$, $\alpha \in \Pi$. W acts on $\mathfrak{a}$ by isometries and can be alternatively be given as $W = M^*/M$ where M^* and M are the normalizer and the centralizer of A in K, respectively. We write $\mathfrak{m}$ for the Lie algebra of M.

2.1 Subalgebras defined by simple roots

Associated to a subset of simple roots $\Theta \subset \Sigma$ there are several Lie algebras and groups (cf. [15], Section 1.2.4): We write $\mathfrak{g}(\Theta)$ for the (semi-simple) Lie subalgebra generated by $\mathfrak{g}_\alpha$, $\alpha \in \Theta$, put $\mathfrak{k}(\Theta) = \mathfrak{g}(\Theta) \cap \mathfrak{k}$ and $\mathfrak{a}(\Theta) = \mathfrak{g}(\Theta) \cap \mathfrak{a}$. The simple roots of $\mathfrak{g}(\Theta)$ are given by Θ, more precisely, by restricting the functionals of Θ to $\mathfrak{a}(\Theta)$. Also, the root spaces of $\mathfrak{g}(\Theta)$ are given by $\mathfrak{g}_\alpha$, for $\alpha \in \langle\Theta\rangle$. Let $G(\Theta)$ and $K(\Theta)$ be the connected groups with Lie algebra, respectively, $\mathfrak{g}(\Theta)$ and $\mathfrak{k}(\Theta)$. Then $G(\Theta)$ is a connected semi-simple Lie group.

Let $\mathfrak{a}_\Theta = \{H \in \mathfrak{a} : \alpha(H) = 0, \alpha \in \Theta\}$ be the orthocomplement of $\mathfrak{a}(\Theta)$ in $\mathfrak{a}$ with respect to the $\langle\cdot,\cdot\rangle_\theta$-inner product. We let K_Θ be the centralizer of $\mathfrak{a}_\Theta$ in K. It is well known that

$$K_\Theta = M(K_\Theta)_0 = MK(\Theta).$$

Let $\mathfrak{n}_\Theta^\pm = \sum_{\alpha \in \Pi^\pm - \langle\Theta\rangle} \mathfrak{g}_\alpha$ and $N_\Theta^\pm = \exp(\mathfrak{n}_\Theta^\pm)$. We have that K_Θ normalizes $\mathfrak{n}_\Theta^\pm$ and that $\mathfrak{g} = \mathfrak{n}_\Theta^- \oplus \mathfrak{p}_\Theta$. The standard parabolic subalgebra of type $\Theta \subset \Sigma$ with respect to chamber $\mathfrak{a}^+$ is defined by

$$\mathfrak{p}_\Theta = \mathfrak{n}^-(\Theta) \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+.$$

The corresponding standard parabolic subgroup P_Θ is the normalizer of $\mathfrak{p}_\Theta$ in G. It has the Iwasawa decomposition $P_\Theta = K_\Theta AN^+$. The empty set $\Theta = \emptyset$ gives the minimal parabolic subalgebra $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$ whose minimal parabolic subgroup $P = P_\emptyset$ has Iwasawa decomposition $P = MAN^+$.

Let $d = \dim(\mathfrak{p}_\Theta)$ and consider the Grassmanian of d-dimensional subspaces of $\mathfrak{g}$, where G acts by its adjoint representation. The flag manifold of type Θ is the G-orbit of the base point $b_\Theta = \mathfrak{p}_\Theta$, which we denote by $\mathbb{F}_\Theta$. This orbit identifies with the homogeneous space G/P_Θ. Since the adjoint action of G factors trough $\mathrm{Int}(\mathfrak{g})$, it follows that the flag manifolds of G depends only on its Lie algebra $\mathfrak{g}$. The empty set $\Theta = \emptyset$ gives the maximal flag manifold $\mathbb{F} = \mathbb{F}_\emptyset$ with basepoint $b = b_\emptyset$.

2.2 Subalgebras defined by elements in $\mathfrak{a}$

The above subalgebras of $\mathfrak{g}$, which are defined by the choice of a Weyl chamber of $\mathfrak{a}$ and a subset of the associated simple roots, can be defined alternatively by the choice of an element $H \in \mathfrak{a}$ as follows. First note that the eigenspaces of $\mathrm{ad}(H)$ in $\mathfrak{g}$ are the weight spaces $\mathfrak{g}_\alpha$. Now define the negative and positive nilpotent subalgebras of type H given by

$$\mathfrak{n}_H^- = \sum\{\mathfrak{g}_\alpha : \alpha(H) < 0\}, \qquad \mathfrak{n}_H^+ = \sum\{\mathfrak{g}_\alpha : \alpha(H) > 0\},$$

and the parabolic subalgebra of type H which is given by

$$\mathfrak{p}_H = \sum\{\mathfrak{g}_\alpha : \alpha(H) \geq 0\}.$$

Denote by $N_H^\pm = \exp(\mathfrak{n}_H^\pm)$ and by P_H the normalizer in G of $\mathfrak{p}_H$. Let $d = \dim(\mathfrak{p}_H)$ and consider the Grassmanian of d-dimensional subspaces of $\mathfrak{g}$, where G acts by its adjoint representation. The flag manifold of type H is the G-orbit of the base point $\mathfrak{p}_H$, which we denote by $\mathbb{F}_H$. This orbit identifies with the homogeneous space G/P_H, where P_H is the normalizer of $\mathfrak{p}_H$ in G.

Now choose a chamber $\mathfrak{a}^+$ of $\mathfrak{a}$ which contains H in its closure, consider the simple roots Σ associated to $\mathfrak{a}^+$ and consider

$$\Theta(H) = \{\alpha \in \Sigma : \alpha(H) = 0\},$$

the set of simple roots which annihilate H. Since a root $\alpha \in \Theta(H)$ if, and only if, $\alpha|_{\mathfrak{a}_{\Theta(H)}} = 0$, we have that

$$\mathfrak{n}_H^\pm = \mathfrak{n}_{\Theta(H)}^\pm \quad \text{and} \quad \mathfrak{p}_H = \mathfrak{p}_{\Theta(H)}.$$

Denoting by K_H the centralizer of H in K, we have that $K_H = K_{\Theta(H)}$. So it follows that

$$\mathbb{F}_H = \mathbb{F}_{\Theta(H)},$$

and that the isotropy of G in $\mathfrak{p}_H$ is

$$P_H = P_{\Theta(H)} = K_{\Theta(H)}AN^+ = K_H AN^+,$$

since $K_{\Theta(H)} = K_H$. Denoting by $G(H) = G(\Theta(H))$ and by $K(H) = K(\Theta(H))$, it is well know that

$$K_H = M(K_H)_0 = MK(H).$$

We remark that the map

$$\mathbb{F}_H \to \mathfrak{s}, \qquad k\mathfrak{p}_H \mapsto kH, \quad \text{where } k \in K, \tag{1}$$

gives an embeeding of $\mathbb{F}_H$ in $\mathfrak{s}$ (see Proposition 2.1 of [3]). In fact, the isotropy of K at H is $K_H = K_{\Theta(H)}$ which is, by the above comments, the isotropy of K at $\mathfrak{p}_H$.

2.3 Connected components of K_H

We assume from now on that G is the adjoint group $\mathrm{Int}\,(\mathfrak{g})$. There is no loss of generality in this assumption because the action on the flag manifolds of any locally isomorphic group factors through $\mathrm{Int}\,(\mathfrak{g})$. The advantage of taking the adjoint group is that it has a complexification $G_{\mathbb{C}} = \mathrm{Aut}_0\,(\mathfrak{g}_{\mathbb{C}})$ with Lie algebra $\mathfrak{g}_{\mathbb{C}}$ in such a way that G is the connected subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}$.

For a root α, let $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$ so that $\langle \alpha^\vee, \alpha \rangle = 2$. Also, let H_α be defined by $\alpha\,(Z) = \langle H_\alpha, Z \rangle$, $Z \in \mathfrak{a}$, and write $H_\alpha^\vee = 2H_\alpha/\langle \alpha, \alpha \rangle$ for the corresponding co-root. Finally, let

$$\gamma_\alpha = \exp(i\pi H_\alpha^\vee),$$

where the exponential is taken in $\mathfrak{g}_{\mathbb{C}}$, and put

$$F = \text{ group generated by } \{\gamma_\alpha : \alpha \in \Pi\},$$

that is $F = \{\exp(i\pi H) : H \in \mathcal{L}\}$, where $\mathcal{L}$ is the lattice spanned by $H_\alpha^\vee$, $\alpha \in \Pi$.

