第36卷第3期 中国科学技术大学学报 Vol. 36, No. 3
2006年3月 JOURNAL OF UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA Mar. 2006

Article ID:0253-2778(2006)03-0241-03

Connectivity of strong product graphs*

SUN Li, XU Jun-ming

(Department of Mathematics, University of Science and Technology of China, Hefei 230026, China)

Abstract: The symbols κ_i, δ_i are used to denote the connectivity and the minimum degree of a graph G_i for $i = 1,2$. $\kappa(G_1 \boxtimes G_2) \geqslant \min\{\kappa_1(1+\delta_2), \kappa_2(1+\delta_1)\}$ is established if G_1 and G_2 are connected undirected graphs, where $G_1 \boxtimes G_2$ is the strong product of G_1 and G_2.

Key words: graph; connectivity; strong product graphs; minimum degree

CLC number: O157.5; TP302.1 **Document code**: A

AMS Subject Classification(2000): 05C40

强乘积图的连通度

孙 犁, 徐俊明

(中国科学技术大学数学系, 安徽合肥 230026)

摘要: 用 $\kappa_i > 0$ 和 δ_i 表示图 $G_i(i=1,2)$ 的连通度和最小度, 给出了无向图强乘积的连通度一个下界:

$$\kappa(G_1 \boxtimes G_2) \geqslant \min\{\kappa_1(1+\delta_2), \kappa_2(1+\delta_1)\}.$$

关键词: 图; 连通度; 强乘积图; 最小度

0 Introduction

The strong product of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \boxtimes G_2$ with vertex set $V_1 \times V_2$ and two distinct pairs (x_1, x_2) and (y_1, y_2) are adjacent in $G_1 \boxtimes G_2$ for each $i = 1,2$ either $x_i = y_i$ or $x_i y_i \in E_i$. Many properties of strong products have been studied in [1,2,3]. The strong product is also an important method to construct a big graph from two small graphs, and plays an important role in the design and analysis of networks[4].

The connectivity of graph G is the minimum number of vertices in a vertex-cut, and will be denoted by $\kappa(G)$. The complete graph K_n has no vertex-cuts, but it is conventional to define $\kappa(K_n)$ to be $n-1$.

For the connectivity of the strong products, up to now, no results have been reported. In this paper, we will give the following bounds of connectivity of strong products.

1 Theorem

Theorem 1.1 Let G_i be a connected undirected graph, κ_i, δ_i be the connectivity and the minimum degree of a graph G_i for each $i=1,2$. Then

$$\kappa(G_1 \boxtimes G_2) \geqslant \min\{\kappa_1(\delta_2+1), \kappa_2(\delta_1+1)\}.$$

* **Received**: 2004-12-09; **Revised**: 2005-10-17

Foundation item: Supported by NNSF of China (10271114).

Biography: SUN Li, male, born in 1978, Master. Research field: graphs and combinatorics. E-mail: lewis@ustc.edu.cn

Corresponding author: XU Jun-ming, Prof. E-mail: xujm@ustc.edu.cn

2 Proof of Theorem 1.1

For terminology and notation on graph theory not given here, the reader should refer to [5]. Let $G=(V,E)$ be a finite simple undirected graph with vertex set $V=V(G)$ and edge set $E=E(G)$. For a vertex x in G, $d_G(x)$ denotes the degree of x in G. The symbol $\delta(G)$ denotes the minimum degree of vertex of G and the symbol xG denotes $\{x\}\boxtimes G$. By the strong product of graphs, the following results are clearly true.

Lemma 2.1 Let G_i be a connected undirected graph for each $i=1,2$ and $G=G_1\boxtimes G_2$. Then

(ⅰ) $d_G(xy) = d_{G_1}(x)d_{G_2}(y) + d_{G_1}(x) + d_{G_2}(y)$ for any $x\in V(G_1)$ and $y\in V(G_2)$,

(ⅱ) $\delta(G) = \delta(G_1)\delta(G_2)+\delta(G_1)+\delta(G_2)$.

