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DEFINING RELATIONS OF ALMOST AFFINE (HYPERBOLIC) LIE SUPERALGEBRAS

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For all almost affine (hyperbolic) Lie superalgebras, the defining relations are computed in terms of their Chevalley generators.

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1. Introduction

For the definition of Lie superalgebras with Cartan matrix, in particular, of the almost affine Lie superalgebras, see [CCLL]. We recall the definition of Chevalley generators and the defining relations expressed in terms of these generators, see [GL2] and a review [BGLL] which also contains the modular case.

Here we list defining relations of almost affine Lie superalgebras with indecomposable Cartan matrices classified in [CCLL]. For the Lie superalgebras whose Cartan matrices are symmetrizable and without zeros on the main diagonal, the relations were known; they are described by almost the same rules as for Lie algebras and they are only of Serre type if the off-diagonal elements of such Cartan matrices are non-positive.

Nothing was known about Lie superalgebras of the type we study in this note: almost affine Lie superalgebras with indecomposable Cartan matrices with zeros on the diagonal and without zeros but non-symmetrizable.

Although presentation — description in terms of generators and relations — is one of the accepted ways to represent a given algebra, it seems that an *explicit* form of the presentation is worth the trouble to obtain only if this presentation is often in need, or (which is usually the same) is sufficiently neat. The Chevalley generators of simple finite dimensional Lie algebras over \mathbb{C} satisfy

simple and neat relations (“Serre relations”) and are often needed for various calculations and theoretical discussions. Relations between their analogs in the super case, although not so neat (certain “non-Serre relations” appear), are still tolerable, at least, for most Cartan matrices. One Lie superalgebra may have several inequivalent Cartan matrices, and although for some Cartan matrices of a given Lie superalgebra the relations between Chevalley generators are rather complicated (non-Serre ones), for some other Cartan matrices of the same algebra there are only Serre relations, see [GL2]; see [RU] where this fact is used.

What we usually need to know about defining relations is that there are finitely many of them; hence the fact that some simple loop superalgebras with Cartan matrix are **not** finitely presentable in terms of Chevalley generators was unexpected (although obvious as an afterthought, see [GL2], where this was first published).

The non-Serre relations — a bit too complicated to be used by humans — were of purely theoretic interest until recently Grozman’s package `SuperLie` ([Gr]) made the task of finding the explicit expression of the defining relations for many types of Lie algebras and superalgebras a routine exercise for anybody capable to use `Mathematica`. Unfortunately, such calculations, performed up to certain degree in generators, are not conclusive if not supported by cohomological arguments of the type elucidated in [Zus]: there might be some relations of higher orders left unobserved.

There are several reasons for deriving the relations even if they look awful:

- (1) Computers easily swallow, often, what is non-palatable to humans (and the relation we derive in this note is precisely the case);
- (2) even a rough description of the relations (the value of the highest degree, etc.) might be sufficient for deriving the next result based on this description (e.g., the description of the real forms of the Lie algebra $\mathfrak{gl}(\lambda)$ of “matrices of complex size”, see [LSg] based on [GL1]).

Incompleteness of relations is irrelevant if it is clear that there are infinitely many relations and their explicit form is not, actually, needed, as in some cases (like $\mathfrak{psl}(n|n)^{(1)}$) considered in [GL1].

2. Recapitulation

Let the elements $X_i^\pm, d_i,$ and $h_i,$ where $i = 1, \dots, n,$ generate the Lie superalgebra denoted $\check{\mathfrak{g}}(A, I),$ where $I = (i_1, \dots, i_n) \in (\mathbb{Z}/2)^n$ is a collection of parities ($p(e_j^\pm) = i_j$ and $A = (A_{ij})_{i,j=1}^n$), see [CCLL].

