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Nonlinear Heat Equations With Multi-Singularities*

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Abstract: Using Kato class functions and the Green tight function, the existence of weak solutions are obtained for both initial-value problems and initial-boundary-value problems of nonlinear heat equations with muti-singularities.

Key word: singular potential; singular coefficients; Green tight; Kato class; 3G theorem

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0 Introduction

This paper considers the initial—value problems of the nonlinear heat equations with singular potentials and singular coefficients in the form

$$\begin{cases} u_t - \dfrac{1}{t^\sigma}\Delta u = V(x)f(u) + g(x), & (x,t) \in R^n \times (0,\infty), \\ u(x,t) = u_0(x), & x \in R^n, \end{cases} \tag{1}$$

and the initial-boundary-value problem

$$\begin{cases} u_t - \dfrac{1}{t^\sigma}\Delta u = V(x)f(u) + g(x), & x \in \Omega,\ 0 < t \leqslant T, \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(x,t)\,|_{\partial\Omega\times(0,\,T)} = 0, \end{cases} \tag{2}$$

where $n \geqslant 3$, $V(x)$, $g(x)$, $u_0(x)$ are given functions in Ω, and $\Omega \subset R^n$ is a smooth domain containing the origin.

In the case $\sigma = 0$, $f(u) = u^p$, $p > 1$. (1) reduces to the following problem:

$$\begin{cases} u_t - \Delta u = u^p, & (x,t) \in R^n \times (0,\infty), \\ u(x,0) = u_0(x), & x \in R^n. \end{cases}$$

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There have been a number of papers devoted to the study of this problem [1~6]. Recently, JIAN *et al*.[7~8] studied (1) in the case where $V(x)=1$, $\sigma>0$, $f(u)=|u|^{p-1}u$, $p>1$ and $u_0(x)=0$. They got the existence and non-existence of global solutions; ZHANG and ZHAO[9] studied (2) in the case where $\sigma=0$, $f(u)=|u|^{p-1}u$ and $V(x)$ possesses singularities and they got the existence of singuar solutions, and CHEN, ZENG[10] discussed the singular case when $f(u)=|u|^{p-1}u$.

In this paper we investigate the existence of weak solutions of problems (1) and (2) in the case where $\sigma>0$, $V(x)$ is a singular function, and the rightside possesses a general form. Our results intensively extend the results metioned above to more general and more difficult situations, i. e., the non-singulairity or mono-singularity case is extended to multi-singulairity case, and the special nonlinear term form of singularity is extended to the more general form with singularity.

1 Preliminares

Given an open set $\Omega\subseteq R^n$ and a function $f(x)$, we denote

$$K_\Omega(f)\equiv\sup_{x\in\Omega}\int_\Omega\frac{|f(y)|}{|x-y|^{n-2}}\,\mathrm{d}y.$$

Definition 1[11] A Borel measueable function f is said to belong to the Kato class K_n, if

$$\lim_{r\to0}\left[\sup_{x\in R^n}\int_{|x-y|\leqslant r}\frac{|f(y)|}{|x-y|^{n-2}}\,\mathrm{d}y\right]=0.$$

Definition 2[12] A Borel measueable function f is called a Green tight function in R^n, if $f\in K_n$ and

$$\lim_{M\to\infty}\left[\sup_{x\in R^n}\int_{|y|\geqslant M}\frac{|f(y)|}{|x-y|^{n-2}}\,\mathrm{d}y\right]=0.$$

Replacing R^n by Ω, we get the Green tight function f in Ω.

Remark From [11] we know that $K_n(\Omega)\supset L_{loc}{}^q(\Omega)$ $(q>n/2)$, and from [13] we have that if f is a Green tight function in R^n, then $K_{R^n}(f)<\infty$.

We denote by

$$G_0(x,y)=c|x-y|^{2-n},\ x\neq y,\ c=\int_0^\infty\frac{e^{-s^2}s^{n-3}}{(4\pi)^{n/3}}\mathrm{d}s$$

the fundmental solution of $H_0\equiv-\Delta$ in R^n, and by $G(x,y)$ the Green function of the operator $H_0\equiv-\Delta$ in Ω with the homogeneous-Dirichlet boundary condition.