Lemma 2.2 Let $G_i=(V_i,E_i)$ be a connected undirected graph with $\delta(G_i)=\delta_i$ and $\kappa(G_i)=\kappa_i$ for each $i=1,2$ and $G=G_1\boxtimes G_2$. Then

(ⅰ) $\kappa(G;xa,xb)\geqslant\kappa_2(\delta_1+1)$ for any $x\in V(G_1)$ and $a,b\in V(G_2)$, $a\neq b$;

(ⅱ) $\kappa(G;xa,ya)\geqslant\kappa_1(\delta_2+1)$ for any $x,y\in V(G_1)$, $x\neq y$ and $a\in V(G_2)$.

Proof To prove the first conclusion, it is sufficient to prove that there is an (xa,xb)-path in $G-S$ for any $S\subseteq V(G)\setminus\{xa,xb\}$ with $|S|=\kappa_2(\delta_1+1)-1$.

Let us consider any vertex $x\in G_1$ and all the vertices in graph G_1 which are adjacent to vertex x. Assume that vertex set H is composed of all these vertices, $H=\{x\}\cup N_{G_1}(x)$. It is obvious that $|H|\geqslant\delta_1+1$. Note that for any $y\in H$, yG_2 is a subgraph of G and any two such subgraphs have no common vertex. Since $|H|\geqslant\delta_1+1$ and the vertex cut set $|S|=\kappa_2(\delta_1+1)-1$, there must be at least one $y\in H$ such that yG_2 contains at most κ_2-1 vertices in S. Since yG_2 is κ_2-connected, there is a (ya,yb)-path P in yG_2. Thus, if $y=x$ then P is an (xa,xb)-path in $G-S$; if $y\neq x$ then $(xa,ya)\cup P\cup(yb,xb)$ is an (xa,xb)-path in $G-S$.

In conclusion, $\kappa(G;xa,xb)\geqslant\kappa_2(\delta_1+1)$ for any $x\in V(G_1)$ and $a,b\in V(G_2)$, $a\neq b$.

Similarly, we can prove the second conclusion. □

Proof of Theorem 1.1 To prove the theorem, it is sufficient to prove that $G_1\boxtimes G_2-S$ is connected for any $S\subset V(G_1\boxtimes G_2)$ with $|S|=\min\{\kappa_1(\delta_2+1),\kappa_2(\delta_1+1)\}-1$. To this end, assume that $x=x_1x_2$ and $y=y_1y_2$ are two distinct vertices in $G_1\boxtimes G_2-S$, where $x_1,y_1\in V(G_1)$ and $x_2,y_2\in V(G_2)$. Note that each vertex has at least $(\delta_1+1)(\delta_2+1)-1$ neighbors in graph $G_1\boxtimes G_2$. For simplicity, arbitrarily taking δ_1 vertices of $N(x_1)$ in graph $G_1\boxtimes\{x_2\}$ and adding the vertex x_1x_2, we have vertex set $N^*_{G_1}(x_1)$; arbitrarily taking δ_2 vertices of $N(x_2)$ in graph $\{x_1\}\boxtimes G_2$ and adding the vertex x_1x_2, we have vertex set $N^*_{G_2}(x_2)$. It is obvious that $N^*_{G_1}(x_1)\cup N^*_{G_2}(x_2)\setminus\{x_1x_2\}$ consists of $(\delta_1+1)\cdot(\delta_2+1)-1$ neighbors of vertex x in $G_1\boxtimes G_2$. Similarly, we define $N^*_{G_1}(y_1)$, $N^*_{G_2}(y_2)$ and $N^*_{G_1}(y_1)\cup N^*_{G_2}(y_2)\setminus\{y_1y_2\}$ consists of $(\delta_1+1)(\delta_2+1)-1$ neighbors of vertex y in $G_1\boxtimes G_2$. It is obvious that

$$|N^*_{G_1}(x_1)|=|N^*_{G_1}(y_1)|=\delta_1+1,$$
$$|N^*_{G_2}(y_2)|=|N^*_{G_2}(y_2)|=\delta_2+1.$$