2.1. Serre relations, see [GL2]. Let A be an $n \times n$ matrix. We find the defining relations by induction on n with the help of the Hochschild–Serre spectral sequence (for its description for Lie superalgebras, which has certain subtleties, see [Po]). For the basis of induction consider the following cases of Dynkin diagrams with one vertex and no edges:

- or ● no relations, i.e., \mathfrak{g}^\pm are free Lie superalgebras;
 - $\text{ad}_{X^\pm}^2(X^\pm) = 0;$
 - ⊗ $[X^\pm, X^\pm] = 0.$
- (2.1)

Set $\text{deg } X_i^\pm = 0$ for $1 \leq i \leq n-1$ and $\text{deg } X_n^\pm = \pm 1.$ Let $\mathfrak{g}^\pm = \bigoplus \mathfrak{g}_i^\pm$ and $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ be the corresponding \mathbb{Z} -gradings. Set $\mathfrak{g}_\pm = \mathfrak{g}^\pm / \mathfrak{g}_0^\pm.$ From the Hochschild–Serre spectral sequence for the pair $\mathfrak{g}_0^\pm \subset \mathfrak{g}^\pm$ we get:

$$H_2(\mathfrak{g}_\pm) \subset H_2(\mathfrak{g}_0^\pm) \oplus H_1(\mathfrak{g}_0^\pm; H_1(\mathfrak{g}_\pm)) \oplus H_0(\mathfrak{g}_0^\pm; H_2(\mathfrak{g}_\pm)). \tag{2.2}$$

It is clear that

$$H_1(\mathfrak{g}_\pm) = \mathfrak{g}_1^\pm, \quad H_2(\mathfrak{g}_\pm) = \wedge^2(\mathfrak{g}_1^\pm) / \mathfrak{g}_2^\pm. \tag{2.3}$$

So, the second summand in (2.2) provides us with relations of the form:

$$\begin{aligned} (\text{ad}_{X_n^\pm})^{k_{ni}}(X_i^\pm) &= 0 \text{ if the } n\text{-th simple root is not } \otimes \\ [X_n, X_n] &= 0 \text{ if the } n\text{-th simple root is } \otimes. \end{aligned} \tag{2.4}$$

while the third summand in (2.2) is spanned by the \mathfrak{g}_0^\pm -lowest vectors in

$$\wedge^2(\mathfrak{g}_1^\pm)/(\mathfrak{g}_2^\pm + \mathfrak{g}^\pm \wedge^2(\mathfrak{g}_1^\pm)). \tag{2.5}$$

Let the matrix $B = (B_{kj})$ be given by the formula

$$B_{kj} = \begin{cases} -\frac{2A_{kj}}{A_{kk}} & \text{if } A_{kk} \neq 0 \text{ and } -\frac{2A_{kj}}{A_{kk}} \in \mathbb{Z}_+, \\ 1 & \text{if } i_k = \bar{1}, A_{kk} = 0, A_{kj} \neq 0, \\ 0 & \text{if } i_k = \bar{1}, A_{kk} = A_{kj} = 0, \\ 0 & \text{if } i_k = \bar{0}, A_{kk} = \bar{0}, A_{kj} = 0. \end{cases} \tag{2.6}$$

The following proposition, its proof being straightforward, illustrates the usefulness of our normalization of Cartan matrices as compared with other options:

2.1.1. Proposition. *The numbers k_{in} and k_{ni} in (2.4) are expressed in terms of B_{kj} as follows:*

$$\begin{aligned} (\text{ad}_{X_k^\pm})^{1+B_{kj}}(X_j^\pm) &= 0 \text{ for } k \neq j \\ [X_i^\pm, X_i^\pm] &= 0 \text{ if } A_{ii} = 0. \end{aligned} \tag{2.7}$$

The relations (??) and (2.7) will be called *Serre relations* of the Lie superalgebra $\mathfrak{g}(A)$.

