

# Inflationary Perturbations in Palatini Generalised Gravity

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We examine the generation of primordial perturbations during an inflationary epoch in generalised theories of gravity when the equations of motion are derived using the Palatini variational principle. Both  $f(R)$  and Scalar-Tensor theories are considered and we compare our results with those obtained under the conventional metric formalism. Non-linear generalisations of the action lead to different theories under the two variational choices and we obtain distinct results for scalar and tensor spectral indices and their ratio. We find the following general result; inflation driven solely by  $f(R)$  modifications alone do not result in suitable curvature perturbations whilst Scalar-Tensor theories generate nearly scalar invariant curvature perturbations but no tensor modes.

## I. INTRODUCTION

Today, more than ever, there is an element of doubt underlying the confidence with which we claim to understand the nature and behaviour of the universe. The so-called standard  $\Lambda$  Cold Dark Matter ( $\Lambda$ CDM) model produces extraordinarily good fits to numerous observations in the era of precision cosmology we live in. However, the theory of General Relativity (GR) underpinning the dynamical workings of the standard model requires a number of supporting partners throughout the entire history of the universe to bring about this remarkable agreement with data. A scalar field is needed to drive the inflationary period after the Big Bang; a strange form of energy, called Dark Energy, is required for the late time observed acceleration; while an unknown kind of pressureless matter, called Dark Matter, is postulated to account for the missing mass of galaxies and galaxy clusters. As such, beyond solar system scales, where the theory has been well tested and confirmed, GR always requires the introduction of some exotic form of energy-density to stand up against confrontation with observations. This uncomfortable situation has led to many attempts at modifying GR itself in the search of a generalised theory that can explain the observed universe without requiring the addition of dark companions. Although none of the modifications seem to have the general and unifying character of GR itself, it can be argued that some models can address specific problems in a more natural way.

In this work we deal with a class of theories which has been denoted in the literature under the name of *Generalised* or *Extended Gravity* [8, 25]. The starting point of these theories is a general modification of the Einstein-Hilbert action consisting both in the introduction of a scalar field  $\phi$  and the addition of terms which are non-linear in the curvature scalar  $R$ . The general form such an action can assume is given in Section II. This class of gravitational theories incorporates both subclasses given by  $f(R)$  theories [31, 32] and Scalar-Tensor

theories [16, 18]. The first treats only non-linear terms in  $R$  given by an arbitrary function  $f(R)$ , without introducing an extra scalar field. The second considers instead all the possible modifications given by  $\phi$ , but  $R$  appears only linearly in the action (though its coefficient is a general function of  $\phi$ ). Further constraints on the form of  $f$  or on the scalar field terms can lead to specific theories which have been used extensively in literature trying to solve many different problems. For example, some of the very first models of inflation were an  $f(R)$  theory based on an  $R^2$  term [3] and a minimally coupled Scalar-Tensor theory [4]. The well-known Brans-Dicke theory [1] can also be regarded as the first Scalar-Tensor theory of gravitation. More recently  $f(R)$  theories have been invoked as a possible explanation for the late time acceleration of the universe [31, 32] without the need to introduce Dark Energy. On the other hand Scalar-Tensor theories seem to be even more promising as models for inflation [12, 13]. Nevertheless there have been several attempts using  $f(R)$  and Scalar-Tensor theories to characterize both inflation and late time acceleration, respectively [20, 32, 34]. Some models go even further in trying to solve the dynamical riddle of early-time inflation and late-time acceleration in a unified manner [21, 22, 26].

Once a modified action has been chosen there is a further degree of freedom when generalising gravity. The variational principle used to obtain the field equations for the model must be chosen. There are two main variational methods widely used in the literature [2] (see also [5] for a historical perspective on their origin). The first is the usual metric variational principle used by Hilbert in the first derivation of the Einstein equation with a variational method. The metric formalism obtains the gravitational field equations by varying the action only with respect to the space-time metric tensor  $g_{\mu\nu}$ , which is assumed to be symmetric. An alternative is known as the Palatini variational principle and consists in varying the action with respect to the metric and the torsionless connection  $\hat{\Gamma}^\lambda_{\mu\nu}$  independently. Here the independent connection  $\hat{\Gamma}^\lambda_{\mu\nu}$  is not assumed to be, a priori, the more commonly used Levi-Civita connection  $\Gamma^\lambda_{\mu\nu}$  for which the additional metric condition  $\nabla_\mu g_{\alpha\beta} = 0$  holds. In the Palatini formalism this condition is in fact obtained dynamically by varying the action rather than being im-

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posed axiomatically from the beginning. Crucially, if the action is non-linear in the curvature scalar, the Palatini method of variation leads to a different theory than the metric variation [31]. This means that the choice of variational method is not just a question of formalism, but can in principle lead to different physical outcomes. Both methods have their own advantages and handicaps, but the metric variational principle has been on the whole more utilized in the literature. Because of this, many issues in generalised gravity have already been studied carefully in the metric variation formalism but have not been looked at in depth in the context of the Palatini formalism. The aim of this work is to deal with precisely one of these issues, namely the characterisation of super-horizon, primordial perturbations generated during an epoch of inflation. These have been examined extensively in the metric case [8, 14, 32], but the Palatini approach has yet to be studied in detail.