It is possible that there are some common vertices in vertex sets $N^*_{G_1}(x_1)$ and $N^*_{G_1}(y_1)$. It is also possible that there are some common vertices in vertex sets $N^*_{G_2}(x_2)$ and $N^*_{G_2}(y_2)$. In order to build some pairs of vertices with no common vertex, we partition the two sets of vertices with common vertices into three sets with no common vertex, as follows:

$$C_1=N^*_{G_1}(x_1)\cap N^*_{G_1}(y_1),$$
$$X_1=N^*_{G_1}(x_1)\setminus C_1,\ Y_1=N^*_{G_1}(y_1)\setminus C_1,$$
$$C_2=N^*_{G_2}(x_2)\cap N^*_{G_2}(y_2),$$
$$X_2=N^*_{G_2}(x_2)\setminus C_2,\ Y_2=N^*_{G_2}(y_2)\setminus C_2.$$

Now it is obvious that X_1,C_1 and Y_1 are pairwise disjoint. So are X_2,C_2 and Y_2. Since $|N^*_{G_1}(x_1)|=|N^*_{G_1}(y_1)|$ and $|N^*_{G_2}(x_2)|=|N^*_{G_2}(y_2)|$, we have $|X_1|=|Y_1|$ and $|X_2|=|Y_2|$. Assume $|X_1|=|Y_1|=m$ and $|X_2|=|Y_2|=n$, then $|C_1|=\delta_1-m$ and $|C_2|=\delta_2-n$. Let

$$C_1=\{c_{11},c_{12},\cdots,c_{1(\delta_1-m)}\},$$
$$C_2=\{c_{21},c_{22},\cdots,c_{2(\delta_2-n)}\},$$

$X_1 = \{x_{11}, x_{12}, \cdots, x_{1m}\}, X_2 = \{x_{21}, x_{22}, \cdots, x_{2n}\},$
$Y_1 = \{y_{11}, y_{12}, \cdots, y_{1m}\}, Y_2 = \{y_{21}, y_{22}, \cdots, y_{2n}\}.$

We know that $(X_1 \cup C_1) \cup (X_2 \cup C_2)$ are neighbors of vertex x (including vertex x itself) and $(Y_1 \cup C_1) \cup (Y_2 \cup C_2)$ are neighbor vertices of vertex y (including vertex y itself). Let

$S_1 = C_1 \cup C_2 = \{c_{1i}c_{2j} \mid i = 1,2,\cdots,\delta_1 - m; j = 1,2,\cdots,\delta_2 - n\},$
$S_2 = C_1 \cup X_2 = \{c_{1i}x_{2j} \mid i = 1,2,\cdots,\delta_1 - m; j = 1,2,\cdots,\delta_2 - n\},$
$S_3 = C_1 \cup Y_2 = \{c_{1i}y_{2j} \mid i = 1,2,\cdots,\delta_1 - m; j = 1,2,\cdots,\delta_2 - n\},$
$S_4 = X_1 \cup C_2 = \{x_{1i}c_{2j} \mid i = 1,2,\cdots,m; j = 1,2,\cdots,\delta_2 - n\},$
$S_5 = Y_1 \cup C_2 = \{y_{1i}c_{2j} \mid i = 1,2,\cdots,m; j = 1,2,\cdots,\delta_2 - n\},$
$S_6 = X_1 \cup X_2 = \{x_{1i}x_{2j} \mid i = 1,2,\cdots,m; j = 1,2,\cdots,n\},$
$S_7 = Y_1 \cup Y_2 = \{y_{1i}y_{2j} \mid i = 1,2,\cdots,m; j = 1,2,\cdots,n\},$
$S_8 = X_1 \cup Y_2 = \{x_{1i}y_{2j} \mid i = 1,2,\cdots,m; j = 1,2,\cdots,n\}.$

It is easy to see that $S_i (i = 1,2,\cdots,8)$ are pairwise disjoint and

$(X_1 \cup C_1)(X_2 \cup C_2) = S_1 \cup S_2 \cup S_4 \cup S_6,$
$(Y_1 \cup C_1)(Y_2 \cup C_2) = S_1 \cup S_3 \cup S_5 \cup S_7.$

In order to prove the theorem, we construct some pairs of vertices as follows, from which we can find a surviving path between vertices x and y.

