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On restricted connectivity of some Cartesian product graphs*

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Abstract: A subset $S \subset V(G)$ is called a restricted cut, if it does not contain a neighbor-set of any vertex as its subset and $G-S$ is disconnected. If there exists a restricted cut S in G, the restricted connectivity $\kappa^1(G) = \min\{|S| : S$ is a restricted cut of $G\}$. The Cartesian product graphs are considered and $\kappa^1(G) = 2\sum_{i=1}^{n} k_i - 2$ is obtained if for each $i = 1, 2, \cdots, n (n \geqslant 3)$, G_i is a k_i-regular k_i-connected graph of girth at least 5 and satisfies some given conditions, where $G = G_1 \times G_2 \times \cdots \times G_n$.

Key words: connectivity; restricted connectivity; regular; Cartesian product; hypercube

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一些笛卡尔乘积图的限制连通度

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摘要:子集 $S \subset V(G)$ 称为限制割,若任何点 $v \in V(G)$ 的邻点集 $N_G(v)$ 都不是 S 的子集且 $G-S$ 不连通. 若 G 中存在限制割,则定义限制连通度 $\kappa^1(G) = \min\{|S| : S$ 是 G 的一个限制割$\}$. 考虑了笛卡尔乘积图,证明了:设 $G = G_1 \times G_2 \times \cdots \times G_n$, 若 G_i 是满足某些给定条件的 k_i 连通 k_i 正则且围长至少为 5 的图,其中 $i = 1, 2, \cdots, n$, 则 $\kappa^1(G) = 2\sum_{i=1}^{n} k_i - 2$.

关键词:连通度;限制连通度;正则图;笛卡尔乘积;超立方体

0 Introduction

In this paper, we only consider a simple graph $G = (V, E)$. We refer the reader to [1] or [2] for basic graph-theoretical terminology and notation not defined here.

It is well-known that when the underlying topology of an interconnection network is modeled by a graph G, the classical connectivity $\kappa(G)$ of G, defined as the minimum cardinality $|S|$ of a vertex-

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cut S, has been used as a deterministic measure of reliability and fault-tolerance of the network. The concept of the connectivity, however, has an obvious deficiency. That is, in general, this concept imposes absolutely no restriction on the set S. To compensate for this shortcoming, Esfahanian and Hakimi[3] have proposed the concept of the restricted connectivity $\kappa^1(G)$ of G. A subset $S \subset V(G)$ is called a restricted cut, if S does not contain a neighbor-set of any vertex as its subset and $G-S$ is disconnected. If there exists a restricted cut S in G, then the restricted connectivity $\kappa^1(G) = \min\{|S| : S$ is a restricted cut of $G\}$.

For a given integer k, we say a graph G to have the property $\mathcal{P}_k$ if G satisfies the following two conditions:

(Ⅰ) There are $2k-2$ internally-disjoint paths between any two nonadjacent edges in G.

(Ⅱ) Between every pair of vertex x and edge (u,v) with $x \notin \{u,v\}$, there are $k-1$ internally-disjoint (x,u)-paths, say $H_i, i=1,2,\cdots,k-1$, and $k-1$ internally-disjoint (x,v)-paths, say H'_j, $j=2,3,\cdots,k$, such that H_1 is vertex-disjoint except x with all H'_j and H_k' is vertex-disjoint except x with all H_i, and $(N_G(v)-\{x,u\}) \cap (\bigcup_{i=1}^{k-1} V(H_i)) = \varnothing = (N_G(u)-\{x,v\}) \cap (\bigcup_{j=2}^{k} V(H'_j))$.

For example, K_2, C_d with $d \geqslant 4$, and the Petersen graph have the property $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$, respectively.

In this paper, we show the following theorem.

1 Theorem

Theorem 1.1 If for each $i=1,2,\cdots,n(n \geqslant 3)$, G_i is a k_i-regular k_i-connected graph of girth at least 5 and the property $\mathcal{P}_{k_i}$, then $\kappa^1(G) = 2\sum_{i=1}^{n} k_i - 2$, where $G = G_1 \times G_2 \times \cdots \times G_n$.