The standard theory of inflation is basically a Scalar-Tensor theory where a potential dominated, minimally coupled scalar field drives inflation [15, 19]. However some  $f(R)$  models succeed in building theories where the universe inflates without the introduction of a scalar field. This can be achieved in both metric [3, 32, 34] and Palatini [17, 21] formalisms. In the latter case, models giving bouncing cosmologies have been also proposed recently [30]. It is then interesting to ask how the primordial perturbations have been generated in the various models. To do this we carry out a general analysis of perturbations for Palatini generalised gravity in which the specific choice of the model is left for the end of the calculations. This analysis will include all the  $f(R)$  and Scalar-Tensor models in Palatini formalism, leaving standard inflation as a particular case. The aim is to determine the power spectra and spectral indices for scalar and tensor perturbations.

The motivation for focusing on the characterisation of the primordial perturbations stems from the observational front-line. Over the next decade observations of the Cosmic Microwave Background (CMB) will constrain to unprecedented level the spectral index of primordial scalar (curvature) perturbations and the amplitude of tensor (gravitational wave) perturbations. In particular, if detected, the tensor modes will allow us to infer the energy scale and nature of the inflaton potential. Conversely, in the context of generalised theories, it may allow us to constrain the nature of the modifications.

This *paper* is organized as follows. In Section II we provide a short review of the Palatini approach to generalised gravity, where the gravitational field equations and the background cosmological dynamics are derived. Section III is dedicated to developing formalism and dynamics of cosmological perturbations in these models. The main section of the paper is Section IV where we examine cosmological perturbations generated during an inflationary epoch and derive the form of spectral indices for both scalar and tensor modes and their ratio. In section V we apply these results to  $f(R)$  gravity and scalar-

tensor theories and provide an example with non-minimal inflation. We conclude by summarising our main results in Section VI.

Throughout the paper, unless otherwise specified, we work in units such that  $c = 8\pi G = 1$ . Greek and Latin indices run from 0 to 3 and 1 to 3, respectively. The Minkowski metric is taken to be  $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$ .

## II. PALATINI FORMULATION OF GENERALISED GRAVITY

We begin with a review of the Palatini approach to generalised gravity, where the connection is varied independently from the metric (see e.g. [11, 31] for more in depth reviews). Consider a generalised gravitational theory described by the action

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} f \left( \hat{R}(g_{\mu\nu}, \hat{\Gamma}_{\beta\gamma}^\alpha), \phi \right) + \mathcal{L}_\phi(g_{\mu\nu}, \phi) + \mathcal{L}_M(g_{\mu\nu}, \Xi) \right], \quad (1)$$

where  $f$  is an arbitrary function of  $\hat{R}$  and of a generic scalar field  $\phi$  while  $\mathcal{L}_M$  is the matter Lagrangian depending on some matter fields denoted collectively by  $\Xi$ . Here, the curvature scalar  $\hat{R} = g^{\mu\nu} \hat{R}_{\mu\nu}$  is the contraction of the Ricci tensor defined by the torsionless independent connection  $\hat{\Gamma}_{\beta\gamma}^\alpha$ ,

$$\hat{R}_{\mu\nu} = \hat{\Gamma}_{\mu\nu,\alpha}^\alpha - \hat{\Gamma}_{\mu\alpha,\nu}^\alpha + \hat{\Gamma}_{\alpha\lambda}^\alpha \hat{\Gamma}_{\mu\nu}^\lambda - \hat{\Gamma}_{\mu\lambda}^\alpha \hat{\Gamma}_{\alpha\nu}^\lambda. \quad (2)$$

The Lagrangian for the scalar field  $\phi$  is taken to be

$$\mathcal{L}_\phi = -\frac{1}{2} \omega(\phi) (\partial\phi)^2 - V(\phi), \quad (3)$$

from which we define the energy-momentum tensor

$$T_{\mu\nu}^{(\phi)} = \omega(\nabla_\mu\phi)(\nabla_\nu\phi) - g_{\mu\nu} \left[ \frac{1}{2} \omega(\partial\phi)^2 + V(\phi) \right], \quad (4)$$

where  $\nabla$  is the covariant derivative associated with the usual Levi-Civita connection  $\Gamma_{\beta\gamma}^\alpha$ . In general, quantities formed with the independent ‘‘Palatini’’ connection will be denoted by an *overhat* to distinguish them from quantities formed with the Levi-Civita connection. For example, the curvature scalar and Ricci tensor we know from canonical GR are denoted by  $R$  and  $R_{\mu\nu}$ . Note that in the Palatini variational principle the matter Lagrangian  $\mathcal{L}_M$  does not depend on the independent connection  $\hat{\Gamma}_{\mu\nu}^\lambda$ . Allowing the Lagrangian for this dependence leads to another formalism, namely the metric-affine variational principle [31, 33], which we will not consider in the current work.