2.2. Non-Serre relations. These are relations that correspond to the third summand in (2.2). Let us consider the simplest case: $\mathfrak{sl}(m|n)$ in the realization with the system of simple roots

$$\circ - \dots - \circ - \otimes - \circ - \dots - \circ \tag{2.8}$$

Then $H_2(\mathfrak{g}_\pm)$ from the third summand in (2.2) is just $\wedge^2(\mathfrak{g}_\pm)$. For simplicity, we confine ourselves to the positive roots. Let X_1, \dots, X_{m-1} and Y_1, \dots, Y_{n-1} be the root vectors corresponding to even roots separated by the root vector Z corresponding to the root \otimes .

If $n = 1$ or $m = 1$, then $\wedge^2(\mathfrak{g})$ is an irreducible \mathfrak{g}_0 -module and there are no non-Serre relations. If $n \neq 1$ and $m \neq 1$, then $\wedge^2(\mathfrak{g})$ splits into 2 irreducible \mathfrak{g}_0 -modules. The lowest component of one of them corresponds to the relation $[Z, Z] = 0$, the other one corresponds to the non-Serre-type relation

$$[[X_{m-1}, Z], [Y_1, Z]] = 0. \tag{2.9}$$

If, instead of $\mathfrak{sl}(m|n)$, we would have considered the Lie algebra $\mathfrak{sl}(m+n)$, the same argument would have led us to the two relations, both of Serre type:

$$\text{ad}_Z^2(X_{m-1}) = 0, \quad \text{ad}_Z^2(Y_1) = 0.$$

In what follows we give an explicit description of the defining relations between the generators of the positive part of the almost affine Lie superalgebras in terms of their Chevalley generators.

166 *S. Bouarroudj, P. Grozman, D. Leites***3. Results (the proof is based on [Zus] and the induction as above)**For NS_{333} (CM2), CM4): Serre relations only):

$$\begin{aligned}
CM1) \quad & [x_1, x_1] = 0, \quad [x_2, x_2] = 0, \quad [x_3, x_3] = 0, \\
& 2[[x_1, x_2], [x_2, [x_1, x_3]]] = a [[x_1, x_2], [x_3, [x_1, x_2]]], \\
& [[x_1, x_3], [x_2, [x_1, x_3]]] = 2a [[x_1, x_3], [x_3, [x_1, x_2]]], \\
& [[x_2, x_3], [[x_2, x_3], [x_2, [x_1, x_3]]]] = (3a + 5) [[x_2, x_3], [[x_2, x_3], [x_3, [x_1, x_2]]]]. \tag{3.1} \\
CM3) \quad & [x_2, x_2] = 0, \quad [x_1, [x_1, x_2]] = 0, \quad [x_1, [x_1, x_3]] = 0, \quad [x_3, [x_2, x_3]] = 0, \\
& [x_3, [x_3, [x_1, x_3]]] = 0, \\
& 10[[x_1, x_3], [[x_1, x_2], [x_2, x_3]]] = 20(a + 2) [[x_2, [x_1, x_3]], [x_3, [x_1, x_2]]] - \\
& (15a + 13) [[x_3, [x_1, x_2]], [x_3, [x_1, x_2]]] - (15a + 8) [[x_2, [x_1, x_3]], [x_2, [x_1, x_3]]].
\end{aligned}$$

For NS_{334} (CM2), CM4): Serre relations only):

$$\begin{aligned}
CM1) \quad & [x_1, x_1] = 0, \quad [x_2, x_2] = 0, \quad [x_3, x_3] = 0, \\
& 2[[x_1, x_2], [x_2, [x_1, x_3]]] = a [[x_1, x_2], [x_3, [x_1, x_2]]], \\
& [[x_1, x_3], [x_2, [x_1, x_3]]] = 2a [[x_1, x_3], [x_3, [x_1, x_2]]] \\
CM3) \quad & [x_2, x_2] = 0, \quad [x_1, [x_1, x_2]] = 0, \tag{3.2} \\
& [x_1, [x_1, x_3]] = 0, \quad [x_3, [x_2, x_3]] = 0, \quad [x_3, [x_3, [x_3, [x_1, x_3]]]] = 0, \\
& 14[[x_1, x_3], [[x_1, x_2], [x_2, x_3]]] = 42(a + 4) [[x_2, [x_1, x_3]], [x_3, [x_1, x_2]]] - \\
& 2(14a + 11) [[x_3, [x_1, x_2]], [x_3, [x_1, x_2]]] - (28a + 15) [[x_2, [x_1, x_3]], [x_2, [x_1, x_3]]].
\end{aligned}$$