Corollary 1.2 $\kappa^1(Q_n) = 2n-2$ for $n \geqslant 3$.

Corollary 1.3 Let $G = C_{d_1} \times C_{d_2} \times \cdots \times C_{d_n}$. If $n \geqslant 3$ and for each $i=1,2,\cdots,n$, $d_i \geqslant 5$, then $\kappa^1(G) = 4n-2$.

Corollary 1.4 Let G_i be a copy of the Petersen graph for each $i=1,2,\cdots,n$. If $n \geqslant 3$, then $\kappa^1(G) = 6n-2$, where $G = G_1 \times G_2 \times \cdots \times G_n$.

2 Preliminary

The symbols K_n and C_n denote a complete graph and a cycle of order n, respectively. The girth $g(G)$ of G is the length of a shortest cycle in G. For $S \subset V(G)$ or $S \subset G$, let $N_G(S) = \{y \in V(G-S) : (x,y) \in E(G)$ for some $x \in S\}$, and replace $N_G(\{x\})$ and $N_G(\{x,y\})$ with $N_G(x)$ and $N_G(x,y)$, respectively. The degree $d_G(v)$ of a vertex v in G is the number of neighbors of v in G, i.e. $d_G(v) = |N_G(v)|$. For a vertex $v \in V$ and a vertex set $U \subset V-\{v\}$, a $v-U$ fan is a set of $|U|$ internally-disjoint paths from v to all vertices of U.

The Cartesian product graph G of n graphs G_1, G_2, $\cdots$, G_n, denoted by $G = G_1 \times G_2 \times \cdots \times G_n$, is the graph with the vertex-set $V(G) = V(G_1) \times V(G_2) \times \cdots \times V(G_n)$ specified by putting an edge, called a j^{th} dimensional edge, between $x = x_1 x_2 \cdots x_n$ and $y = y_1 y_2 \cdots y_n$, iff they differ exactly in the j^{th} coordinate and for this coordinate, $(x_j, y_j) \in E(G_j)$. For any $x_i \in V(G_i)$, let $G_i(x_i) = G_1 \times G_2 \times \cdots \times G_{i-1} \times \{x_i\} \times G_{i+1} \times G_{i+2} \times \cdots \times G_n$. An n-dimensional hypercube $Q_n = K_2 \times K_2 \times \cdots \times K_2$. From the definition of $G_1 \times G_2 \times \cdots \times G_n$, if $P = (x', v_1, v_2, \cdots, v_m, y')$ is an (x', y')-path in $G_2 \times G_3 \times \cdots \times G_n$, then for any $x_1 \in V(G_1)$, $x_1 P = (x_1 x', x_1 v_1, x_1 v_2, \cdots, x_1 v_m, x_1 y')$ is an $(x_1 x', x_1 y')$-path in $G_1 \times G_2 \times \cdots \times G_n$. Similarly, if $w = (x_1, u_1, u_2, \cdots, u_l, y_1)$ is an (x_1, y_1)-path in G_1, then for any $u' \in V(G_2 \times G_3 \times \cdots \times G_n)$, $wu' = (x_1 u', u_1 u', u_2 u', \cdots, u_l u', y_1 u')$ is an $(x_1 u', y_1 u')$-path in $G_1 \times G_2 \times \cdots \times G_n$.

If P is an (x,y)-path in G and $u, v \in V(P)$, then the (u,v)-section of P, denoted by $P(u,v)$, is a (u,v)-path in G. If for each $i=1,2,\cdots,n$, P_i is an (x_{i-1}, x_i)-path and $P = P_1 \cup P_2 \cup \cdots \cup P_n$ an (x_0, x_n)-path, then P can be expressed as

$$P = x_0 \xrightarrow{P_1} x_1 \xrightarrow{P_2} x_2 \xrightarrow{P_3} \cdots \xrightarrow{P_{n-1}} x_{n-1} \xrightarrow{P_n} x_n, \tag{1}$$

and P_i can be omitted in (1) if P_i is an edge, where $i=1,2,\cdots,n$. A path P between an edge (x,y) and a vertex z with $z\notin\{x,y\}$ can be expressed as

$$P=(x,y)\xrightarrow{P}z \text{ or } P=z\xrightarrow{P}(x,y), \quad (2)$$

and the symbol P above the arrow in (2) can be omitted if P is the unique path of length 1 between (x,y) and z.

