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Galerkin boundary element method for solving the boundary integral equation with hypersingularity

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Abstract: A Galerkin boundary elements method is applied to solve the integral equation with hypersingularity, which can be deduced from the double layer solution for the Neumann problem of Laplace equation. The scheme of integration by parts in the sense of distributions is performed to reduce the hypersingularity integral into a weak one, which shifts the partial derivatives of hypersingular kernel to the unknown function in the variational formulation. Thus, the boundary rotation of an unknown function is used to substitute for the original unknown function in the variational equation. When linear boundary elements are used in two-dimensional cases, the boundary rotation can be discretized into a constant vector on each element, so that the integrations can be performed in a simple way. The numerical tests illustrate the effectiveness and practicality of the scheme presented.

Key words: Galerkin boundary element method; double layer potential; hyper singular integral; Laplace equation; Neumann problem

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带强奇异边界积分方程的迦辽金边界元解法

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摘要: 采用双层位势来表示二维 Laplace 方程 Neumann 问题的解, 导致求解含超强奇异性的边界积分方程, 将其转换为边界上的 Galerkin 变分方程求解. 针对超强奇异积分的计算, 运用分步积分, 详细地推导了基于边界旋度的变分公式及边界旋度的表达式, 最终把超强奇异的积分计算转化为弱奇异积分的数值计算. 当采用线性边界单元来离散 Galerkin 变分公式时, 在每个离散的单元上边界旋度成为常向量, 因此, 数值积分变得很简单. 数值算例验证了方法的有效性和实用性.

关键词: Galerkin 边界元法; 双层位势; 超强奇异积分; Laplace 方程; Neumann 问题

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0 Introduction

In boundary element methods, we often encounter the calculation of hypersingular integrations. Such as the following Neumann problem of Laplace equation,

$$\left.\begin{aligned}&\Delta u(x)=0, x\in\Omega;\\&\frac{\partial u(x)}{\partial \boldsymbol{n}(x)}\Big|_{\Gamma}=g(x), x\in\Gamma;\end{aligned}\right\}\tag{1a}$$

or

$$\left.\begin{aligned}&\Delta u(x)=0, x\in\Omega';\\&\frac{\partial u(x)}{\partial \boldsymbol{n}(x)}\Big|_{\Gamma}=g(x), x\in\Gamma.\end{aligned}\right\}\tag{1b}$$

Here Ω is an open bounded domain with boundary Γ in $\mathbb{R}^2$, and Ω' is the infinite complementary of $\bar{\Omega}=\Omega+\Gamma$. When a double layer solution is used (where c is a constant, $\varphi=[u]=u^+-u^-$ is an intermediary unknown function on the boundary to be determined),

$$u(x)=\frac{1}{2\pi}\int_{\Gamma}\varphi(y)\,\frac{\partial}{\partial \boldsymbol{n}_y}\Big(\ln\frac{1}{|x-y|}\Big)\mathrm{d}s_y+c,\tag{2}$$

from which the corresponding boundary integral equation has a hypersingular kernel.

$$\frac{1}{2\pi}\int_{\Gamma}\varphi(y)\,\frac{\partial^2}{\partial \boldsymbol{n}_x\partial \boldsymbol{n}_y}\Big(\ln\frac{1}{|x-y|}\Big)\mathrm{d}s_y=g(x).\tag{3}$$

Actually, this expression of the normal derivative of such a potential is not an integral. It is a finite part expression. Therefore, we use a Galerkin variational formulation. By Green's formula, we have[1,12] $\forall v\in(H^1(\Omega)/p_0)\times w_0^1(\Omega')$ and

$$w_0^1(\Omega')=\left\{\frac{v}{r\ln r}\in L^2(\Omega'), Dv\in L^2(\Omega')\right\}.$$

A variational formulation of problem (1) is given by

$$\int_{\Omega\cup\Omega'}\mathrm{grad}\,u\cdot\mathrm{grad}\,v\,\mathrm{d}x=\langle g,\mu\rangle_{H_0^{-\frac{1}{2}}(\Gamma)\times(H^{\frac{1}{2}}(\Gamma)/p_0)},\tag{4}$$

where $\mu=v^+-v^-\in H^{\frac{1}{2}}(\Gamma)/p_0$, and p_0 is a constant.

Further, one can obtain an explicit expression of the bilinear form in the left hand side of Eq. (4), which needs to do double integration on Γ,

$$b(\varphi(x),\mu(y))=\frac{1}{4\pi}\int_{\Gamma}\int_{\Gamma}(\varphi(x)-\varphi(y))(\mu(x)-\mu(y))\cdot\frac{\partial^2}{\partial \boldsymbol{n}(x)\partial \boldsymbol{n}(y)}\Big(\ln\frac{1}{|x-y|}\Big)\mathrm{d}s_x\mathrm{d}s_y.\tag{5}$$

The following variational equation has a unique solution (allowing a difference of constant) in $H^{\frac{1}{2}}(\Gamma)/p_0$,

$$\left.\begin{aligned}&b(\varphi,\mu)=\int_H g(x)\mu(x)\mathrm{d}s_x,\\&\forall\,\mu\in(H^1(\Omega)/p_0)\times w_0^1(\Omega').\end{aligned}\right\}\tag{6}$$

The variational formulation (4) is the key starting point to the numerical solution, but since Eq. (5) is inconvenient for implementation, we shall use another expression for the bilinear form $b(\varphi,\mu)$ deduced directly from Eq. (4). According to the idea of integration by parts in the sense of distributions, one can shift the partial derivatives of hypersingular kernel to the unknown function in the variational formulation. Then, we try using the boundary rotation of unknown function to substitute for the original unknown function in the variational equation. Finally the calculation of coefficients of the corresponding matrix becomes very simple in spite of the complicated deduction.