It is known that F is a subgroup of M normalized by M^* and that $M = FM_0$ (see Proposition 7.53 and Theorem 7.55 of [9]). Also, γ_α leaves invariant each root space $\mathfrak{g}_\beta$ and its restriction to $\mathfrak{g}_\beta$ has the only eigenvalue $\exp(i\pi\langle \alpha^\vee, \beta \rangle)$. The next result shows that F intersects each connected component of the centralizer K_H.

Lemma 2.1 *For $H \in \mathfrak{a}$, we have that $K_H = F(K_H)_0$. In particular, $K_\Theta = F(K_\Theta)_0$.*

Proof: Take $w \in \mathcal{W}$ such that $Z = wH \in \mathrm{cl}\mathfrak{a}^+$. Thus, since $K_Z = M(K_Z)_0$ and $M = FM_0$, we have that $K_Z = F(K_Z)_0$. Now

$$K_H = w^{-1}K_Z w = w^{-1}Fw(w^{-1}K_Z w)_0 = F(K_H)_0,$$

since M^* normalizes F. The last assertion follows, since $K_\Theta = K_{H_\Theta}$, where $H_\Theta \in \mathrm{cl}\mathfrak{a}^+$ is such that $\Theta(H_\Theta) = \Theta$. $\square$

2.4 Stable and unstable bundles over the fixed points

Take $H \in \mathrm{cl}\mathfrak{a}^+$. The one-parameter group $\exp(tH)$ acts on a flag manifold $\mathbb{F}_\Theta$, defining a flow, whose behavior was described in Duistermat-Kolk-Varadarajan [3]. This is the flow of a gradient vector field, and the connected components of its fixed points are given by the orbits $\mathrm{fix}_\Theta(H,w) = K_H w b_\Theta$, where w runs trough $\mathcal{W}$, b_Θ is the origin of the flag manifold $\mathbb{F}_\Theta$ and $wb_\Theta = \overline{w}b_\Theta$, where $\overline{w}$ is any representative of w in M^*. Since $K_H = K(H)M$ and the group M fixes wb_Θ, it follows that

$$\mathrm{fix}_\Theta(H,w) = K(H)wb_\Theta.$$

It can be checked that each $\mathrm{fix}_\Theta(H,w)$ is a flag manifold of the semisimple group $G(H)$ (see [12]). The stable set of each $\mathrm{fix}_\Theta(H,w)$ is given by

$$\mathrm{st}_\Theta(H,w) = N_H^- w b_\Theta,$$

and the stable bundle, denoted by $V_\Theta^-(H,w)$, is the subbundle of the tangent bundle to $\mathrm{st}_\Theta(H,w)$ transversal to the fixed point set.

In order to write $V_\Theta^-(H,w)$ explicitly in terms of root spaces we use the following notation: Given a vector subspace $\mathfrak{l} \subset \mathfrak{g}$ and $x \in \mathbb{F}_\Theta$ denote by $\mathfrak{l} \cdot x$ the subspace of the tangent space $T_x\mathbb{F}_\Theta$ given by the infinitesimal action of $\mathfrak{l}$, namely

$$\mathfrak{l} \cdot x = \{\widetilde{X}(x) \in T_x\mathbb{F}_\Theta : X \in \mathfrak{l}\},$$

where $\widetilde{X}(x) = \frac{d}{dt}(\exp tX)_{|t=0}(x)$ is the vector field induced by $X \in \mathfrak{g}$. With this notation the tangent space $T_{b_\Theta^w}\mathbb{F}_\Theta$ at $b_\Theta^w \approx wH_\Theta$ is

$$T_{b_\Theta^w}\mathbb{F}_\Theta = \mathfrak{n}_{wH_\Theta}^- \cdot b_\Theta^w.$$

Now, $V_\Theta^-(H,w) \to \mathrm{fix}_\Theta(H,w)$ (which we write simpler as $V^- \to \mathrm{fix}_\Theta(H,w)$) is given by the following expressions:

1. At b_Θ^w we put $V_{b_\Theta^w}^- = \left(\mathfrak{n}_{wH_\Theta}^- \cap \mathfrak{n}_H^-\right) \cdot b_\Theta^w$.

2. At $x = gb_\Theta^w \in K_H \cdot b_\Theta^w$, $g \in K_H$ put

$$V_x^- = \left(\mathrm{Ad}(g)\left(\mathfrak{n}_{wH_\Theta}^- \cap \mathfrak{n}_H^-\right)\right) \cdot x. \tag{2}$$

   This is the same as $dg_{b_\Theta^w}\left(V_{b_\Theta^w}\right)$ due to the well known formula $g_*\widetilde{X} = \widetilde{(\mathrm{Ad}(g)X)}$. Also, the right hand side of (2) depends only on x because $\mathfrak{n}_{wH_\Theta}^- \cap \mathfrak{n}_H^-$ is invariant under the isotropy subgroup $K_H \cap K_{wH_\Theta}$ of $\mathrm{fix}_\Theta(H,w) = K(H)/(K(H) \cap K_{wH_\Theta})$.

For future reference we note that, by taking derivatives, the action of $K(H)$ on $\mathrm{fix}_\Theta(H,w)$ lifts to a linear action on $V_\Theta^-(H,w)$. Also, in terms of root spaces we have

$$\mathfrak{n}_{wH_\Theta}^- \cap \mathfrak{n}_H^- = \sum_{\beta\in\Pi_\Theta^-(H,w)} \mathfrak{g}_\beta$$

where

$$\Pi_\Theta^-(H,w) = \{\beta\in\Pi : \beta(H) < 0,\ \beta(wH_\Theta) < 0\}.$$

In a similar way we can define the unstable bundles $V_\Theta^+(H,w) \to \mathrm{fix}_\Theta(H,w)$ that are tangent to the unstable sets $N_H^+ wb_\Theta$ and transversal to the fixed point set $\mathrm{fix}_\Theta(H,w)$. The construction is the same unless that $\mathfrak{n}_H^-$ is replaced by $\mathfrak{n}_H^+$, and hence $\Pi_\Theta^-(H,w)$ is replaced by

$$\Pi_\Theta^+(H,w) = \{\beta\in\Pi : \beta(H) > 0,\ \beta(wH_\Theta) < 0\}.$$

Remark: The stable and unstable bundles $V_\Theta^\pm(H,w) \to \mathrm{fix}_\Theta(H,w)$ can be easily obtained by using the general device to construct a vector bundle from a principal bundle $Q \to X$ and a representation of the structural group G on a vector space V. The resulting associated bundle $Q \times_G V$ is a vector bundle. For the stable and unstable bundles we can take the principal bundle $K(H) \to \mathrm{fix}_\Theta(H,w)$, defined by identification of $\mathrm{fix}_\Theta(H,w) = K(H)/(K(H)\cap K_{wH_\Theta})$, whose structural group is $K(H)\cap K_{wH_\Theta}$. Its representation on $\mathfrak{l}^\pm = \mathfrak{n}_{wH_\Theta}^- \cap \mathfrak{n}_H^\pm$ yields $V_\Theta^\pm(H,w)$, respectively.

3 Vector bundles over homogeneous spaces

We state a general criterion of orientability of vector bundles acted by Lie groups. Let $V \to B$ be a n-dimensional vector bundle and denote by FV the bundle of frames $p : \mathbb{R}^n \to V$. It is well known that the vector bundle V is orientable if and only if FV has exaclty two connected components, and is connected otherwise.

Let K be a connected Lie group acting transitively on the base space B in such a way that the action lifts to a fiberwise linear action on V. This linear action in turn lifts to an action on FV by composition with the frames.

Fix a base point $x_0 \in B$ with isotropy subgroup $L \subset K$. Then each $g \in L$ gives rise to a linear operator of the fiber $\mathfrak{l} = V_{x_0}$. Denote by $\det(g|_\mathfrak{l})$, $g \in L$, the determinant of this linear operator.