Lastly, we list three fundamental facts used in this paper which are well-known and can be found in the literature.

Fact 1[2] $G_1\times G_2\times\cdots\times G_n$ is k-regular if each G_i is k_i-regular, and is k-connected if each G_i is k_i-connected, where $i=1,2,\cdots,n$ and $k=k_1+k_2+\cdots+k_n$.

Fact 2[4] Let G be a k-connected graph and x, $y\in V(G)$ with $x\neq y$, then there are k internally-disjoint (x,y)-paths in G.

Fact 3[4] Let G be a k-connected graph. Let U be any vertex set of $V(G)$ such that $|U|\leqslant k$ and let v be any vertex in $V(G)-U$. Then there is a $v-U$ fan.

3 Proof of Theorem 1.1

First, we show that $\kappa^1(G)\leqslant 2\sum_{i=1}^{n}k_i-2$. Let (a,b) be an arbitrary edge in G and $S=N_G(a,b)$. Then $|S|=2\sum_{i=1}^{n}k_i-2$, since G contains no triangles, and $G-S$ is disconnected since $|V(G)-S-\{a,b\}|\geqslant\prod_{i=1}^{n}(k_i+1)-2\sum_{i=1}^{n}k_i\geqslant 2$ for $n\geqslant 3$. It is easy to show that S is a restricted cut of G. So $\kappa^1(G)\leqslant 2\sum_{i=1}^{n}k_i-2$.

To complete the proof of Theorem 1.1, it suffices to show that between any two nonadjacent edges of G there are $2\sum_{i=1}^{n}k_i-2$ internally-disjoint paths. We prove this by induction on n ①.

The argument for $n=1$ is trivial, since for each $i=1,2,\cdots,n$, G_i has the property $\mathcal{P}_{k_i}$. Assume the induction hypothesis for $n-1$ with $n\geqslant 2$. Let $(x,y)=(x_1x_2\cdots x_n,y_1y_2\cdots y_n)$ and $(u,v)=(u_1u_2\cdots u_n,v_1v_2\cdots v_n)$ be two nonadjacent edges in G, then there is some $j\in\{1,2,\cdots,n\}$ such that $x_i=y_i$ for all $i\neq j$ but $(x_j,y_j)\in E(G_j)$. Without loss of generality, assume $x_1=y_1$, then $x,y\in V(G_1(x_1))$. Let $x=x_1x'$, $y=y_1y'$, $u=u_1u'$ and $v=v_1v'$. Let $N_{G_1}(x_1)=\{w_1^i: i=1,2,\cdots,k_1\}$, since G_1 is k_1-regular. Let $s=\sum_{i=2}^{n}k_i$ and $t=2s-2$.

We proceed to the induction step and construct $2\sum_{i=1}^{n}k_i-2$ internally-disjoint paths between (x,y) and (u,v) by considering several cases and subcases. To save space, we have to omit some discussions for some subcases according to the referee's suggestions.

Case 1 $x_1\in\{u_1,v_1\}$. Without loss of generality, assume $u_1=x_1$. There are two subcases, $v_1\neq x_1$ and $v_1=x_1$.

The case of $v_1\neq x_1$ is omitted. We only consider the case of $v_1=x_1$. Since $G_1(x_1)\cong G_2\times\cdots\times G_n$ is s-connected by Fact 1, by the induction hypothesis, between (x,y) and (u,v) in $G_1(x_1)$, there are t internally-disjoint paths, denoted by $P_i=x_1P'_i$, $i=1,2,\cdots,t$. Clearly, among P_i, $i=1,2,\cdots,t$, there are two paths which have different start and end vertices.