In the following arguments we shall use the $3G$ theorem in a substantial way:

Lemma 1($3G$ Theorem) Let Ω be a domain with the Lipschtiz boundary. Then, there exists a constant $C=C(n)$, such that

$$\frac{G(x,y)G(y,z)}{G(x,z)}\leqslant C\left[\frac{1}{|x-y|^{n-2}}+\frac{1}{|y-z|^{n-2}}\right],\ \forall\,x,y,z\in\Omega. \tag{3}$$

If G is replaced by G_0 and Ω by R^n, then (3) still holds. For the proof we refer the reader to

[13].

From the 3G theorem, we get immediately

Corollary 1 Let f be a Green tight function in Ω or R^n. Then there exists a constant C, such that

$$\frac{1}{G(x,0)}\int_{\Omega} G(x,y)\,|f(y)|\,G(y,0)\ \mathrm{d}y \leqslant CK_{\Omega}(f) \tag{4}$$

or

$$\frac{1}{G_0(x,0)}\int_{R^n} G_0(x,y)\,|f(y)|\,G_0(y,0)\ \mathrm{d}y \leqslant CK_{R^n}(f). \tag{5}$$

Let $e^{-tH_0}(x,y)$ be the fundmental solution of the operator $\partial_t - H_0$, i. e. ,

$$e^{-tH_0}(x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}}.$$

Then the fundmental solution $G_0(x,y)$ of the operator H_0 and the fundmental solution of $\partial_t - H_0$ have the following relation:

Lemma 2

$$\int_0^t e^{-(t-s)H_0}(x,y)\mathrm{d}s \leqslant G_0(x,y),\ x\neq y \tag{6}$$

$$\int_0^\infty e^{-sH_0}(x,y)\mathrm{d}s = G_0(x,y),\ x\neq y \tag{7}$$

Proof (6) follows easily from the non-negativity of the intergrand. To prove (7) we make the change of variables $t = \dfrac{x-y}{2\sqrt{s}}$, and get

$$\int_0^\infty e^{-sH_0}(x,y)\mathrm{d}s = \int_0^\infty \frac{e^{-s^2} s^{n-3}}{(4\pi)^{n/3}} |x-y|^{2-n}\mathrm{d}s = c\ |x-y|^{2-n} \equiv G_0(x,y),\ \text{if } x\neq y.$$

This completes the proof.

Denote by $G(x,y)$ the Green function of the operator H_0 in Ω with homogeneous-Dirichlet boundary condition, and by $\Gamma(x,t;y,s)(t>s)$ the heat kernel of the operator $\partial_t - H_0$ with the homogeneous-Dirichlet boundary condition on $\Omega\times(0,\infty)$[14]. We then have

Lemma 3[10]

$$\int_0^t \Gamma(x,t;y,s)\mathrm{d}s \leqslant G(x,y), \tag{8}$$

$$\int_0^\infty \Gamma(x,t;y,0)\mathrm{d}t = G(x,y). \tag{9}$$

We denote $h(x) \equiv V(x)/|x|^{(n-2)(p-1)}$. Note that Ω is a ball and $V = V(|x|)$ is a radial function. The following lemma gives a sufficient condition of $h(x)$ being in K_n.

Lemma 4[9] (a) Let $V = V(|x|)$ be a radial function, for some $r_0 > 0$,

$$\int_0^{r_0} r^{n-1-p(n-2)} V(r)\mathrm{d}r < \infty. \tag{10}$$

Then the function $h(x) \equiv V(x)/|x|^{(n-2)(p-1)}$ defined in $B(0,r_0)$ belongs to the class K_n.

Especially, if $V(x)=\frac{C}{|x|^l}$, $l<2$, $1<p<\frac{n-l}{n-2}$, then (10) holds, i. e., $h(x)\equiv C/|x|^{(n-2)(p-1)+l}\in K_n$.