The following statement gives a simple criterion for the orientability of V.

Proposition 3.1 *The vector bundle V is orientable if and only if* $\det(g|_{\mathfrak{l}}) > 0$*, for every $g \in L$.*

Proof: Suppose that $\det(g|_{\mathfrak{l}}) > 0$, $g \in L$, and take a basis $\beta = \{e_1, \ldots, e_k\}$ of V_{x_0}. Let $g_1, g_2 \in G$ be such that $g_1 x_0 = g_2 x_0$. Then the bases $g_i \beta = \{g_i e_1, \ldots, g_i e_k\}$, $i = 1, 2$, obtained by the linear action on V, have the same orientation since $\deg\left(g_1^{-1} g_2|_{\mathfrak{l}}\right) > 0$. These translations orient each fiber consistently and hence V.

Conversely, denote by FV the bundle of frames of V. If V is orientable then FV splits into two connected components. Each one is a $\mathrm{Gl}^+(k, \mathbb{R})$-subbundle, $k = \dim V$, and corresponds to an orientation of V. The linear action of G on V lifts to an action on FV. Since G is assumed to be connected, both connected components of FV are G-invariant. Hence if $g \in L$ and β is a basis of V_{x_0} then β and $g\beta$ have the same orientation, that is, $\det(g|_{\mathfrak{l}}) > 0$. $\square$

Remark: Clearly, $\det(g|_{\mathfrak{l}})$ does not change sign in a connected component of L. Hence to check the condition of the above proposition it is enough to pick a point on each connected component of L.

4 Vector bundles over flag manifolds

Now we are ready to get criteria for orientability of an stable vector bundle $V_\Theta^-(H, w) \to \mathrm{fix}_\Theta(H, w)$ and for the tangent bundle of a flag manifold $\mathbb{F}_\Theta$. These two cases have the following properties in common:

1. The vector bundles is acted by a connected subgroup of G whose action on the base space is transitive. Hence Proposition 3.1 applies.

2. There is a subgroup S of the lattice group F of G that touches every connected component of the isotropy subgroup, furthermore S is generated by
$$\{\gamma_\alpha : \alpha \in \langle\Lambda\rangle\}$$
with roots belonging to a certain subset $\Lambda \subset \Sigma$.

3. The action of the isotropy subgroup on the fiber above the origin reduces to the adjoint action on a space

$$\mathfrak{l} = \sum_{\alpha\in\Gamma} \mathfrak{g}_\alpha$$

spanned by root spaces, with roots belonging to a certain subset $\Gamma \subset \Pi$.

Now, a generator of S, given by

$$\gamma_\alpha = \exp\left(i\pi H_\alpha^\vee\right), \qquad \alpha \in \langle\Lambda\rangle,$$

acts on a root space $\mathfrak{g}_\beta$ by $\exp(i\pi\langle\alpha^\vee,\beta\rangle)\cdot\mathrm{id}$. Hence the determinant of γ_α restricted to $\mathfrak{l} = \sum_{\alpha\in\Gamma}\mathfrak{g}_\alpha$ is given by

$$\det(\gamma_\alpha|_{\mathfrak{l}}) = \exp\left(i\pi\sum_{\beta\in\Gamma} n_\beta\langle\alpha^\vee,\beta\rangle\right).$$

So that $\det(\gamma_\alpha|_{\mathfrak{l}}) = \pm 1$ with the sign depending whether the sum

$$\sum_{\beta\in\Gamma} n_\beta\langle\alpha^\vee,\beta\rangle$$

is even or odd. Here, as before n_β is the multiplicity $\dim\mathfrak{g}_\beta$ of the root β. From this we get the following criterion for orientability in terms of roots: The vector bundle is orientable if and only if for every root $\alpha \in \langle\Lambda\rangle$ the sum

$$\sum_{\beta\in\Gamma} n_\beta\langle\alpha^\vee,\beta\rangle \equiv 0 \qquad (\mathrm{mod} 2)$$

where the sum is extended to $\beta \in \Gamma$. Finally, we note that it is enough to check the above condition for every simple root $\alpha \in \Lambda$. This is because the set of co-roots $\langle\Lambda\rangle^\vee = \{\alpha^\vee : \alpha \in \langle\Lambda\rangle\}$ is also a root system having $\Lambda^\vee = \{\alpha^\vee : \alpha \in \Lambda\}$ as a simple system of roots. Thus the elements of $\langle\Lambda\rangle^\vee$ can be written as a sum of elements of $\Lambda^\vee$ and the above condition holds for any root $\alpha \in \langle\Lambda\rangle$ if and only if it holds for the simple roots in Λ.

4.1 Flag manifolds

In case of orientability of a flag manifold $\mathbb{F}_\Theta$ (its tangent bundle) the subspace to be considered is

$$\mathfrak{l} = \mathfrak{n}_\Theta^- = \sum_{\beta\in\Pi^-\setminus\langle\Theta\rangle} \mathfrak{g}_\beta,$$

that identifies with the tangent space to $\mathbb{F}_\Theta$ at the origin, and the acting group is K.

Theorem 4.1 *The flag manifold $\mathbb{F}_\Theta$ is orientable if and only if*

$$\sum_\beta n_\beta \langle\alpha^\vee, \beta\rangle \equiv 0 \pmod 2 \tag{3}$$

where the sum is extended to $\beta \in \Pi^- \setminus \langle\Theta\rangle$ (or equivalently to $\beta \in \Pi^+ \setminus \langle\Theta\rangle$). This condition must be satisfied for any simple root α.

Proof: In order to apply the determinant criterion, note that the isotropy of the base $\mathbb{F}_\Theta$ at b_Θ is K_Θ which decomposes as $K_\Theta = F(K_\Theta)_0$ (see Lemma 2.1). It follows that that F touches every connected component of the isotropy K_Θ. Hence we can apply the determinant criterion with $\Lambda = \Sigma$ (so that $S = F$) and $\Gamma = \Pi^- \setminus \langle\Theta\rangle$ to get the above result. □

Now we derive some consequences of the criteria stated above. First we prove that any maximal flag manifold is orientable, a result already obtained by Kocherlakota [10] as a consequence that the top $\mathbb{Z}$-homology groups are nontrivial.

Theorem 4.2 *Any maximal flag manifold $\mathbb{F}$ is orientable.*

Proof: We write, for a simple root α, $\Pi_\alpha = \{\alpha, 2\alpha\} \cap \Pi^+$, $\Pi_0^\alpha = \{\beta \in \Pi^+ : \langle\alpha^\vee, \beta\rangle = 0\}$ and $\Pi_1^\alpha = \{\beta \in \Pi^+ : \langle\alpha^\vee, \beta\rangle \neq 0,\ \beta \notin \Pi_\alpha\}$. Let r_α be the reflection with respect to α. It is known that $r_\alpha(\Pi^+ \setminus \Pi_\alpha) = \Pi^+ \setminus \Pi_\alpha$. Moreover, for a root β we have

$$\langle\alpha^\vee, r_\alpha(\beta)\rangle = \langle\alpha^\vee, \beta - \langle\alpha^\vee, \beta\rangle\alpha\rangle = \langle\alpha^\vee, \beta\rangle - \langle\alpha^\vee, \alpha\rangle\langle\alpha^\vee, \beta\rangle = -\langle\alpha^\vee, \beta\rangle.$$

Hence the subsets Π_0^α and Π_1^α are r_α-invariant and $\langle\alpha^\vee, \beta + r_\alpha(\beta)\rangle = 0$.