### Field Equations

Variation of the action (1) with respect to the metric gives

$$F(\hat{R}, \phi)\hat{R}_{\mu\nu} - \frac{1}{2}f(\hat{R}, \phi)g_{\mu\nu} = T_{\mu\nu} + T_{\mu\nu}^{(\phi)}, \quad (5)$$

where  $T_{\mu\nu}$  is the usual matter energy-momentum tensor and the function  $F$  is defined by

$$F := \frac{\partial f(R, \phi)}{\partial R}. \quad (6)$$

Varying with respect to the independent connection  $\hat{\Gamma}_{\mu\nu}^\lambda$  gives instead the condition

$$\hat{\nabla}_\mu(\sqrt{-g}g^{\alpha\beta}F) = 0. \quad (7)$$

Equations for the scalar field are obtained varying action (1) with respect to  $\phi$ . However they are not of interest in what follows and we omit them here. In order to recover GR we can simply choose  $f(\hat{R}, \phi) = \hat{R}$  in (1). In this case from (7) we find that the connection coincides with the usual Levi-Civita  $\hat{\Gamma}_{\mu\nu}^\lambda \equiv \Gamma_{\mu\nu}^\lambda$  and (5) reduces to the standard Einstein field equation.

If we define a new metric  $h_{\mu\nu}$  conformally connected to  $g_{\mu\nu}$  by  $h_{\mu\nu} = F g_{\mu\nu}$ , (7) becomes

$$\hat{\nabla}_\mu(\sqrt{-h}h^{\alpha\beta}) = 0, \quad (8)$$

signifying that  $\hat{\Gamma}_{\mu\nu}^\lambda$  is the Levi-Civita connection with respect to the new metric  $h_{\mu\nu}$ . We then find

$$\begin{aligned} \hat{\Gamma}_{\mu\nu}^\lambda &= \frac{1}{2}h^{\lambda\sigma}(h_{\nu\sigma,\mu} + h_{\mu\sigma,\nu} + h_{\mu\nu,\sigma}) \\ &= \Gamma_{\mu\nu}^\lambda + \frac{1}{2F}[2\delta^\lambda_{(\mu}\partial_{\nu)}F + g_{\mu\nu}g^{\lambda\sigma}\partial_\sigma F]. \end{aligned} \quad (9)$$

It is useful to use (9) to replace the Palatini connection wherever it appears in our equations. The hatted curvature scalar and Ricci tensor then become [17]

$$\begin{aligned} \hat{R}_{\mu\nu} &= R_{\mu\nu} + \frac{3}{2F^2}(\nabla_\mu F)(\nabla_\nu F) \\ &\quad - \frac{1}{F}\nabla_\mu\nabla_\nu F - \frac{1}{2F}g_{\mu\nu}\nabla_\sigma\nabla^\sigma F, \end{aligned} \quad (10)$$

and

$$\hat{R} = R - \frac{3}{F}\nabla_\mu\nabla^\mu F + \frac{3}{2F^2}(\nabla_\mu F)(\nabla^\mu F). \quad (11)$$

We can also write the field equation (5) as the canonical Einstein field equation plus an effective energy-momentum tensor containing the dependence on  $f$  [23]

$$G_{\mu\nu} = T_{\mu\nu} + T_{\mu\nu}^{(\phi)} + T_{\mu\nu}^{(\text{eff})}, \quad (12)$$

with

$$\begin{aligned} T_{\mu\nu}^{(\text{eff})} &= (1 - F)R_{\mu\nu} - \frac{3}{2F}(\nabla_\mu F)(\nabla_\nu F) + \nabla_\mu\nabla_\nu F \\ &\quad + \frac{1}{2}g_{\mu\nu} \left[ (f - R) + \left(1 - \frac{3}{F}\right)\nabla_\sigma\nabla^\sigma F \right. \\ &\quad \left. + \frac{3}{2F}(\nabla_\sigma F)(\nabla^\sigma F) \right]. \end{aligned} \quad (13)$$

Since the Palatini connection does not appear explicitly in the final form of the field equation, it plays the role of an auxiliary field. The physically relevant connection, i.e. the one determining the free-falling motion, is still  $\Gamma_{\mu\nu}^\lambda$ . All the energy-momentum tensors in the right hand side of (12) are indeed covariantly conserved by  $\nabla$  [24].

### Background Cosmological Equations

Assuming the universe is described by the general Friedmann-Robertson-Walker (FRW) metric, we can find equations for the background cosmological evolution. This assumption is completely independent of the theory of gravity we choose since it relies only on treating the universe as homogeneous and isotropic.

The (unperturbed) general FRW metric reads

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \right], \quad (14)$$

where  $a(t)$  is the scale factor of the universe,  $K = 0, 1, -1$  is the spatial curvature and  $d\Omega^2 := d\theta^2 + \sin^2\theta d\varphi^2$ . Moreover we take the matter energy-momentum tensor to be of the form of a perfect fluid

$$T_{\mu\nu} = (\rho_M + p_M)u_\mu u_\nu + p_M g_{\mu\nu}, \quad (15)$$

with  $u^\mu$  denoting the four-velocity of an observer comoving with the fluid.  $\rho_M$  is the energy-density and  $p_M$  is the pressure of the matter fluid. If we further consider the scalar field  $\phi$  to be homogeneous in space, i.e. depending only on the physical time  $t$ , we can rewrite its energy-momentum tensor (4) in the form (15) defining

$$\rho_\phi = \frac{1}{2}\omega\dot{\phi}^2 + V(\phi) \quad p_\phi = \frac{1}{2}\omega\dot{\phi}^2 - V(\phi), \quad (16)$$

as its energy-density and pressure, respectively. Here a dot denotes differentiation with respect to the physical time  $t$ .