(b) Let $V=V(|x|)$ be a radial function and satisfy

$$\int_0^\infty r^{n-1-p(n-2)}V(r)\mathrm{d}r<\infty. \tag{11}$$

Then the function $h(x)\equiv V(x)/|x|^{(n-2)(p-1)}$ defined in R^n is a Green tight function. Especially, if $V(x)=\frac{C}{|x|^l(1+|x|^\beta)}$, $l<2$, $\beta>1$, $1<p<\frac{n-l}{n-2}$, then (11) holds, i. e., $h(x)\equiv\frac{C}{|x|^{(n-2)(p-1)+l}(1+|x|^\beta)}$ is a Green tight function in R^n.

Lemma 5[9] Let $h(x)\equiv V(x)/|x|^{(n-2)(p-1)}\in K_n$. *Assume that* $w(x,t)$ is a bounded function in $R^n\times[0,\infty)$. Then the funtion

$$u(x,t)=\int_0^t\int_{R^n\setminus\{0\}}e^{-(t-s)H_0}(x,y)V(y)[w(y,s)G(y,0)]^p\mathrm{d}y\mathrm{d}s$$

is continuous in $R^n\times[0,\infty)$.

Remark Lemma 5 still holds if we replace $e^{-(t-s)H_0}$ by $\Gamma(x,t;y,s)$.

2 Main Results

Suppose that

(H1) $u_0(x)\in C^2(R^n)$ is nonnegative and for some $M_1>0$, $M_2>0$, it satisfies

$$u_0(x)\leqslant M_1G_0(x,0),\ x\in R^n\setminus\{0\}, \tag{12}$$

$$|\Delta u_0(x)|\leqslant M_2G_0(x,0)h_1(x),\ x\in R^n\setminus\{0\}, \tag{13}$$

where $h_1(x)$ is a Green tight function in R^n;

(H2) $g(x)\in C^0(R^n)$ and there exists a constant $M_3>0$ such that

$$|g(x)|\leqslant M_3G_0(x,0)h_2(x),\ x\in R^n\setminus\{0\}, \tag{14}$$

where $h_2(x)$ is a Green tight function in R^n;

(H3) The function $h(x)\equiv\frac{V(x)}{|x|^{(n-2)(p-1)}}$ is a Green tight function in R^n;

(H4) $f(u)\in C^1(R^n)$ and there exist C_1, C_2 such that $|f(u)|\leqslant C_1|u|^p$ and $|f'(u)|\leqslant C_2|u|^{p-1}$.

We then have

Theorem 1 Assume that (H1)～(H4) hold. If $M_i>0(i=1,2,3)$ in (12)～(14) are small enough, then there exists a sufficiently small number $\alpha>0$ such that the Cauchy problem (1) has a weak solution $u(x,t)$ satisfying $|u(x,t)|\leqslant\alpha G_0(x,0)$, $x\in R^n\setminus\{0\}$, $0<t\leqslant T$, where T is an arbitray fixed number.

For the problem (2), we suppose that

(H1)′ $u_0\in C^2(\Omega)$ is nonnegative and there exist $m_1>0$, $m_2>0$ such that

$$u_0(x)\leqslant m_1G(x,0),\ x\in\Omega\setminus\{0\}, \tag{15}$$

$$|\Delta u_0(x)| \leqslant m_2 G(x,0) h_3(x),\ x \in \Omega \setminus \{0\}. \tag{16}$$

where $h_3(x) \in K_n$ in R^n.

(H2)′ $g(x) \in C^0(\Omega)$ and there exists $m_3 > 0$ such that

$$|g(x)| \leqslant m_3 G(x,0) h_4(x),\ x \in \Omega \setminus \{0\}, \tag{17}$$

where $h_4(x) \in K_n$ in R^n;

(H3)′ $h(x) = V(x)/|x|^{(n-2)(p-1)} \in K_n$ in R^n.

Theorem 2 Assume that (H1)′～(H3)′ and (H4) hold, If $m_i > 0$ ($i = 1,2,3$) in (15)～(17) are small enough, then there exists a sufficiently small number $\alpha > 0$ such that problem (2) has a weak solution $u(x,t)$ satisfying

$$|u(x,t)| \leqslant \alpha G(x,0),\ x \in \Omega \setminus \{0\},\ 0 < t \leqslant T.$$

Remark If we keep all other conditions in theroem 1 and theroem 2 and replace $h(x)$ by $\dfrac{C}{|x|^{(n-2)(p-1)+l}}$ and $\dfrac{C}{|x|^{(n-2)(p-1)+l}(1+|x|^{\beta})}$, $\left(1<l<2, \beta>1, 1<p<\dfrac{(n-l)}{(n-2)}\right)$ respectively, then the results of theroem 1 and theroem 2 still hold.