Now fix $\alpha \in \Sigma$ and split the sum $\sum_{\beta\in\Pi^+} n_\beta\langle\alpha^\vee, \beta\rangle$ into Π_α, Π_0^α and Π_1^α. For Π_α this sum is $2n_\alpha + 4n_{2\alpha}$, with $n_{2\alpha} = 0$ if 2α is not a root. For

Π_0^α the sum is zero. In Π_1^α the roots are given in pairs $\beta \neq r_\alpha(\beta)$ with $\langle \alpha^\vee, \beta + r_\alpha(\beta) \rangle = 0$, since Π_1^α is r_α-invariant and $\beta = r_\alpha(\beta)$ if and only if $\langle \alpha^\vee, \beta \rangle = 0$. Since $n_{r_\alpha(\beta)} = n_\beta$, it follows that $\sum_{\beta \in \Pi_1^\alpha} n_\beta \langle \alpha^\vee, \beta \rangle = 0$. Hence the total sum is even for every $\alpha \in \Sigma$, proving the orientability of $\mathbb{F}$. $\square$

In particular this orientability result applies to the maximal flag manifold of the semi-simple Lie algebra $\mathfrak{g}(\Theta)$. Here the set of roots is $\langle \Theta \rangle$ having Θ as a simple system of roots. Therefore the equivalent conditions of Theorem 4.1 combined with the orientability of the maximal flag manifold of $\mathfrak{g}(\Theta)$ immplies the

Corollary 4.3 *If $\alpha \in \Theta$ then*

$$\sum_\beta n_\beta \langle \alpha^\vee, \beta \rangle \equiv 0 \pmod{2},$$

where the sum is extended to $\beta \in \langle \Theta \rangle^-$ (or equivalently to $\beta \in \langle \Theta \rangle^+$).

This allows to simplify the criterion for a partial flag manifold $\mathbb{F}_\Theta$.

Proposition 4.4 $\mathbb{F}_\Theta$ *is orientable if and only if, for every root $\alpha \in \Sigma \setminus \Theta$, it holds*

$$\sum_\beta n_\beta \langle \alpha^\vee, \beta \rangle \equiv 0 \pmod{2}, \tag{4}$$

where the sum is extended to $\beta \in \langle \Theta \rangle^-$ (or equivalently to $\beta \in \langle \Theta \rangle^+$).

Proof: Applying Corollary 4.3 with $\Theta = \Sigma$, we have that $\sum_{\beta \in \Pi^-} n_\beta \langle \alpha^\vee, \beta \rangle$ is even. Hence, by Theorem 4.1, $\mathbb{F}_\Theta$ is orientable if and only if, for every root $\alpha \in \Sigma$, the sum $\sum_{\beta \in \langle \Theta \rangle^-} n_\beta \langle \alpha^\vee, \beta \rangle$ is even. By Corollary 4.3, it is enough to check this for every root $\alpha \in \Sigma \setminus \Theta$. $\square$

Finally we observe that if G is a complex group then the real multiplicities are $n_\beta = 2$ so that any flag $\mathbb{F}_\Theta$ is orientable. This is well known since the $\mathbb{F}_\Theta$ are complex manifolds. In Section 5 we make a detailed analysis of the orientability of the partial flag manifolds for the split real forms (normal real forms) of the simple complex Lie algebras.

4.2 Stable and unstable bundles in flag manifolds

For the stable bundles $V_{\Theta}^{-}(H,w)$ we take

$$\mathfrak{l}=\mathfrak{n}_{wH_{\Theta}}^{-}\cap\mathfrak{n}_{H}^{-}=\sum_{\beta\in\Pi_{\Theta}^{-}(H,w)}\mathfrak{g}_{\beta},$$

where

$$\Pi_{\Theta}^{-}(H,w)=\{\beta\in\Pi:\beta(H)<0,\ \beta(wH_{\Theta})<0\}.$$

and the acting Lie group is $K(H)$.

Theorem 4.5 *The vector bundle $V_{\Theta}^{-}(H,w)$ is orientable if and only if*

$$\sum_{\beta}n_{\beta}\langle\alpha^{\vee},\beta\rangle\equiv0\pmod{2},$$

where the sum is extended to $\beta\in\Pi_{\Theta}^{-}(H,w)$. Here the condition must be verified for every $\alpha\in\Theta(H)$.

Proof: In order to apply the determinant criterion we look at the isotropy of the base $\mathrm{fix}_{\Theta}(H,w)$ at wb_{Θ} in a suitable way. Looking at $\mathbb{F}_{\Theta}$ as the adjoint orbit KH_{Θ} we have that $b_{\Theta}=H_{\Theta}$, $wb_{\Theta}=wH_{\Theta}$ and

$$\mathrm{fix}_{\Theta}(H,w)=K(H)wH_{\Theta}.$$

Let $\pi_{H}:\mathfrak{a}\to\mathfrak{a}(H)$ be the orthogonal projection parallel to $\mathfrak{a}_{H}$. Since $K(H)$ centralizes $\mathfrak{a}_{H}$, it follows that the isotropy of $K(H)$ at wH_{Θ} is the centralizer $K(H)_{Z}$, where $Z=\pi_{H}(wH_{\Theta})$. Since $K(H)$ is the compact component of the semisimple Lie group $G(H)$, applying Lemma 2.1, we have that the isotropy is

$$K(H)_{Z}=F(H)(K(H)_{Z})_{0},$$

where $F(H)$ is the lattice group of $G(H)$. It follows that that $F(H)$ touches every connected component of the isotropy $K(H)_{Z}$. Since the root system of $G(H)$ is given by the restriction of $\langle\Theta(H)\rangle$ to $\mathfrak{a}(H)$, it follows that $F(H)$ is the subgroup of F generated by

$$\{\gamma_{\alpha}=\exp(i\pi H_{\alpha}^{\vee}):\ \alpha\in\langle\Theta(H)\rangle\}.$$

Hence it is enough to check the determinant condition for the simple roots in $\Theta(H)$. Therefore we can apply the determinant criterion with $\Lambda=\Theta(H)$

(so that $S = F(H)$) and $\Gamma = \Pi_{\Theta}^{-}(H, w)$ to get the above result. $\square$

Remark: The same result holds for the unstable vector bundles $V_{\Theta}^{+}(H, w)$ with $\Pi_{\Theta}^{+}(H, w)$ instead of $\Pi_{\Theta}^{-}(H, w)$.

We have the following result in the special case when $\Theta = \emptyset$ and w is the principal involution w^{-}.

Corollary 4.6 *For every $H \in \mathrm{cl}\mathfrak{a}^{+}$, the vector bundles $V^{-}(H, 1)$ and $V^{+}(H, w^{-})$ are orientable.*

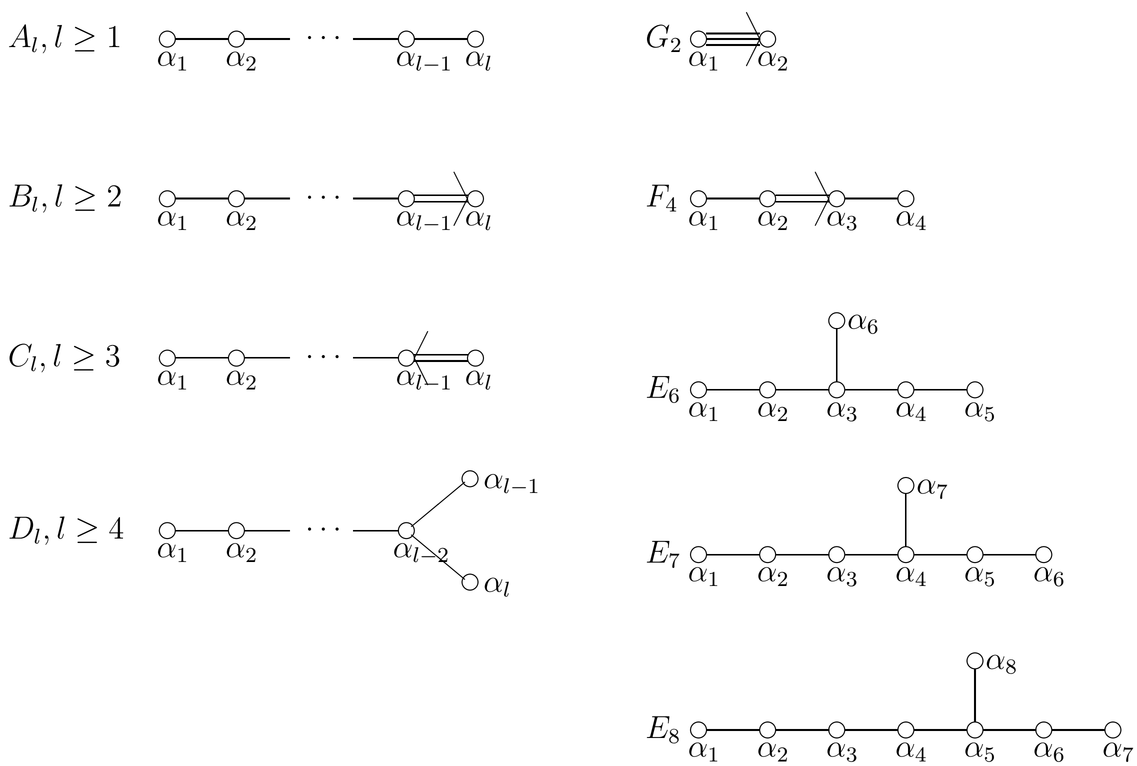
Proof: Applying Corollary 4.3 with $\Theta = \Sigma$ and $\Theta = \Theta(H)$, it follows that both
$$\sum_{\beta \in \Pi^{+}} n_{\beta} \langle \alpha^{\vee}, \beta \rangle \qquad \text{and} \qquad \sum_{\beta \in \langle \Theta(H) \rangle^{+}} n_{\beta} \langle \alpha^{\vee}, \beta \rangle$$
are even for $\alpha \in \Theta(H)$. Hence, for every $\alpha \in \Theta(H)$, it holds that $\sum_{\beta} n_{\beta} \langle \alpha^{\vee}, \beta \rangle$ is even, where the sum is extended to $\beta \in \Pi^{+} \setminus \langle \Theta(H) \rangle$. If $\Theta = \emptyset$, then H_{Θ} is regular and $\beta(w^{-} H_{\Theta}) < 0$ if and only if $\beta \in \Pi^{+}$. Thus $\Pi^{+}(H, w^{-}) = \Pi^{+} \setminus \langle \Theta(H) \rangle$ and the result follows from Theorem 4.5.