Having made these assumptions we can derive equations for the scale factor  $a(t)$  in the Palatini formalism. Using the FRW metric (14), the non vanishing components of (10) are [17]

$$\hat{R}_{00} = -3\frac{\ddot{a}}{a} + \frac{3\dot{F}^2}{2F^2} - \frac{3}{2F}\nabla_0\nabla_0 F, \quad (17)$$

$$\hat{R}_{ij} = [a\ddot{a} + 2\dot{a}^2 + 2K + \frac{a^2}{2F}\nabla_0\nabla_0 F]\delta_{ij} + \frac{\dot{F}}{F}\Gamma_{ij}^0. \quad (18)$$

Substituting these two equations back in (5), we find

$$6F \left( H^2 + \frac{\dot{F}^2}{4F^2} + \frac{H\dot{F}}{2F} + \frac{K}{a^2} \right) - f = \rho + 3p, \quad (19)$$

$$2F \left( \dot{H} + \frac{K}{a^2} \right) - \frac{3\dot{F}^2}{2F} - H\dot{F} + \ddot{F} = -\rho - p, \quad (20)$$

where  $\rho = \rho_\phi + \rho_M$  and  $p = p_\phi + p_M$ . Equation (19) can be considered as the modified Friedmann equation, whilst (20) is the modified acceleration equation (or second Friedmann equation). The first one governs the evolution of the scale factor in the cosmology of generalised Palatini gravity theories [17]; the second one will turn out useful later on.

### III. COSMOLOGICAL PERTURBATIONS

In this section we develop the general formalism of cosmological perturbations, presenting equations for scalar and tensor perturbations in Palatini generalised gravity (see [8, 32] for the metric case).

#### Preliminaries

The general perturbed FRW metric is

$$ds^2 = -(1 + 2\alpha)dt^2 - 2a(t)(\partial_i\beta - b_i) dt dx^i + a^2(t)(\delta_{ij} + 2\psi\delta_{ij} + 2\partial_i\partial_j\gamma + 2\partial_j c_i + h_{ij}) dx^i dx^j, \quad (21)$$

where  $\alpha, \beta, \psi, \gamma$  are scalar perturbations,  $b_i, c_i$  are vector perturbations and  $h_{ij}$  is the tensor perturbation. As usual, we have decomposed the independent degrees of freedom of the metric perturbations according to their transformation properties under spatial rotations. Because of the assumed background homogeneity and isotropy, all the physical quantities have to be invariant under purely spatial gauge transformations. Consequently, we want to derive gauge invariant equations for the perturbations. In general there are several gauge invariants we can build, but among them, we will need only four quantities. The two gauge invariants

$$\chi := a(\beta + a\dot{\gamma}), \quad A := 3(H\alpha - \dot{\psi}) - \frac{\Delta}{a^2}\chi, \quad (22)$$

where  $\Delta$  denotes the three-space Laplacian, will be useful to simplify the equations later; while the two invariants

$$\mathcal{R}_{\delta\phi} := \psi - \frac{H}{\dot{\phi}}\delta\phi, \quad \mathcal{R}_{\delta F} := \psi - \frac{H}{\dot{F}}\delta F, \quad (23)$$

will play a role in the discussion of the gauge choice. Here  $\delta\phi$  and  $\delta F$  are the perturbations of  $\phi$  and  $F$ , respectively.

Before writing down equations for the cosmological perturbations we discuss the choice of a specific gauge. To date, several choices have been considered in the literature. The most frequent choices are the Synchronous gauge  $\alpha = 0$ , the Comoving gauge  $v = 0$  (where  $v$  is the matter fluid velocity perturbation) and the Newtonian gauge  $\chi = 0$ . A more convenient choice for this work is to choose either the *Uniform-F gauge*  $\delta F = 0$  or the *Uniform-field gauge*  $\delta\phi = 0$ .

The first gauge is the more useful to work with when we are restricted to  $f(R)$  theories of gravity. In this class of theories all the contributions of the scalar field  $\phi$  (and of its perturbation  $\delta\phi$ ) are absent. From the definitions (23) we see immediately that setting  $\delta F = \delta\phi = 0$  gives  $\mathcal{R}_{\delta F} = \mathcal{R}_{\delta\phi} = \psi$ . This means that  $\psi$  becomes then a gauge invariant and we can try to find an equation governing its evolution.

The Uniform-field gauge is instead used in Scalar-Tensor theories of gravity. In this class of theories  $F$  depends only on  $\phi$ , i.e. we have  $f(R, \phi) \equiv F(\phi)R$ , and we get  $\delta F = F_{,\phi}\delta\phi$ . Choosing  $\delta\phi = 0$  immediately gives  $\delta F = 0$ . These are the same conditions we had in the  $f(R)$  theories with the Uniform-F gauge. All the results obtained setting  $\delta F = \delta\phi = 0$  will then be valid for both the classes of theories.

Unfortunately this analysis does not hold for more general theories, where both a scalar field and a non-linear coupling of  $R$  appear. Setting  $\delta\phi = 0$  does not imply  $\delta F = 0$  since in that case we have  $\delta F = F_{,\phi}\delta\phi + F_{,R}\delta R$ . Another choice of gauge is probably more appropriate in the general case but we will restrict ourselves to the pure  $f(R)$  or  $F(\phi)$  case in this work.