For NS_{335} (CM2), CM4): Serre relations only):

$$\begin{aligned}
CM1) \quad & [x_1, x_1] = 0, \quad [x_2, x_2] = 0, \quad [x_3, x_3] = 0, \\
& 2[[x_1, x_2], [x_2, [x_1, x_3]]] = a [[x_1, x_2], [x_3, [x_1, x_2]]] \\
& [[x_1, x_3], [x_2, [x_1, x_3]]] = 2a [[x_1, x_3], [x_3, [x_1, x_2]]]. \\
CM3) \quad & [x_2, x_2] = 0, \quad [x_1, [x_1, x_2]] = 0, \quad [x_1, [x_1, x_3]] = 0, \quad [x_3, [x_2, x_3]] = 0, \tag{3.3} \\
& [x_3, [x_3, [x_3, [x_3, [x_1, x_3]]]] = 0 \\
& 6[[x_1, x_3], [[x_1, x_2], [x_2, x_3]]] = 24(a + 2) [[x_2, [x_1, x_3]], [x_3, [x_1, x_2]]] - \\
& (15a + 11) [[x_3, [x_1, x_2]], [x_3, [x_1, x_2]]] - (15a + 8) [[x_2, [x_1, x_3]], [x_2, [x_1, x_3]]].
\end{aligned}$$

For NS_{336} (CM2), CM3), CM4): Serre relations only):

$$\begin{aligned}
CM1) \quad & [x_1, x_1] = 0, \quad [x_2, x_2] = 0, \quad [x_3, x_3] = 0, \\
& 3[[x_1, x_2], [[x_1, x_2], [x_2, [x_1, x_3]]]] = a [[x_1, x_2], [[x_1, x_2], [x_3, [x_1, x_2]]]], \\
& [[x_1, x_3], [[x_1, x_3], [x_2, [x_1, x_3]]]] = 3a [[x_1, x_3], [[x_1, x_3], [x_3, [x_1, x_2]]]], \\
& [[x_2, x_3], [x_2, [x_1, x_3]]] = (2a + 5) [[x_2, x_3], [x_3, [x_1, x_2]]], \\
& 16[[x_2, x_3], [[x_1, x_2], [x_1, x_3]]] = -(15a + 37) [[x_2, [x_1, x_3]], [x_2, [x_1, x_3]]] + \\
& (10a + 14) [[x_2, [x_1, x_3]], [x_3, [x_1, x_2]]] - 5(3a + 1) [[x_3, [x_1, x_2]], [x_3, [x_1, x_2]]]. \tag{3.4}
\end{aligned}$$

For NS_{37} (CM2), CM3, CM4): Serre relations only):

$$\begin{aligned}
 CM1) \quad & [x_1, x_1] = 0, \quad [x_2, x_2] = 0, \quad [x_3, x_3] = 0, \\
 & 3[[x_1, x_2], [[x_1, x_2], [x_2, [x_1, x_3]]]] = a [[x_1, x_2], [[x_1, x_2], [x_3, [x_1, x_2]]]] \\
 & [[x_1, x_3], [[x_1, x_3], [x_2, [x_1, x_3]]]] = 3a [[x_1, x_3], [[x_1, x_3], [x_3, [x_1, x_2]]]] \\
 & [[x_2, x_3], [[x_2, x_3], [x_2, [x_1, x_3]]]] = (3a + 8) [[x_2, x_3], [[x_2, x_3], [x_3, [x_1, x_2]]]]
 \end{aligned} \tag{3.5}$$

For the remaining almost affine Lie superalgebras, the defining relations are only Serre ones regardless of symmetrizability of their Cartan matrix.

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