The proof for $V^{+}(H, w^{-})$ is analogous. $\square$

Remark: The above result is not true in a partial flag manifold. An example is given in $G = \mathrm{Sl}(3, \mathbb{R})$ with $H = \mathrm{diag}(2, -1, -1)$ acting on the projective plane, which is a partial flag manifold of G. Then it can be seen that the repeller component of H is a projective line and its unstable bundle a Möbius strip.

5 Split real forms

When $\mathfrak{g}$ is a split real form every root β has multiplicity $n_{\beta} = 1$. Hence, the criterion of Corollary 4.4 reduces to
$$S(\alpha, \Theta) = \sum_{\beta \in \langle \Theta \rangle^{+}} \langle \alpha^{\vee}, \beta \rangle \equiv 0 \qquad (\mathrm{mod} 2), \tag{5}$$
that can be checked by looking at the Dynkin diagrams. In the sequel we use a standard way of labelling the roots in the diagrams as in the picture below.



For the diagram G_2 there are three flag manifolds: the maximal $\mathbb{F}$, which is orientable, and the minimal ones $\mathbb{F}_{\{\alpha_1\}}$ and $\mathbb{F}_{\{\alpha_2\}}$, where α_1 and α_2 are the simple roots with α_1 the longer one. These minimal flag manifolds are not orientable since in both cases (5) reduces to the Killing numbers $\langle \alpha_1^\vee, \alpha_2 \rangle = -1$ and $\langle \alpha_2^\vee, \alpha_1 \rangle = -3$. From now on we consider only simple and double laced diagrams.

Our strategy consists in counting the contribution of each connected component Δ of Θ to the sum $S(\alpha, \Theta)$ in (5). Thus we keep fixed α and a connected subset $\Delta \subset \Sigma$. If α is not linked to Δ then $S(\alpha, \Delta) = 0$ and we can discard this case. Otherwise, α is linked to exactly one root of Δ, because a Dynkin diagram has no cycles. We denote by δ the only root in Δ linked to α.

A glance at the Dynkin diagrams show the possible subdiagrams Δ properly contained in Σ. We exhibit them in table 1. For these subdiagrams we can write down explicitly the roots of $\langle \Delta \rangle^+$ and then compute $S(\alpha, \Delta)$, when

Δ	Σ
A_k $(k \geq 1)$	any diagram
B_k $(k \geq 2)$	B_l $(l > k)$, C_l $(k = 2)$ and F_4 $(2 \leq k \leq 3)$
C_k $(k \geq 3)$	C_l $(l > k)$ and F_4 $(k = 3)$
D_k $(k \geq 4)$	D_l $(l > k)$, E_6 $(4 \leq k \leq 5)$, E_7 $(4 \leq k \leq 6)$ and E_8 $(4 \leq k \leq 7)$
E_6	E_7 and E_8
E_7	E_8

Table 1: Connected subdiagrams

$\Delta = A_k$	
links	$S(\alpha, \Delta)$
α — δ	$-k$
$\alpha \Rightarrow \delta$	$-k$
$\alpha \Leftarrow \delta$	$-2k$

Table 2: A_l subdiagrams

α is linked to Δ. In fact, if $\beta \subset \langle \Delta \rangle^+$ then $\beta = c\delta + \gamma$ where δ is the only root in Δ which is linked to α and $\langle \gamma, \alpha^\vee \rangle = 0$, so that $\langle \beta, \alpha^\vee \rangle = c\langle \delta, \alpha^\vee \rangle$. Hence it is enough to look at those roots $\beta \in \Delta$ whose coefficient c in the direction of δ is nonzero. In the sequel we write down the values of $S(\alpha, \Delta)$ and explain how they were obtained.

In the diagram A_k with roots $\alpha_1, \ldots, \alpha_k$ the positive roots are $\alpha_i + \cdots + \alpha_j$, $i \leq j$. Hence if $\Delta = A_k$ then the possibilities for δ are the extreme roots α_1 and α_k. In case $\delta = \alpha_1$ the sum $S(\alpha, \Delta)$ extends over the k positive roots $\alpha_1 + \cdots + \alpha_j$, $j = 1, \ldots, k$, that have nonzero coefficient in the direction of α_1. (It is analogous for $\delta = \alpha_k$.)

In the standard realization of B_k the positive roots are $\lambda_i \pm \lambda_j$, $i \neq j$, and λ_i, where $\{\lambda_1, \ldots, \lambda_k\}$ is an orthonormal basis of the k-dimensional space. The possibilities for δ are extreme roots $\lambda_1 - \lambda_2$ (to the left) and λ_k (to the right). If $\delta = \lambda_1 - \lambda_2$ then α and δ are linked by one edge, that is, $\langle \delta, \alpha^\vee \rangle = -1$. Also, the positive roots in B_k having nonzero coefficient c in the direction of $\lambda_1 - \lambda_2$ are the $2k - 2$ roots $\lambda_1 \pm \lambda_j$, $j > 1$ together with λ_1. For all of them $c = 1$, hence the contribution of Δ to $S(\alpha, \Delta)$ is $-(2k - 1)$. Analogous computations with $\delta = \lambda_k$ yields the table

For C_k the positive roots are $\lambda_i \pm \lambda_j$, $i \neq j$, and $2\lambda_i$. If $\delta = \lambda_1 - \lambda_2$ then

$\Delta = B_k$

Σ	$S(\alpha, \Delta)$
B_l $(2 \leq k < l)$	$-(2k-1)$
C_l $(k = 2)$	-4
F_4 $(k = 2)$	-3 or -4
F_4 $(k = 3)$	-9

Table 3: B_l subdiagrams

$\Delta = C_k$

Σ	$S(\alpha, \Delta)$
C_l $(3 \leq k < l)$	$-2k$
F_4 $(k = 3)$	-6

Table 4: C_l subdiagrams

$\langle \delta, \alpha^\vee \rangle = -1$, and we must count the $2k-2$ roots $\lambda_1 \pm \lambda_j$, $j > 1$, having coefficient $c = 1$ and $2\lambda_1$ with $c = 2$. Then the contribution to $S(\alpha, \Delta)$ is $-2k$. This together with a similar computation for the other δ gives table

For D_k the positive roots are $\lambda_i \pm \lambda_j$, $i \neq j$. If $\delta = \lambda_1 - \lambda_2$ then $\langle \delta, \alpha^\vee \rangle = -1$, and we must count the $2k-2$ roots $\lambda_1 \pm \lambda_j$, $j > 1$, all of them having coefficient $c = 1$. Then the contribution to $S(\alpha, \Delta)$ is $-2k-2$. We leave to the reader the computation of the other entries of table