#### Perturbation Equations

Since we deal with perturbations during the inflationary phase of the universe, we do not consider any kind of matter (a part from the scalar field  $\phi$ ). From now on we then set  $\mathcal{L}_M = \rho_M = p_M = 0$ . Cosmological perturbation equations in the Palatini context have been derived in [23] (see also [27] in which some typos are corrected). In (spatial) Fourier space, the equations governing the *scalar* perturbations read

$$\begin{aligned}
\mathcal{G}A + \frac{3K - k^2}{a^2}\psi + \frac{1}{2F} \left[ \frac{3\dot{F}^2}{2F} + 3H\dot{F} - \omega\dot{\phi}^2 \right] \alpha \\
= \frac{1}{2F} \left\{ -\omega\dot{\phi}\delta\dot{\phi} - \frac{1}{2} \left[ \omega_{,\phi}\dot{\phi}^2 + (2V - f)_{,\phi} \right] \delta\phi \right. \\
\left. + \left[ 3H^2 - \frac{3\dot{F}^2}{4F^2} - \frac{R}{2} + \frac{3K^2 + k^2}{a^2} \right] \delta F + 3\mathcal{G}\delta\dot{F} \right\}, \tag{24}
\end{aligned}$$

$$\mathcal{G}\alpha - \dot{\psi} + \frac{K}{a^2}\chi = \frac{1}{2F} \left[ \omega\dot{\phi}\delta\phi - \left( H + \frac{3\dot{F}}{2F} \right) \delta F + \delta\dot{F} \right], \tag{25}$$

$$\dot{\chi} + \left( H + \frac{\dot{F}}{F} \right) \chi - \alpha - \psi = \frac{1}{F} \delta F, \tag{26}$$

$$\begin{aligned}
\dot{A} + (H + \mathcal{G})A + \frac{3\dot{F}}{2F}\dot{\alpha} + \left[ 3\dot{H} + \frac{3H\dot{F}}{2F} + \frac{3\ddot{F}}{F} - \frac{3\dot{F}^2}{F^2} + \frac{2}{F}\omega\dot{\phi}^2 - \frac{k^2}{a^2} \right] \alpha \\
= \frac{1}{2F} \left\{ 4\omega\dot{\phi}\delta\dot{\phi} + \left[ 2\omega_{,\phi}\dot{\phi}^2 + (f - 2V)_{,\phi} \right] \delta\phi \right. \\
\left. + \left[ \frac{3\dot{F}^2}{F^2} + 6(H^2 + \dot{H}) - R + \frac{k^2}{a^2} \right] \delta F + \left( 3H - \frac{6\dot{F}}{F} \right) \delta\dot{F} + 3\delta\ddot{F} \right\}. \tag{27}
\end{aligned}$$

Where we have defined  $\mathcal{G} = H + \dot{F}/2F$ . These equations are obtained by perturbing the  $G_0^0$ ,  $G_i^i$ ,  $G_j^j - \frac{1}{3}\delta_j^i G_k^k$  and  $G_k^k - G_0^0$  components of the field equations, respectively.

The equations for vector and tensor perturbations in Palatini formalism are the same of the corresponding ones in the metric formulation. This is due to the fact that both  $R$  and  $\phi$  do not have either vector or tensor components being pure scalars. The equation for *vector* perturbations is

$$\frac{k^2 - 2K}{2a^2} \Psi_i = 0, \tag{28}$$

where we have introduced the gauge invariant  $\Psi_i = b_i + a\dot{c}_i$ . Equation (28) is exactly the same equation we find in canonical GR (without matter). It tells us that vector perturbations will not be generated during inflation, making these modes uninteresting in generalised gravity too.

The gravitational wave equation, the equation for *tensor* perturbations, is given by

$$\ddot{h}_{ij} + \left( 3H + \frac{\dot{F}}{F} \right) \dot{h}_{ij} + \frac{k^2 + 2K}{a^2} h_{ij} = 0. \tag{29}$$

In this case the equation is modified from the corresponding equation in canonical GR. The only difference ap-

pears in an additional damping term modulated by the form of  $f$ .

#### IV. PERTURBATIONS FROM INFLATION

We now focus on the main subject of this paper. Cosmological perturbations generated during the inflationary stage have already been considered in the metric formulation for generalised gravity models. Power spectra amplitudes and indices were calculated for both scalar and tensor modes in [14, 32]. An analysis of the growth of matter density perturbations has been discussed in both metric and Palatini formalisms [27]. However, the calculation of scalar and tensor power spectra in the Palatini formalism has not been looked at in detail. In obtaining our results we follow the procedure carried out in [14, 32] and find some important differences in the observational consequences between the two formalisms.

##### Scalar Perturbations

Current observations suggest that the universe is spatially flat and we therefore restrict our calculation to the  $K = 0$  case. Furthermore imposing the gauge choice

discussed above ( $\delta\phi = \delta F = 0$ ) the right hand sides of equations (24)-(27) vanish and from (25) we then obtain

$$\alpha = \frac{\dot{\psi}}{\mathcal{G}}, \quad (30)$$

Inserting (30) into (24) gives

$$A = \frac{1}{S} \left[ \frac{k^2}{a^2} \psi - \frac{\dot{\psi}}{2FS} \left( \frac{3\dot{F}^2}{2F} + 3H\dot{F} - \omega\dot{\phi}^2 \right) \right]. \quad (31)$$

Putting (30) and (31) into (26) and using the background equation (20), we obtain a second-order differential equation for the gauge invariant perturbation  $\psi$

$$\ddot{\psi} + \left( 3H + \frac{\dot{Q}}{Q} \right) \dot{\psi} + \frac{k^2}{a^2} \psi = 0, \quad (32)$$

where we have defined

$$Q := \frac{\omega\dot{\phi}^2}{S^2}. \quad (33)$$

The background term defined above is to be compared with (7.38) of [32] which is its analogue for the metric formalism case. The crucial difference being that in our case the term is proportional to  $\dot{\phi}$  and vanishes in the absence of a scalar field.