Without loss of generality, assume that P_1 and P_2 are an (x,u)- and a (y,v)-path, respectively. Let

$$P_{t+m}=x\to w_1^m x'\xrightarrow{w_1^m P'_1}w_1^m u'\to u,$$
$$P_{t+k_1+m}=y\to w_1^m y'\xrightarrow{w_1^m P'_2}w_1^m v'\to v,$$
$$m=1,2,\cdots,k_1.$$

Case 2 $x_1\notin\{u_1,v_1\}$. There are two subcases, $v_1=u_1$ and $v_1\neq u_1$.

If $v_1=u_1$, then there are k_1 internally-disjoint (x_1,u_1)-paths in G_1, denoted by T_i, $i=1,2,\cdots,k_1$. Without loss of generality, assume $\{w_1^i\}=V(T_i)\cap N_{G_1}(x_1)$ for each $i=1,2,\cdots,k_1$, since G_1 is k_1-regular, and T_m is of length at least 3 for each

① We can start the induction step from $n=1$, since the condition $n\geqslant 3$ is only used to prove that S is a restricted cut before.

$m=2,3,\cdots,k_1$, since $g(G_1)\geqslant 5$.

Assume $\{x',y'\}=\{u',v'\}$. Without loss of generality, assume $x'=u'$ and $y'=v'$. Obviously, $N_{G_1(x_1)}(x)\cap N_{G_1(x_1)}(y)=\emptyset$. So, let $N_{G_1(x_1)}(x,y)=\{x_1d'_{(i)}:i=1,2,\cdots,t\}$. Let

$$P_i=(x,y)\to x_1d'_{(i)}\xrightarrow{T_1d'_{(i)}}u_1d'_{(i)}\to(u,v),\quad i=1,2,\cdots,t;$$

$$P_{t+l}=x\xrightarrow{T_lx'}u,\ P_{t+k_1+l}=y\xrightarrow{T_ly'}v,l=1,2,\cdots,k_1.$$

Similarly, we can consider the cases of $\{x',y'\}\cap\{u',v'\}=\emptyset$ and $|\{x',y'\}\cap\{u',v'\}|=1$. The details are omitted.

If $v_1\neq u_1$, then $v'=u'$ and $v_1\in N_{G_1}(u_1)-\{x_1\}$.

Assume $u'\notin\{x',y'\}$. Then there are $s(s\geqslant 2)$ internally-disjoint paths, denoted by $x_1P'_i,i=1,2,\cdots,s$, between the edge (x,y) and the vertex x_1u' such that each $x_1P'_i$ contains only a vertex of $N_{G_1(x_1)}(x,y)$. Without loss of generality, assume $x_1P'_1$ and $x_1P'_2$ are an (x,x_1u')-and a (y,x_1u')-path, respectively, and for each $i=2,3,\cdots,s,x_1P'_i$ is of length at least two, since G is triangle free. Let $\{x_1q'_{(i)}\}=N_{G_1(x_1)}(x_1u')\cap V(x_1P'_i)$, where $i=2,3,\cdots,s$. Then $x_1q'_{(i)}\notin\{x,y\}$ for each $i=2,3,\cdots,s$. Since G_1 has the property $\mathscr{P}_{k_1}$, between the vertex x_1 and the edge (u_1,v_1), there are k_1-1 internally-disjoint (x_1,u_1)-paths, say $H_i,i=1,2,\cdots,k_1-1$, and k_1-1 internally-disjoint (x_1,v_1)-paths, say $H'_j,j=2,3,\cdots,k_1$, such that H_1 is vertex-disjoint except x_1 with all H'_j and H'_{k_1} is vertex-disjoint except x_1 with all H_i, and $(N_{G_1}(v_1)-\{x_1,u_1\})\cap(\bigcup_{i=1}^{k_1-1}V(H_i))=\emptyset=(N_{G_1}(u_1)-\{x_1,v_1\})\cap(\bigcup_{i=2}^{k_1}V(H'_i))$. Obviously, either H_1 or H'_{k_1} is of length at least 2, since G_1 is triangle free. Without loss of generality, assume H'_{k_1} is of length at least 2, and $\{w_1^{k_1}\}=V(H'_{k_1})\cap N_{G_1}(x_1)$, since G_1 is k_1-regular. Clearly, H_m and H'_m are of length at least 2 for each $m=2,3,\cdots,k_1-1$. Let $\{h_1^m\}=N_{G_1}(u_1)\cap V(H_m)$ and $\{g_1^m\}=N_{G_1}(v_1)\cap V(H'_m)$, where $m=2,3,\cdots,k_1-1$, since G_1 is k_1-regular. By the property $\mathscr{P}_{k_1}$ of $G_1,h_1^m\notin\bigcup_{i=2}^{k_1}V(H'_i)$ and $g_1^m\notin\bigcup_{i=1}^{k_1-1}V(H_i)$ for each $m=2,3,\cdots,k_1-1$. Let $\{x_1q'_{(i)}:i=s+4,s+5,\cdots,2s+1\}=N_{G_1(x_1)}(x,y)-\bigcup_{i=1}^{s}(N_{G_1(x_1)}(x,y)\cap V(x_1P'_i))$. Let $D=\{v_1x',v_1y'\}\cup\{v_1q'_{(i)}:i=s+4,s+5,\cdots,2s+1\}$, then $|D|=s$. By Fact 3, there is a v-D fan in $G_1(v_1)$. Let $v_1P'_i,i=s+2,s+3,\cdots,2s+1$, be all paths in the fan. Without loss of generality, assume $v_1P'_{s+2}$, $v_1P'_{s+3}$ and $v_1P'_i$, $i=s+4,s+5,\cdots,2s+1$, are a (v_1x',v)-, a (v_1y',v)- and a $(v_1q'_{(i)},v)$-path, respectively. Let