The results for the exceptional cases are included in table 6. To do the computations we used the realization of Freudenthal of the split real form of E_8 in the vector space $\mathfrak{sl}(9, \mathbb{R}) \oplus \bigwedge^3 \mathbb{R}^9 \oplus \left(\bigwedge^3 \mathbb{R}^9\right)^*$. The roots of E_8 are

$\Delta = D_k$

Σ	$S(\alpha, \Delta)$
D_l $(4 \leq k < l)$	$-2(k-1)$
E_l $(k = 4)$	-6
E_l $(k = 5)$	-8, $\delta = \alpha_1$
E_l $(k = 5)$	-10, $\delta = \alpha_5$
E_l $(k = 6)$	-6, $\delta = \alpha_1$
E_l $(k = 6)$	-15, $\delta = \alpha_6$
E_8 $(k = 7)$	-21

Table 5: D_l subdiagrams

$\Delta = E_k$

Σ	$S(\alpha, \Delta)$
E_l $(k=6)$	-16
E_8 $(k=7)$	-27

Table 6: E_l subdiagrams

the weights of the representation of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{sl}(9, \mathbb{R})$ of the diagonal matrices (see Fulton-Harris [5] and [13]). The roots are $\lambda_i - \lambda_j$, $i \neq j$ (with root spaces in $\mathfrak{sl}(9, \mathbb{R})$) and $\pm(\lambda_i + \lambda_j + \lambda_k)$, $i < j < k$ (with root spaces in $\bigwedge^3 \mathbb{R}^9 \oplus \left(\bigwedge^3 \mathbb{R}^9\right)^*$). From the realization of E_8 one easily obtains E_6 and E_7, and the computations can be performed.

5.1 Classical Lie algebras

The split real forms of the classical Lie algebras are $A_l = \mathfrak{sl}(l+1, \mathbb{R})$, $B_l = \mathfrak{so}(l, l+1)$, $C_l = \mathfrak{sp}(l, \mathbb{R})$ and $D_l = \mathfrak{so}(l, l)$. Their associated flag manifolds are concretely realized as manifolds of flags $(V_1 \subset \cdots \subset V_k)$ of vector subspaces $V_i \subset \mathbb{R}^n$. For A_l one take arbitrary subspaces of $\mathbb{R}^n$, $n = l+1$. Given integers $1 \leq d_1 < \cdots < d_k \leq l$ we denote by $\mathbb{F}(d_1, \ldots, d_k)$ the manifold of flags $(V_1 \subset \cdots \subset V_k)$ with $\dim V_i = d_i$.

For the other classical Lie algebras we take similar manifolds of flags, but now the subspaces V_i are isotropic w.r.t. a quadratic form for B_l and D_l, and w.r.t. a symplectic form in C_l. Again the flag manifolds are given by integers $1 \leq d_1 < \cdots < d_k \leq l$ and we write $\mathbb{F}^I(d_1, \ldots, d_k)$ for the manifold of flags of isotropic subspaces with $\dim V_i = d_i$. Here $V_i \subset \mathbb{R}^n$ with $n = 2l+1$ in B_l and $n = 2l$ in the C_l and D_l cases.

The way we order the simple roots Σ in the Dynkin diagrams allows a direct transition between the dimensions $d_1, \ldots, d_k$ and the roots $\Theta \subset \Sigma$ when $\mathbb{F}(d_1, \ldots, d_k)$ or $\mathbb{F}^I(d_1, \ldots, d_k)$ is $\mathbb{F}_\Theta$. In fact, except for some flags of D_l the dimensions $d_1, \ldots, d_k$ coincide with the indices of the roots $\alpha_j \notin \Theta$. (For example, the Grassmannian $\mathbb{F}(r)$ is the flag manifold $\mathbb{F}_\Theta$ with $\Theta = \Sigma \setminus \{\alpha_r\}$.) We detail this correspondence below.

The orientability criteria for the split real groups uses several times the following

Condition: We say that the numbers $0 = d_0, d_1, \ldots, d_k$ satisfy the mod2 condition if the differences $d_{i+1} - d_i$, $i = 0, \ldots, k$, are congruent mod2, that is, they are simultaneously even or simultaneously odd.

5.1.1 $A_l = \mathfrak{sl}(l+1,\mathbb{R})$

The flag manifolds are $\mathbb{F}(d_1,\ldots,d_k) = \mathbb{F}_\Theta$ such that $j \in \{d_1,\ldots,d_k\}$ if and only if j is the index of a simple root $\alpha_j \notin \Theta$. If we write $\mathbb{F}(d_1,\ldots,d_k) = \mathrm{SO}(n)/K_\Theta$ then $K_\Theta = \mathrm{SO}(d_1)\times\cdots\times\mathrm{SO}(n-d_k)$ is a group of block diagonal matrices, having blocks of sizes $d_{i+1}-d_i$.

Proposition 5.1 *A flag manifold $\mathbb{F}(d_1,\ldots,d_k)$ of A_l is orientable if and only if $d_1,\ldots,d_k,d_{k+1}$ satisfy the* mod2 *condition. Here we write $d_{k+1} = n = l+1$. Alternatively orientability holds if and only if the sizes of the blocks in K_Θ are congruent* mod2.

Proof: By the comments above, the simple roots outside Θ are $\alpha_{r_1},\ldots,\alpha_{r_k}$, where $d_1,\ldots,d_k$ are the dimensions determining the flag. For an index i there either $d_{i+1} = d_i + 1$ or $d_{i+1} > d_i + 1$. In the second case the set $\Delta = \{\alpha_{r_i+1},\ldots,\alpha_{r_{i+1}-1}\}$ is a connected component of Θ, having $d_{i+1}-d_i-1$ elements. We consider two cases:

1. If the second case holds for every $\alpha \notin \Theta$ then the connected components of $\Sigma\backslash\Theta$ are singletons. If this holds and $\alpha \notin \Theta$ is not one of the extreme roots α_1 or α_l then α is linked to exactly two connected components of Θ. By the first row of table 2 these connected components of Θ must have the same mod2 number of elements if $\mathbb{F}(d_1,\ldots,d_k)$ is to be orientable. Hence if $\{\alpha_1,\alpha_l\} \subset \Theta$ then $\mathbb{F}(d_1,\ldots,d_k)$ is orientable if and only if the number of elements in the components of Θ are mod2 congruent. This is the same as the condition in the statement because a connected component has $d_{i+1}-d_i-1$ elements. On the other hand if α_1 or α_l is not in Θ then orientability holds if and only if all the number of elements of the components of Θ are even. In this case $d_{i+1}-d_i$ is odd and $d_1 - d_0 = 1$ or $d_{k+1}-d_k = 1$. Hence the result follows.

2. As in the first case one can see that if some of the components of $\Sigma\setminus\Theta$ is not a singleton then all the components of Θ must have an even number of elements. Therefore the integers $d_{i+1}-d_i$ are odd.

$\square$

Example: A Grassmannian $\mathrm{Gr}_k(n)$ of k-dimensional subspaces in $\mathbb{R}^n$ is orientable if and only if n is even.

Remark: The orientability of the flag manifolds of $\mathrm{Sl}(n,\mathbb{R})$ can be decide also via Stiefel-Whitney classes as in Conde [2].

5.1.2 $B_l = \mathfrak{so}(l, l+1)$

Here the flag manifolds are $\mathbb{F}^I(d_1,\ldots,d_k) = \mathbb{F}_\Theta$ such that $j \in \{d_1,\ldots,d_k\}$ if and only if j is the index of a simple root $\alpha_j \notin \Theta$. The subgroup K_Θ is a product $\mathrm{SO}(n_1) \times \cdots \times \mathrm{SO}(n_s)$ with the sizes n_i given as follows:

1. If $d_k = l$, or equivalently $\alpha_l \notin \Theta$ then $K_\Theta = \mathrm{SO}(d_1) \times \cdots \times \mathrm{SO}(d_{k-1} - d_{k-2})$.
2. If $d_k < l$, or equivalently $\alpha_l \in \Theta$ then
   (a) $K_\Theta = \mathrm{SO}(d_1) \times \cdots \times \mathrm{SO}(d_k - d_{k-1}) \times \mathrm{SO}(2)$ if $d_k = l-1$, that is, $\{\alpha_l\}$ is a connected component of Θ.
   (b) $K_\Theta = \mathrm{SO}(d_1) \times \cdots \times \mathrm{SO}(d_k - d_{k-1}) \times \mathrm{SO}(l - d_k) \times \mathrm{SO}(l - d_k + 1)$ if $d_k < l - 1$, that is, the connected component of Θ containing α_l is a B_{l-d_k}.