The second-order differential equation can then be re-written in a useful form by defining suitable Mukhanov-Sasaki variables [6, 9],  $z_s := a\sqrt{Q}$  and  $u_s := z_s\psi$ . This allows us to write (32) as the equation for an oscillator with a time dependent frequency which depends only on background quantities

$$u_s'' + \left( k^2 - \frac{z_s''}{z_s} \right) u_s = 0, \quad (34)$$

where a prime denotes differentiation with respect to the conformal time  $\eta := \int \frac{dt}{a}$ . In analogy with the standard and metric cases [10, 14], we define the so-called slow-roll parameters

$$\epsilon_1 := -\frac{\dot{H}}{H^2} \quad \epsilon_2 := \frac{\ddot{\phi}}{H\dot{\phi}} \quad \epsilon_3 := \frac{\dot{F}}{2HF}, \quad (35)$$

where the additional  $\epsilon_3$  parametrises the shape of the  $f(R)$  function in analogy with the relation of the first two parameters with the shape of the inflaton potential. The background term can also be re-written in terms of the new parameters as

$$Q = \frac{\omega\dot{\phi}^2}{H^2(1+\epsilon_3)^2}. \quad (36)$$

If  $\dot{\epsilon}_1 = 0$  we can write the conformal time as [10]

$$\eta = -\frac{1}{aH(1-\epsilon_1)}. \quad (37)$$

If in addition  $\dot{\epsilon}_2 = \dot{\epsilon}_3 = 0$  the time dependent background contribution to the frequency term can be written as

$$\frac{z_s''}{z_s} = \frac{\nu_s^2 - 1/4}{\eta^2}, \quad (38)$$

with

$$\nu_s^2 := \frac{1}{4} + \frac{(1+\epsilon_1+\epsilon_2)(2+\epsilon_2)}{(1-\epsilon_1)^2}. \quad (39)$$

The solution of eq. (34) is then

$$u_s = \frac{1}{2} \sqrt{\pi|\eta|} e^{i\pi(1+2\nu_s)/4} H_{\nu_s}^{(1)}(k|\eta|), \quad (40)$$

which we have normalised by taking the early time, short wavelength ( $k\eta \rightarrow \infty$ ) limit of the general solution which reduces to the adiabatic vacuum case for a field in an expanding background  $u_s \rightarrow e^{-ik\eta}/\sqrt{2k}$ . The power spectrum of the curvature perturbation can then be defined as

$$\begin{aligned} \mathcal{P}_s &:= \frac{k^3}{2\pi^2} |\psi|^2 \\ &\simeq \frac{1}{Q} \left( (1-\epsilon_1) \frac{\Gamma(\nu_s)}{\Gamma(3/2)} \frac{H}{2\pi} \right)^2 \left( \frac{|k\eta|}{2} \right)^{3-2\nu_s}, \end{aligned} \quad (41)$$

where we have taken the super-horizon limit ( $k\eta \rightarrow 0$ ) of the solution.

Since the scalar perturbation is conserved after exiting the Hubble radius we can evaluate the power spectrum (41) at the time when  $k = aH$  and examine its spectral index  $n_s$  defined as

$$n_s - 1 := \left. \frac{d \ln \mathcal{P}_s}{d \ln k} \right|_{k=aH} = 3 - 2\nu_s. \quad (42)$$

During inflation we have  $\epsilon_i \ll 1$  ( $i = 1, 2, 3$ ), hence at first order in the  $\epsilon$ 's the scalar spectral index becomes

$$n_s - 1 \simeq -4\epsilon_1 - 2\epsilon_2. \quad (43)$$

This result is identical to the standard case arising from the Einstein-Hilbert action minimally coupled to an inflaton field in the metric formalism. It is however distinct from the generalised gravity case with metric formalism in that the scalar spectral index  $n_s$  has no explicit dependence, at linear order, on  $\epsilon_3$ , or rather, the choice of function  $f(R)$ .