Proof of Theroem 1

Case Ⅰ $\sigma > 0$ and $\sigma \neq 1$.

Let $u(x,t)$ be a weak solution of the problem (1) and let $w_1(x,t) = u(x,t) - u_0(x)$. Then $w_1(x,t)$ satisfies

$$\left.\begin{aligned} & w_{1,t} - \frac{1}{t^\sigma}\Delta w_1 = V(x) f(w_1(x,t) + u_o(x)) + g(x) + \frac{1}{t^\sigma}\Delta u_0,\ (x,t) \in R^n \times (0,\infty) \\ & w_1(x,0) = 0,\ x \in R^n \end{aligned}\right\}. \tag{18}$$

We call $w_1(x,t)$ a weak solution of the problem of (18) if it satisfies the following integration equation

$$w_1(x,t) = \int_0^t T(\xi(s) - \xi(t)) \left[V(x) f(w_1(x,s) + u_o(x)) + g(x) + \frac{1}{s^\sigma}\Delta u_0(x)\right] ds, \tag{19}$$

where $$T(\tau) g(x) = \int_{R^n} e^{-\tau H_0}(x,y) g(y) dy, \quad \xi(t) = \frac{1}{(\sigma-1) t^{\sigma-1}}.$$

Set $$\tau = \xi(s) - \xi(t) = \frac{1}{\sigma-1}\left(\frac{1}{s^{\sigma-1}} - \frac{1}{t^{\sigma-1}}\right),\ 0 < \varepsilon \leqslant s \leqslant t.$$

Then, $s = s(\tau,t) = t[1 + (\sigma-1)t^{\sigma-1}\tau]^{\frac{1}{1-\sigma}}$, $ds = -t^\sigma[1 + (\sigma-1)t^{\sigma-1}\tau]^{\frac{\sigma}{1-\sigma}} d\tau = -s^\sigma d\tau$. so that

$$\begin{aligned} w_1(x,t) = & \lim_{\varepsilon \to 0^+} \int_\varepsilon^t T(\xi(s) - \xi(t)) \left[V(x) f(w_1(x,s) + u_o(x)) + g(x) + \frac{1}{s^\sigma}\Delta u_0(x)\right] d\tau = \\ & t^\sigma \lim_{\varepsilon \to 0^+} \int_0^{\tau(\varepsilon)} T(\tau) [V(x) f(w_1(x, (s(\tau,t)) + u_0(x)) + g(x)][1 + (\sigma-1)t^{\sigma-1}\tau]^{\frac{\sigma}{1-\sigma}} d\tau + \\ & \lim_{\varepsilon \to 0^+} \int_0^{\tau(\varepsilon)} T(\tau) \Delta u_0(x)\, d\tau, \end{aligned} \tag{20}$$

where $$\tau(\varepsilon) = \frac{1}{\sigma-1}\left(\frac{1}{\varepsilon^{\sigma-1}} - \frac{1}{t^{\sigma-1}}\right).$$

We now consider two different situations.

(a) Assume that $\sigma > 1$. Then $\lim_{\varepsilon\to 0^+}\tau(\varepsilon) = +\infty$ and (20) can be written as

$$w_1(x,t) = \int_0^\infty T(\tau)\Delta u_0(x)\, d\tau + t^\sigma \int_0^\infty T(\tau)[V(x)f(w_1(x,(s(\tau,t)) + u_0(x)) + g(x)] \cdot [1+(\sigma-1)t^{\sigma-1}\tau]^{\frac{\sigma}{1-\sigma}}\, d\tau. \tag{21}$$