Proposition 5.2 *The following two cases give necessary and sufficient conditions for flag manifold $\mathbb{F}^I(d_1,\ldots,d_k)$ of B_l to be orientable.*

1. *Suppose that $d_k = l$, that is, $\alpha_l \notin \Theta$. Then $\mathbb{F}^I(d_1,\ldots,d_k)$ is orientable if and only if $d_1,\ldots,d_{k-1}$, up to $k-1$, satisfy the* mod2 *condition. Equivalently, the sizes of the* $\mathrm{SO}(n_i)$*-components of K_Θ are congruent* mod2.
2. *Suppose that $d_k < l$, that is, $\alpha_l \in \Theta$. Then $\mathbb{F}^I(d_1,\ldots,d_k)$ is orientable if and only if $d_1,\ldots,d_k$ together with $l - d_k$ satisfy the* mod2 *condition.*

Proof: If $\alpha_l \notin \Theta$ then Θ is contained in the A_{l-1}-subdiagram $\{\alpha_1,\ldots,\alpha_{l-1}\}$. Hence the condition is the same as in the A_l case. Furthermore, $S(\alpha_l,\Delta)$ is even for any Δ because α_l is a short root. Therefore no further condition comes in.

In the second case, if Δ is the connected component of Θ containing α_l then the contribution $S(\alpha,\Delta)$ of Δ to the total sum is the number of elements of Δ by tables 2 and 3. Again, the conclusion is as in the A_l case. $\square$

Example: A Grassmannian $\mathrm{Gr}_k^I(n) = \mathbb{F}^I(k)$ of k-dimensional isotropic subspaces in $\mathbb{R}^{2l+1}$ is orientable if and only if either i) $k = l$ or ii) $k < l$ and l is even.

5.1.3 $C_l = \mathfrak{sp}(l, \mathbb{R})$

Again the flag manifolds are $\mathbb{F}^I(d_1, \ldots, d_k) = \mathbb{F}_\Theta$ such that $j \in \{d_1, \ldots, d_k\}$ if and only if j is the index of a simple root $\alpha_j \notin \Theta$. The subgroup K_Θ is

1. $\mathrm{SO}(d_1) \times \cdots \times \mathrm{SO}(d_{k-1} - d_{k-2})$ if $d_k = l$.
2. $\mathrm{SO}(d_1) \times \cdots \times \mathrm{SO}(d_{k-1} - d_{k-2}) \times \mathrm{SO}(2)$ if $d_k = l - 1$.
3. $\mathrm{SO}(d_1) \times \cdots \times \mathrm{SO}(d_{k-1} - d_{k-2}) \times \mathrm{U}(l - d_k)$ if $d_k < l - 1$.

Proposition 5.3 *For C_l a necessary and sufficitent condition for the orientability of $\mathbb{F}^I(d_1, \ldots, d_k)$ is that $d_1, \ldots, d_k$ satisfy the* mod2 *condition.*

Proof: There are two possibilities:

1. If $d_k = l$, that is, $\alpha_l \notin \Theta$ then Θ is contained in the A_{l-1} and the condition, up to $k-2$, comes from the A_l case. The difference $d_k - d_{k-1}$ also enters in the condition because α_l is a large root.

2. If $d_k < l$, that is, $\alpha_l \in \Theta$ then the conditions are necessary as in the A_l case. To see that no further condition appears look at the connected component Δ containing α_l. If $\Delta = \{\alpha_l\}$ then $S(\alpha_{l-1}, \Delta)$ is even because α_{l-1} is a short root. Otherwise, Δ is a C_k and its contribution is also even by table 4.

□

5.1.4 $D_l = \mathfrak{so}(l,l)$

The flag manifolds of $\mathfrak{so}(l,l)$ are also realized as flags of isotropic subspaces with a slight difference from the odd dimensional case $B_l = \mathrm{SO}(l,l+1)$. First a minimal flag manifold $\mathbb{F}_{\Sigma\setminus\{\alpha_i\}}$ is the Grassmannian of isotropic subspaces of dimension i if $i \leq l-2$. However, both $\mathbb{F}_{\Sigma\setminus\{\alpha_{l-1}\}}$ and $\mathbb{F}_{\Sigma\setminus\{\alpha_l\}}$ are realized as subsets of l-dimensional isotropic subspaces. Each one is a closed orbit of the identity component of $\mathrm{SO}(l,l)$ in the Grassmannian $\mathrm{Gr}_l^I(2l)$ of l-dimensional isotropic subspaces. We denote these orbits by $\mathrm{Gr}_{l^+}^I(2l) = \mathbb{F}_{\Sigma\setminus\{\alpha_l\}}$ and $\mathrm{Gr}_{l^-}^I(2l) = \mathbb{F}_{\Sigma\setminus\{\alpha_l-1\}}$. (By the way the isotropic Grassmannian $\mathrm{Gr}_{l-1}^I(2l)$ is the flag manifold $\mathbb{F}_{\Sigma\setminus\{\alpha_{l-1},\alpha_l\}}$, which is not minimal.)

Accordingly, the flag manifolds of $\mathfrak{so}(l,l)$ are defined by indices $1 \leq d_1 \leq \cdots \leq d_k \leq l-2$ joined eventually to l^+ and l^-. The elements of $\mathbb{F}^I(d_1\ldots,d_k)$ are flags of isotropic subspaces $V_1 \subset \cdots \subset V_k$ with $\dim V_i = d_k$. When l^+ or l^- are present then one must include an isotropic subspace in $\mathrm{Gr}_{l^+}^I(2l)$ or $\mathrm{Gr}_{l^-}^I(2l)$, respectively, containing V_k, and hence the other subspaces.

The group K_Θ is a product of $\mathrm{SO}(d)$'s components each one for a connected component of Θ unless a D_k component appears. Such a component contributes to K_Θ with a $\mathrm{SO}(k) \times \mathrm{SO}(k)$.

Proposition 5.4 *The orientability of the flag manifolds of $D_l = \mathfrak{so}(l,l)$ is given as follows:*

1. *For a flag $\mathbb{F}^I(d_1,\ldots,d_k)$ there are the possibilities:*

   (a) *If $d_k \leq l-4$ then orientability holds if and only if $d_1,\ldots,d_k$ satisfy the* mod2 *condition.*

   (b) *If $d_k = l-3$ then orientability holds if and only if the differences $d_{i+1}-d_i$, $i=0,\ldots,k-1$, are even numbers.*

   (c) *If $d_k = l-2$ then orientability holds if and only if the differences $d_{i+1}-d_i$, $i=0,\ldots,k-1$, are odd numbers.*

2. *For the flag manifolds $\mathbb{F}^I(d_1,\ldots,d_k,l^+)$ and $\mathbb{F}^I(d_1,\ldots,d_k,l^-)$ we have:*

   (a) *If $d_k = l-2$ then the condition is that $d_{i+1}-d_i$, $i=0,\ldots,k-2$, are even numbers.*

(b) *If* $d_k < l-2$ *then the condition is that* $d_{i+1} - d_i$*,* $i = 0, \ldots, k-2$*, are odd numbers and* $d_k - d_{k-1}$ *is even.*

3. *For the flag manifolds* $\mathbb{F}^I(d_1, \ldots, d_k, l^+, l^-)$ *we have:*

   (a) *If* $d_k = l-2$ *then* $d_1, \ldots, d_{k-2}$ *satisfy the* mod2 *condition.*

   (b) *If* $d_k < l-2$ *then* $d_{i+1} - d_i$*,* $i = 0, \ldots, k-2$*, are odd numbers.*

Proof: If $d_k \leq l-4$ then Θ contains a connected component Δ which is a D_k (at the right side of the diagram). By table 5 the contribution of Δ is even, so that orientability depends on the roots in the A_{l-4} diagram $\{\alpha_1, \ldots, \alpha_{l-4}\}$ where the condition is as in the statement. If $d_k = l-3$ then the differences $d_{i+1} - d_i$, $i = 0, \ldots, k-1$, must be congruent mod2 to have orientability. But the root α_{l-3} is linked to the $A_3 = \{\alpha_{l-2}, \alpha_{l-1}, \alpha_l\}$, so that the number of elements of the components of Θ are odd, that is, the differences $d_{i+1} - d_i$ are even. The same argument applies to $d_k = l-2$, but now α_{l-2} is linked to the two A_1's $\{\alpha_{l-1}\}$ and $\{\alpha_l\}$.