It is clear that S_1 consists of $(\delta_1 - m)(\delta_2 - n)$ vertices, which are common neighbors of x and y, as follows:

$$S_1 = \bigcup_{i=1}^{\delta_1 - m} \bigcup_{j=1}^{\delta_2 - n} \{c_{1i}c_{2j}\};$$

$S_2 \cup S_3$ consists of $(\delta_1 - m)n$ pairs of vertices as follows:

$$S_2 \cup S_3 = \bigcup_{i=1}^{\delta_1 - m} \bigcup_{j=1}^{n} \{c_{1i}x_{2j}, c_{1i}y_{2j}\};$$

$S_4 \cup S_5$ consists of $m(\delta_2 - n)$ pairs of vertices as follows:

$$S_4 \cup S_5 = \bigcup_{i=1}^{m} \bigcup_{j=1}^{\delta_2 - n} \{x_{1i}c_{2j}, y_{1i}c_{2j}\};$$

$S_6 \cup S_7 \cup S_8$ consists of mn pairs of vertices as follows:

$$S_6 \cup S_7 \cup S_8 = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} \{x_{1i}x_{2j}, y_{1i}y_{2j}, x_{1i}y_{2j}\}.$$

It is easy to calculate that there are $(\delta_1 + 1) \cdot (\delta_2 + 1)$ pairs of vertices in the above. And it is obvious that all these pairs of vertices are pairwise disjoint since $S_i, i = 1, 2, \cdots, 8$ are pairwise disjoint.

Now we consider the surviving graph of $G_1 \boxtimes G_2$ of vertex cut S with cardinality $\min\{\kappa_1(\delta_2 + 1), \kappa_2(\delta_1 + 1)\} - 1$. Since $\kappa_1 \leqslant \delta_1$ and $\kappa_2 \leqslant \delta_2$, we have

$$\min\{\kappa_1(\delta_2 + 1), \kappa_2(\delta_1 + 1)\} - 1 < (\delta_1 + 1)(\delta_2 + 1)$$

It indicates that there is at least one pair of vertices in which all vertices are surviving after deleting $|S|$ vertices.

If there is a surviving pair of vertices in S_1, without loss of generality, assume that such a pair of vertices is $\{c_{1s}c_{2t}\}$, then vertex $c_{1s}c_{2t}$ is the vertex x or adjacent to the vertex x and $c_{1s}c_{2t}$ is the vertex y or adjacent to the vertex y, so vertices x and y are connected.

If there is a surviving pair of vertices in $S_2 \cup S_3$, without loss of generality, assume that such a pair of vertices is $\{c_{1s}x_{2t}, c_{1s}y_{2t}\}$. By Lemma 2.2,

$$\kappa(G_1 \boxtimes G_2; c_{1s}x_{2t}, c_{1s}y_{2t}) \geqslant \min\{\kappa_1(\delta_2 + 1), \kappa_2(\delta_1 + 1)\},$$

so the vertices $c_{1s}x_{2t}$ and $c_{1s}y_{2t}$ is connected in graph $G - S$. Since the vertex $c_{1s}x_{2t}$ is the vertex x or adjacent to the vertex x and the vertex $c_{1s}y_{2t}$ is the vertex y or adjacent to the vertex y, the vertices x and y are connected in $G - S$.

If there is a surviving pair of vertices in $S_4 \cup S_5$, without loss of generality, assume that such a pair of vertices is $\{x_{1s}c_{2t}, y_{1s}c_{2t}\}$. By Lemma 2.2,

$$\kappa(G_1 \boxtimes G_2; x_{1s}c_{2t}, y_{1s}c_{2t}) \geqslant \min\{\kappa_1(\delta_2 + 1), \kappa_2(\delta_1 + 1)\},$$

so the vertices $x_{1s}c_{2t}$ and $y_{1s}c_{2t}$ are connected in $G - S$. Since the vertex $x_{1s}c_{2t}$ is the vertex x or adjacent to the vertex x and the vertex $y_{1s}c_{2t}$ is the vertex y or adjacent to the vertex y, the vertices x and y are connected in $G - S$.

(下转第 248 页)