Let $w_1(x,t) = w(x,t)G_0(x,0)$. Then by lemma 2,

$$\int_0^\infty T(\tau)\Delta u_0(x)\, d\tau = \int_0^\infty \int_{R^n} e^{-\tau H_0}(x,y)\Delta u_0(y)\,dy d\tau = \int_{R^n} G_0(x,y)\Delta u_0(y)\,dy$$

$$w(x,t) = \frac{1}{G_0(x,0)}\int_{R^n} G_0(x,y)\Delta u_0(y)dy + \frac{t^\sigma}{G_0(x,0)}\int_0^\infty T(\tau)g(x)\cdot[1+(\sigma-1)t^{\sigma-1}\tau]^{\frac{\sigma}{1-\sigma}}d\tau + \frac{t^\sigma}{G_0(x,0)}\int_0^\infty T(\tau)[V(x)f(w(x,s(\tau,t))G_0(x,0)+u_0(x))]\cdot[1+(\sigma-1)t^{\sigma-1}\tau]^{\frac{\sigma}{1-\sigma}}d\tau \equiv I_1(x) + I_2(x,t) + F_1w(x,t) \equiv Fw(x,t). \tag{22}$$

Set

$$S_{\alpha,T} = \{\, w \in C^0(R^n\backslash\{0\} \times [0,T]) \mid |w(x,t)| \leqslant \alpha\}. \tag{23}$$

Since $s = s(\tau,t) = t[1+(\sigma-1)t^{\sigma-1}\tau]^{\frac{1}{1-\sigma}}$, We have $0 \leqslant s(\tau,t) \leqslant t$, $\forall \tau > 0$. Thus, $w(x,s(\tau,t)) \in S_{\alpha,T}$ supposed $w(x,t) \in S_{\alpha,T}$. We show below that the mapping F has a fixed point in $S_{\alpha,T}$.

By (13) and (5), we have

$$|I_1(x)| \leqslant \frac{M_2}{G_0(x,0)}\int_{R^n} G_0(x,y)h_1(y)G_0(y,0)dy \leqslant CM_2K_{R^n}(h_1). \tag{24}$$

Since $t^\sigma[1+(\sigma-1)t^{(\sigma-1)}\tau]^{\frac{\sigma}{1-\sigma}} \leqslant t^\sigma$, for $\sigma > 1$, $\forall \tau > 0$. By lemma 2, (H2) and (5), we have,

$$|I_2(x,t)| \leqslant \frac{t^\sigma}{G_0(x,0)}\int_{R^n} G_0(x,y)|g(y)|dy \leqslant \frac{M_3t^\sigma}{G_0(x,0)}\int_{R^n} G_0(x,y)h_2(y)G_0(y,0)dy \leqslant CM_3K_{R^n}(h_2)t^\sigma. \tag{25}$$

From (5), (H1), (H3) and (H4), we obtain

$$\begin{aligned} |F_1w| &\leqslant \frac{C_1t^\sigma}{G_0(x,0)}\int_0^\infty\int_{R^n} e^{-\tau H_0}(x,y)|V(y)|w(y,s(\tau,t))G_0(y,0)+u_0(y)|^p)dyd\tau \leqslant \\ &\frac{C\,2^{p-1}t^\sigma}{G_0(x,0)}(\alpha^p + M_1{}^p)\int_{R^n} G_0(x,y)|V(y)|\; G_0^{p-1}(y,0)G_0(y,0)dy \leqslant \\ &\frac{C2^{p-1}t^\sigma}{G_0(x,0)}(\alpha^p + M_1{}^p)\int_{R^n} G_0(x,y)h(y)G_0(y,0)dyds \leqslant \\ &C2^{p-1}t^\sigma(\alpha^p + M_1{}^p)K_{R^n}(h). \end{aligned} \tag{26}$$

We get, from (24)~(26), that

$$|Fw| \leqslant CM_2K_{R^n}(h_1) + CM_3K_{R^n}(h_2)T^\sigma + C2^{p-1}(\alpha^p + M_1^p)K_{R^n}(h)\; T^\sigma.$$

For a fixed T, by virtue of $p > 1$, we might as well suppose $M_1 < \alpha$, it's easy to find α, $M_i(i=1,2,3)$ such that $|Fw| \leqslant \alpha$. To prove the continuity of $Fw(x,t)$, it is sufficient to prove the continuity of $F_1 w(x,t)$.