The other cases are checked the same way. □

6 Vector bundles over flag bundles

In this final section we consider vector bundles over flag bundles. The orientability of vector bundles over the flag manifolds carry over to vector bundles over flag bundles in case the latter are bundles associated to trivial principal bundles.

With the notations of Section 3, let R be a K-principal bundle. Since K acts continuously on V and B, the associated bundle $R \times_K V$ is a finite dimensional vector bundle over $R \times_K B$ whose fibers are the same as the fibers of V.

Proposition 6.1 *Assume that* R *is trivial. Then the vector bundle*

$$R \times_K V \to R \times_K B$$

is orientable if, and only if, the vector bundle $V \to B$ *is orientable.*

Proof: Since the K-principal bundle $R \to Y$ is trivial, we have that $R \times_K V \to R \times_K B$ is homeomorphic as a vector bundle to $Y \times V \to Y \times B$. Since the frame bundle of $Y \times V$ can be given by $Y \times FV$, the orientation bundle of $Y \times V$ can be given by $Y \times \mathcal{O}V$. If $\sigma : B \to \mathcal{O}V$ is a continuous section, then $(y, x) \mapsto (y, \sigma(x))$ is a continuous section of $Y \times \mathcal{O}V$. Reciprocally, if $\sigma : Y \times B \to Y \times \mathcal{O}V$ is a continuous section, then $x \mapsto \sigma(y_0, x)$ is a continuous section of $\mathcal{O}V$, where $y_0 \in Y$. $\square$

Let G be a Lie group acting on its Lie algebra $\mathfrak{g}$ by the adjoint action. The vector bundles we will consider in the sequel arise as associated bundles of the L-principal bundle $K \to K/L$, where K is a subgroup of G. For an L-invariant subspace $\mathfrak{l}$ of $\mathfrak{g}$, we will consider the associated vector bundle

$$V = K \times_L \mathfrak{l},$$

whose typical fiber is $\mathfrak{l}$.

Corollary 6.2 *The associated vector bundle V is orientable if and only if* $\det(g|_{\mathfrak{l}}) > 0$, *for every $g \in L$.*

Proof: We only need to show that V satisfies the hypothesis of Proposition 3.1. First we note that its frame bundle is given by $FV = K \times_L \mathrm{Gl}(\mathfrak{l})$. Defining an action $k \in K$ on $m \cdot X \in FV$ by

$$k(m \cdot X) = km \cdot X,$$

where $m \in K$, $X \in \mathfrak{l}$, we have that the action of K on K/L lifts to a continuous action of automorphisms on the frame bundle FV. $\square$

To conclude we apply our results to the situation of [12], where flows on flag bundles and their Conley indices are considered. In [12] one starts with a principal bundle $Q \to B$ whose structural group G is semi-simple, and a flow ϕ_t, $t \in \mathbb{Z}$ or $\mathbb{R}$, of continuous automorphisms of Q which is chain transitive on the base B. There are induced flows on the associated bundles $Q \times_G F$, where the typical fiber F is acted by G on the left. In particular, in [12] it is taken as a typical fiber F a flag manifold $\mathbb{F}_\Theta$ of G yielding the flag bundle $\mathbb{E}_\Theta = Q \times_G \mathbb{F}_\Theta$.

According to the results of [11] and [12], each Morse component $\mathcal{M}_\Theta(w)$ of ϕ^t is a flag bundle of a certain subbundle Q_ϕ of Q. Moreover, the unstable set $\mathcal{V}_\Theta^+(w)$ of the Morse component $\mathcal{M}_\Theta(w)$ is an associated vector bundle of Q_ϕ whose base is $\mathcal{M}_\Theta(w)$ and whose typical fiber is the same as the fiber of $V_\Theta^+(H_\phi, w)$, where is H_ϕ is a certain element of $\mathrm{cl}\mathfrak{a}^+$, called the parabolic type of ϕ^t.

When the base B is a point, the flow of automorphisms ϕ^t is given by g^t for some $g \in G$, when $t \in \mathbb{Z}$, or by $\exp(tX)$ for some $X \in \mathfrak{g}$, when $t \in \mathbb{R}$. In [4], it is shown that the parabolic type H_ϕ of these flows is given by the hyperbolic component of g or X under the Jordan decomposition.

In [12], we show that the Conley index of the attractor component in the maximal flag bundle and, under certain hypothesis, the Conley index of each Morse component, is the Thom space of its unstable vector bundle. The orientability of the unstable vector bundle then comes to the scene in order to apply Thom isomorphism and detect the homological Conley indices of the Morse components. With these results in mind we state the following criterion of orientability of $\mathcal{V}_\Theta^+(w)$, that follows immediately from Proposition 6.1.

Proposition 6.3 *Assume that the reduction R_ϕ is a trivial bundle. The stable and unstable vector bundles $\mathcal{V}_\Theta^\pm(H, w)$ are orientable if and only if the vector bundles $V_\Theta^\pm(H, w)$ are orientable.*

There are two cases where the hypothesis of the above result are automatically satisfied. Namely for periodic flows, it is shown in [4] that the reduction Q_ϕ is trivial. For the control flow of [1], the reduction Q_ϕ is always trivial since the base space of the control flow is contractible.

References

[1] F. Colonius and W. Kliemann: The dynamics of control. Birkhäuser, Boston (2000).

[2] Conde, A.: Sobre as classes de Atiyah-Hirzebruch, de Thom, o problema do mergulho e variedades flâmulas. Thesis USP-SC (1979). (In portuguese.)

[3] Duistermat, J.J., Kolk, J.A.C., Varadarajan, V.S. *Functions, flows and oscilatory integral on flag manifolds.* Compositio Math. 49, 309-398, (1983).

[4] T. Ferraiol, M. Patrão and L. Seco: *Jordan decomposition and dynamics on flag manifolds*, Discrete Contin. Dyn. Syst. A **26** (2010), 923-947.

[5] Fulton, W. and J. Harris: Representation Theory. A first course. Graduate Texts in Mathematics. Springer-Verlag, 1991.

[6] Helgason, S.: Diferential Geometry, Lie groups and Symmetric spaces. Ac. Press (1978).

[7] Humphreys, J. E.: Reflection groups and Coxeter groups. Cambridge Studies in Advanced Mathematics, **29**. Cambridge University Press (1990).

[8] Johnson, K.D.: The structure of parabolic subgroups. J. Lie Theory **14** (2004), 287-316.

[9] Knapp, A.W. Lie Groups. Beyond an Introduction. Progress in Mathematics **140**, Birkhäuser, 2004.

[10] Kocherlakota, R.R.: Integral homology of real flag manifolds and loop spaces of symmetric spaces. Adv. Math. **110** (1995), 1-46.

[11] Patrão, M., and L.A.B San Martin: *Chain recurrence of flows and semi-flows on fiber bundles*. Discrete Contin. Dynam. Systems A, **17** (2007), 113-139.

[12] Patrão, M., L.A.B. San Martin and Lucas Seco: Conley indexes and stable sets for flows on flag bundles. Dynamical Systems, **24** (2009), 249-276.

[13] San Martin, L.A.B: Álgebras de Lie. Editora Unicamp (1999). (In portuguese.)

[14] San Martin, L.A.B. and L. Seco: *Morse and Lyapunov Spectra and Dynamics on Flag Bundles*. Ergodic Theory & Dynamical Systems, **30** (2009), 893-922.

[15] Warner, G.: Harmonic Analysis on Semi-simple Lie Groups I, Springer-Verlag, 1970.