## Tensor Perturbations

A gravitational wave can be decomposed into its two polarization states  $h_\times$  and  $h_+$  which both obey (29). In the following we will omit the subscripts and denote both states by  $h$  as they are decoupled and follow the same solutions. In defining their power spectra we will take into account the presence of two complex modes to which reality conditions apply. In analogy with the scalar case

(29) can also be redefined as an equation for an oscillator with time dependent frequency by introducing the Mukhanov-Sasaki variable  $u_t := z_t h$  with  $z_t := a\sqrt{F}$  such that

$$u_t'' + \left(k^2 - \frac{z_t''}{z_t}\right) u_t = 0. \quad (44)$$

Following the same route as the previous section we can define the background contribution as

$$\frac{z_t''}{z_t} = \frac{\nu_t^2 - 1/4}{\eta^2}, \quad (45)$$

with

$$\nu_t^2 := \frac{1}{4} + \frac{(1 + \epsilon_3)(2 - \epsilon_1 + \epsilon_3)}{(1 - \epsilon_1)^2}, \quad (46)$$

and obtain the solution for the mode  $u_t$

$$u_t = \frac{1}{2} \sqrt{\pi|\eta|} e^{i\pi(1+2\nu_t)/4} H_{\nu_s}^{(1)}(k|\eta|). \quad (47)$$

The spectrum of tensor perturbations (taking into account polarization states) is then defined as

$$\begin{aligned} \mathcal{P}_t &:= \frac{4k^3}{\pi^2} |h|^2 \\ &\simeq \frac{8}{F} \left( (1 - \epsilon_1) \frac{\Gamma(\nu_t)}{\Gamma(3/2)} \frac{H}{2\pi} \right)^2 \left( \frac{|k\eta|}{2} \right)^{3-2\nu_t}, \end{aligned} \quad (48)$$

where we have once again taken the super-horizon limit ( $k\eta \rightarrow 0$ ). Similarly, we can evaluate the tensor spectral index as

$$n_t := \left. \frac{d \ln \mathcal{P}_t}{d \ln k} \right|_{k=aH} = 3 - 2\nu_t, \quad (49)$$

which during inflation ( $\epsilon_i \ll 1$ ) reduces to

$$n_t \simeq -2(\epsilon_1 + \epsilon_3). \quad (50)$$

Finally, we can evaluate the tensor-to-scalar ratio  $r$ , defined as the ratio between the tensor and scalar spectra respectively. We find

$$r := \frac{\mathcal{P}_t}{\mathcal{P}_s} \simeq \frac{8Q}{F} \simeq 16(\epsilon_1 + \epsilon_3), \quad (51)$$

which also introduces the consistency relation between the tensor spectral index and tensor-to-scalar ratio  $r = -8n_t$ .

## V. PRIMORDIAL SPECTRA FROM INFLATION IN SPECIFIC CLASSES OF THEORIES

We now apply the results derived in the previous section to three classes of theories which have been given particular attention in the literature;  $f(R)$  theories, Scalar-Tensor theories and non-minimally coupled inflation.

### $f(R)$ Theories

We first look at  $f(R)$  theories where the acceleration of the inflationary expansion is driven solely by the  $f(R)$  modifications to the action as opposed to a potential dominated, minimally coupled scalar. In attempting to define such a model in the Palatini formalism an important constraint arises; whereas  $f(R)$  theories in the metric formalism introduce an extra dynamical, scalar degree of freedom, the same theory in the Palatini formalism introduces only a non-dynamical scalar mode with an algebraic constraint. This is most clearly seen by relating both cases to Brans-Dicke theories with an explicit scalar degree of freedom [31] and deriving the scalar equation of motion by varying the action obtained. In the Palatini case the second-order terms in the scalar cancel out exactly resulting in a non-dynamical scalar mode, although this is not obvious when first inspecting the action.

This result is evident in our treatment in that the second-order term in (32) is proportional to  $Q$  which is not defined in the absence of the scalar mode  $\phi$ . Indeed in this case the equation reduces to a first-order constraint

$$\left(3H + \frac{3\dot{F}}{2F}\right) \dot{\psi} + \frac{k^2}{a^2} \psi = 0. \quad (52)$$

This shows that in the Palatini formulation of  $f(R)$  theories any initial curvature perturbation decays during inflation and are not useful for seeding the growth of structure. Thus Palatini inflation driven solely by  $f(R)$  modifications can only be a viable model with the addition of an auxiliary slowly rolling scalar field. This requirement negates the theoretical motivation behind this kind of model and in addition our analysis does not account for this possibility due to the restriction we made for imposing our choice of gauge.

### Scalar-Tensor Theories

We are then constrained to work with Scalar-Tensor theories in obtaining a model where Palatini generalised gravity successfully drives inflation. However, we can also derive some general results for the observables in this case where, as previously noted,  $f(R, \phi) \rightarrow F(\phi)R$ . In this case (20) leads to a constraint between  $\epsilon_1$  and  $\epsilon_3$

$$(\epsilon_1 + \epsilon_3)(1 + \epsilon_3) = \epsilon_3^2 \frac{2\omega F}{F_{,\phi}^2}, \quad (53)$$

where no further  $\epsilon_3$  contribution appears for general choices of  $\omega(\phi)$  and  $F(\phi)$ . To first order in  $\epsilon_1$  and  $\epsilon_3$  this leads to the constraint

$$\epsilon_1 + \epsilon_3 \simeq 0. \quad (54)$$

This has an immediate impact on the observables since both the tensor spectral index  $n_t$  and scalar-to-tensor

ratio  $r$  are proportional to  $\epsilon_1 + \epsilon_3$  (see (50) and (51)). Thus although the Palatini formulation of scalar-tensor theories could drive inflation and produce nearly scalar invariant scalar perturbations, in agreement with observations, they will not produce any tensor modes. This is a general features of Palatini Scalar-Tensor theories, which consequently might have to be discarded as possible models for inflation if even a small amount of tensor modes are observed by future CMB experiments.