$$P_1=x\xrightarrow{H_1x'}u_1x'\xrightarrow{u_1P'_1}u;\ P_2=y\xrightarrow{H_1y'}u_1y'\xrightarrow{u_1P'_2}u;$$

$$P_i=(x,y)\xrightarrow{x_1(P'_i(x'\text{or }y',q'_{(i)}))}x_1q'_{(i)}\xrightarrow{H_1q'_{(i)}}u_1q'_{(i)}\to u,\ i=3,4,\cdots,s;$$

$$P_{s+1}=x\xrightarrow{x_1P'_1}x_1u'\xrightarrow{H_1u'}u;$$

$$P_{s+2}=x\xrightarrow{H'_{k_1}x'}v_1x'\xrightarrow{v_1P'_{s+2}}v;$$

$$P_{s+3}=y\xrightarrow{H'_{k_1}y'}v_1y'\xrightarrow{v_1P'_{s+3}}v;$$

$$P_l=(x,y)\to x_1q'_{(l)}\xrightarrow{(H'_{k_1}(x_1,v_1))q'_{(l)}}v_1q'_{(l)}\xrightarrow{v_1P'_l}v,\quad l=s+4,s+5,\cdots,2s+1=t+3;$$

$$P_{t+4}=y\xrightarrow{x_1(P'_2(y',q'_{(2)}))}x_1q'_{(2)}\to w_1^{k_1}q'_{(2)}\to w_1^{k_1}v'\xrightarrow{(H'_{k_1}(w_1^{k_1},v_1))v'}v;$$

$$P_{t+3+m}=x\xrightarrow{(H_m(x_1,h_1^m))x'}h_1^mx'\xrightarrow{h_1^mP'_1}h_1^mu'\to u,$$

$$P_{t+k_1+1+m}=y\xrightarrow{(H'_m(x_1,g_1^m))y'}g_1^my'\xrightarrow{g_1^mP'_2}g_1^mu'\to v,m=2,3,\cdots,k_1-1.$$

Assume $u'\in\{x',y'\}$. Without loss of generality, assume $u'=x'$. Then $u'=v'=x'$. So $u,v,x\in V(G_2(x_2))$. Then the proof is similar to that in Case 1.

It is just a routine to check that $P_j,j=1,2,\cdots,2\sum_{i=1}^{n}k_i-2$ in all cases are as required.

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