$$F_1 w(x,t) = \frac{t^\sigma}{G_0(x,0)} \int_0^\infty \int_{R^n \setminus \{0\}} e^{-\tau H_0}(x,y)[V(y) f(w(y,s(\tau,t))G_0(y,0) + u_0(y))] \cdot [1+(\sigma-1)t^{(\sigma-1)}\tau]^{\frac{\sigma}{1-\sigma}} \mathrm{d}\tau$$

Since $[1+(\sigma-1)t^{(\sigma-1)}\tau]^{\frac{\sigma}{1-\sigma}} \leqslant 1$ and

$$\left| \frac{f(w(y,s(\tau,t))G_0(y,0)+u_0(y))}{G_0(y,0)^p} \right| \leqslant \frac{C|w(y,s(\tau,t))G_0(y,0)+u_0(y))|^p}{G_0(y,0)^p} \leqslant C2^{(p-1)}(\alpha^p + M_1{}^p)$$

By lemma 5, we find that $F_1 w(x,t)$ is continuous in $R^n \setminus \{0\} \times [0,\infty)$. Thus $Fw(x,t)$ is continuous in $S_{\alpha,T}$.

Therefore, $F: S_{\alpha,T} \to S_{\alpha,T}$ is a continuous mapping. Now we verify that F is a contraction mapping.

For any $w_1, w_2 \in S_{\alpha,T}$, we have

$$\begin{aligned} &|Fw_1(x,t) - Fw_2(x,t)| = |F_1 w_1(x,t) - F_1 w_2(x,t)| = \\ &\frac{t^\sigma}{G_0(x,0)} \left| \int_0^\infty [1+(\sigma-1)t^{\sigma-1}\tau]^{\frac{\sigma}{1-\sigma}} \int_{R^n} e^{-\tau H_0}(x,y) V(y) f'(\theta w_1 G_0 + \right. \\ &\left. (1-\theta) w_2 G_0 + u_0)(w_1 - w_2) G_0 \,\mathrm{d}y\mathrm{d}\tau \right| \leqslant \\ &\frac{Ct^\sigma(\alpha+M_1)^{p-1}}{G_0(x,0)} \int_0^\infty G_0(x,y)|V(y)| G_0^p(y,0) \cdot \\ &|w_1(y,s(\tau,t)) - w_2(y,s(\tau,t))| \cdot [1+(\sigma-1)t^{\sigma-1}\tau]^{\frac{\sigma}{1-\sigma}} \mathrm{d}y\mathrm{d}\tau \leqslant \\ &CT^\sigma(\alpha+M_1)^{p-1} K_{R^n}(h) \mid w_1 - w_2 \mid. \end{aligned} \tag{27}$$

For a fixed T, since $p > 1$, we can choose postive numbers α, M_1 small enough so that

$$CT^\sigma(\alpha+M_1)^{p-1} K_{R^n}(h) < 1,$$

Therefore F is a contration mapping. By the Banach fixed point theroem, F has a unique fixed point $w(x,t)$ in $S_{\alpha,T}$ which is the continuous solution of (22), and then $w_1(x,t) = w(x,t)G_0(x,0)$ ($x \in R^n \setminus \{0\}$) is the solution of (21) or (19), i. e.

$$u(x,t) = w(x,t)G_0(x,0) + u_0(x)$$

is a weak solution of Cauchy problem (1).

(b) Assume that $0 < \sigma < 1$. Then $\lim_{\varepsilon \to 0^+} \tau(\varepsilon) = \frac{t^{(1-\sigma)}}{1-\sigma} < \infty$, and (20) reads

$$w_1(x,t) = \int_0^{\frac{t^{(1-\sigma)}}{1-\sigma}} T(\tau)\Delta u_0(x)\mathrm{d}\tau + t^\sigma \int_0^{\frac{t^{(1-\sigma)}}{1-\sigma}} T(\tau) \cdot [V(x) f(w_1(x,s(\tau,t)) + u_0(x)) + g(x)]\,[1+(\sigma-1)t^{(\sigma-1)}\tau]^{\frac{\sigma}{1-\sigma}} \mathrm{d}\tau. \tag{28}$$

Because of $s = s(\tau,t) = t[1+(\sigma-1)t^{(\sigma-1)}\tau]^{\frac{1}{1-\sigma}}$, we obtain

$$0 \leqslant s(\tau,t) \leqslant t, \ \forall\ 0 < \tau \leqslant \frac{t^{(1-\tau)}}{1-\sigma}.$$

Therefore, the arguments in (a) are still valid in this case and then the desired assertion holds.