### Non-Minimally Coupled Inflation

As an example of Scalar-Tensor gravity we analyse the case of non-minimally coupled inflation [7] which has been recently used as an attempt to identify the inflaton field with the standard model Higgs boson [28]. In these models we have  $F(\phi) = 1 + \xi\phi^2$ ,  $\omega(\phi) = 1$  and  $V(\phi) = \lambda(\phi^2 - v^2)^2$ , where  $\xi$ ,  $\lambda$ ,  $v$  are parameters of the theory. Slow-Roll inflation is obtained in the region  $\xi\phi^2 \gg v^2$ , where the potential becomes  $V(\phi) \simeq \lambda\phi^4$ . The slow-roll parameters (35) at first order in the slow-roll conditions are

$$\epsilon_1 \simeq -4\xi \frac{\xi\phi^2 - 1}{\xi\phi^2 + 1}, \quad (55)$$

$$\epsilon_2 \simeq 4\xi \frac{(\xi\phi^2 + 3)(2\xi\phi^2 - 1)}{\xi\phi^2(\xi\phi^2 + 1)}, \quad (56)$$

$$\epsilon_3 \simeq 4\xi \frac{\xi\phi^2 - 1}{\xi\phi^2 + 1}. \quad (57)$$

According to the general result we obtained above for Scalar-Tensor theories (54), we immediately note that  $\epsilon_1$  is exactly the opposite of  $\epsilon_3$ . As expected then the tensor spectral index and tensor-to-scalar ratio vanish at first order. Using (43) the scalar spectral index can be written as

$$n_s - 1 \simeq \frac{8}{\phi^2} \frac{3 - 7\xi\phi^2}{1 + \xi\phi^2}. \quad (58)$$

In order to constrain  $\xi$  with this result we need a value for  $\phi$  to substitute in. This can be obtained from the minimum number of  $e$ -foldings  $N$  which occurred during inflation,

$$N = \int_{\phi_{\text{start}}}^{\phi_{\text{end}}} \frac{H}{\dot{\phi}} d\phi \simeq -\frac{1}{8\xi} \log(1 - \xi\phi_{\text{start}}^2), \quad (59)$$

where  $\phi_{\text{start}}$  and  $\phi_{\text{end}}$  are the values of the inflaton field at the beginning and end of inflation respectively and we have used the approximation  $\phi_{\text{start}} \gg \phi_{\text{end}}$  in the last step. Considering  $N \simeq 62$  and an observed scalar spectral index of  $n_s \simeq 0.96$ , we obtain the (positive) value for  $\xi$  of

$$\xi \simeq 2.9 \cdot 10^{-3}. \quad (60)$$

The value is small, which means small deviation from minimal coupling. This is in agreement with the metric

case result found in [35]. However, non-minimally coupled inflationary models in the regime  $\xi \gg 1$  are also considered viable [28] (see also [29] for the Palatini approach). The latter are indeed the models which try to unify the inflaton with the Higgs boson [36].

Finally, given the value for  $\xi$  and imposing the COBE normalization for the curvature perturbation power spectrum

$$\mathcal{P}_s \simeq \left( \frac{H^2}{2\pi\dot{\phi}} \right)^2 \Big|_{aH=k} \simeq \frac{(0.027)^4}{24\pi^2}, \quad (61)$$

we can also obtain a value of  $\lambda$ :

$$\lambda \simeq 2.3 \cdot 10^{-14}. \quad (62)$$

Again, this result seems to agree with the metric case [35], where small values of  $\lambda$  are also obtained.

## VI. CONCLUSIONS

We have analysed the generation of primordial scalar and tensor perturbations during inflation in the Palatini formulation of generalised gravity. A general result, derived in this work, is that the scalar spectral index does not depend on the choice of  $f(R)$  function explicitly and, to first order in slow-roll parameters, we recover the same expression for  $n_s$  as the standard case. This is distinct from the metric case where  $n_s$  depends explicitly on the form of  $f(R)$  at the same order. On the other hand, we find that the tensor mode spectral index  $n_t$  and the tensor-to-scalar ratio  $r$  depend on  $\epsilon_3$ , the generalised slow-roll parameter defined by the choice of  $f(R)$ . This result is the same as the metric case.

In restricting the analysis to specific classes of theories we also find that models of inflation driven purely by  $f(R)$  modifications are not viable scenarios in the Palatini framework for generating primordial curvature perturbations that seed structure formation. This is specific to the Palatini treatment and is in contrast to the metric formalism where the extra scalar degree of freedom, dynamical in this case, allows the evolution of perturbations.

The other interesting class of theories, namely Scalar-Tensor models, result in a specific prediction that tensor modes vanish at first order in the slow-roll parameters. This prediction will soon be tested by CMB experiments and can potentially falsify these models. As a particular example we analyse the case of non-minimally coupled inflation where we find that constraints on the scalar spectral index allow only small values of the coupling parameters of the theory. Thus for the case considered the theory is constrained to a regime which is not of interest in effort to identify the inflaton field with the Higgs boson.

It should be stressed that the classes of models analysed in this work are restricted by our choice of gauge to either pure  $f(R)$  or Scalar-Tensor cases. More general



choices of  $f(R, \phi)$  depending on both  $R$  and  $\phi$  have not been constrained here and may provide viable models for

early or late-time acceleration.

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