Case Ⅱ $\sigma = 1$.

We call $w_1(x,t)$ a weak solution of (18) if it satisfies the following integration equation

$$w_1(x,t) = \int_0^t T\left(\ln\frac{t}{s}\right)\frac{\Delta u_0(x)}{s}\mathrm{d}s + \int_0^t T\left(\ln\frac{t}{s}\right)[V(x)f(w_1(x,s)+u_0(x))+g(x)]\mathrm{d}s, \quad (29)$$

where $T(\tau)g(x) = \int_{R^n} e^{-\tau H_0}(x,y)g(y)\mathrm{d}y$.

Let $\tau = \ln(t/s)$, then $s = te^{-\tau}$, $\mathrm{d}s = -te^{-\tau}\mathrm{d}\tau = -s\mathrm{d}\tau$. Therefore, (29) becomes

$$w_1(x,t) = \int_0^\infty \int_{R^n} e^{-\tau H_0}(x,y)\Delta u_0(y)dy\mathrm{d}\tau + t\int_0^\infty e^{-\tau}\int_{R^n} e^{-\tau H_0}(x,y) \cdot [V(y)\cdot f(w_1(y,s(\tau,t))+u_0(y))+g(y)]\mathrm{d}y\mathrm{d}\tau. \quad (30)$$

where $0 \leqslant s(\tau,t) = te^{-\tau} \leqslant t$, $\forall\ \tau > 0$. The above conditons are similar to those of (a), so we can make use of Banach fixed point theroem to obtain the existence of solution of the above integration eqution and hence we obtain a weak solution of problem (1). The proof of theroem 1 is complete.

Proof of Theroem 2

Let $u(x,t)$ be a weak solution of problem (2), and let $w_1(x,t) = u(x,t) - u_0(x)$. By virtue of the fact $u_0(x) \in C^2(R^n)$, we see that $w_1(x,t)$ satisfies the following problem

$$\begin{cases} w_{1,t} - \dfrac{1}{t^\sigma}\Delta w_1 = V(x)f(w_1(x,t)+u_0(x)) + g(x) + \dfrac{1}{t^\sigma}\Delta u_0, (x,t) \in \Omega\backslash\{0\}\times(0,T], \\ w_1|_{\partial\Omega\times(0,T]} = 0, \\ w_1(x,0) = 0, x \in \Omega. \end{cases} \quad (31)$$

We call $w_1(x,t)$ a weak solution of (31) if it satisfies the following integration equation

$$w_1(x,t) = \int_0^t T(\xi(s)-\xi(t))[V(x)f(w_1(x,t)+u_0(x)) + g(x) + \frac{1}{s^\sigma}\Delta u_0(x)]\mathrm{d}s, \quad (32)$$

where $T(\tau)g(x) = \int_\Omega \Gamma(x,\tau;y,0)g(y)dy$, $\Gamma(x,t;y,s)$ is the heat kernel of the operator $\partial_t - \Delta$ with the homogeneous-Dirichlet boundary condition in the region $\Omega\times(0,\infty)$, and

$$\xi(t) = \begin{cases} \dfrac{1}{(\sigma-1)t^{\sigma-1}}, & \text{if } \sigma \neq 1; \\ -\ln t, & \text{if } \sigma = 1. \end{cases}$$

Then, replacing (H1)～(H4) by (H1)′－(H3)′ and (H4), using Lemma 3 instead of Lemma 2, and deriving by similar arguments as in section 2, we get all the assertions in this section. This completes the proof.

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具有多奇性的非线性热方程的定解问题

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摘要:利用 Kato 类函数和 Green 胎紧函数的性质得到了具有多奇性的非线性热方程的初值问题和初边值问题.弱解的存在性.

关键词:奇位势;奇系数;Green 胎紧;Kato 类;3G-定理