The scheme of integration by parts in the sense of distributions is commonly used in order to reduce hyper singularity, but regularization can be accomplished in different ways[14]. Some approaches transform the normal derivative to the tangential derivative first, then perform the integration be parts[6,7,8,15]. Some other techniques were used also, such as using singularity subtraction technique to reduce the hyper-singular operator to a weakly singular one[18]; some calculate the Hadamard finite-part of a hypersingular integral[17]; some use the Maue-type identities to replace the hyper-singular operator with the weakly-singular operator[16]. When Green's function instead of fundamental solution is used, one can develop the hypersingular kernel in series and separate the singular parts[9~11], but the domain should be regularly shaped, otherwise we can

not find the explicit expression of corresponding Green's function. The approach to shifting the partial derivatives of hypersingular kernel to the boundary rotation (curl) of the unknown function in three-dimensional cases was proposed by Nedelec[2~4], Duong gave the numerical implementation later[5]. The approach presented in this paper used the same idea for 2D problems different from the approaches cited above.

1 Variational formulation based on boundary rotation

In $\mathbb{R}^2$, suppose Γ_δ is an open neighborhood of boundary Γ. For a proper δ, any point x in Γ_δ has a local projection $p(x)$ from Γ_δ onto Γ. Then for any smooth function $\varphi(x)$ defined on Γ, one can define a vector field orthogonal to the plan

$$\overrightarrow{\mathrm{rot}}_\Gamma\varphi(x)=\mathrm{grad}\,\tilde{\varphi}(x)\times\boldsymbol{n},\quad\forall x\in\Gamma,\tag{7}$$

where $\boldsymbol{n}(x)$ is the unitary exterior normal to the boundary Γ, and the function $\tilde{\varphi}(x)$ is explicited on Γ_δ by

$$\tilde{\phi}(x)=\varphi(p(x)),\ x\in\Gamma_\delta,\ p(x)\in\Gamma.\tag{8}$$

One can define a scalar function $\mathrm{rot}_\Gamma X(x)$ to a vector field X on Γ by

$$\mathrm{rot}_\Gamma X(x)=\boldsymbol{n}(x)\cdot\overrightarrow{\mathrm{rot}}\widetilde{X}(x).\tag{9}$$

Here, $\overrightarrow{\mathrm{rot}}$ is the ordinary rotation operator in $\mathbb{R}^2$, $\widetilde{X}(x)$ is a vector field defined in Γ_δ by

$$\widetilde{X}(x)=X(p(x)),\ x\in\Gamma_\delta,\ p(x)\in\Gamma.\tag{10}$$

According to the definitions above, one can obtain the Laplace-Beltrami operator on Γ,

$$\Delta_\Gamma\varphi(x)=\mathrm{rot}_\Gamma\overrightarrow{\mathrm{rot}}_\Gamma\varphi(x).\tag{11}$$

Then, for the double layer potential $u(x)$ (Eq. (2)), we have

$$\mathrm{grad}\,u(x)=-\frac{1}{2\pi}\int_\Gamma\overrightarrow{\mathrm{rot}}_\Gamma\varphi(x)\times\mathrm{grad}_y\left(\ln\frac{1}{|x-y|}\right)\mathrm{d}s_y,\ x\in\mathbb{R}^2-\Gamma.\tag{12}$$

After a long deduction, the bilinear form $b(\varphi,\mu)$ given by Eq. (4) has the expression[13],

$$b(\varphi,\mu)=\frac{1}{2\pi}\int_\Gamma\int_\Gamma\ln|x-y|\,\overrightarrow{\mathrm{rot}}_\Gamma\varphi(y)\cdot\overrightarrow{\mathrm{rot}}_\Gamma\mu(x)\mathrm{d}s_y\mathrm{d}s_x.\tag{13}$$

Then, we are going to solve the following variational equation with the left hand term expressed by Eq. (13),

$$b(\varphi,\mu)=\frac{1}{2\pi}\int_\Gamma\int_\Gamma\ln|x-y|\,\overrightarrow{\mathrm{rot}}_\Gamma\varphi(y)\cdot\overrightarrow{\mathrm{rot}}_\Gamma\mu(x)\mathrm{d}s_y\mathrm{d}s_x=\int_\Gamma g(x)\mu(x)\mathrm{d}s_x.\tag{14}$$

2 Implementation of the variational formulation

We now give the approximation process to solve Eq. (14). The boundary Γ is approximated by $\Gamma_h=\bigcup_{i=1}^{N}\Gamma_i$, where boundary elements $\Gamma_i=\overline{P_iP_{i+1}}$ are straight lines, P_i denote boundary nodes ($i=1,2,\cdots,N$). For the closed boundary $P_{N+1}=P_1$. We use linear elements, then, the approximation of a distribution φ (or μ), using linear basis functions, may be written as a linear combination

$$\varphi=\sum_{i=1}^{N}\varphi_i\Psi_i\quad(i=1,2,\cdots,N),\tag{15}$$

where the basis Ψ_i are the linear polynomials, and the coefficients φ_i are determined at the N nodes. When $i=N$, $\Gamma_{N+1}=\Gamma_1$,

$$\Psi_i=\begin{cases}\psi^{(i)}, & x\in\Gamma_i;\\ \psi^{(i+1)}, & x\in\Gamma_{i+1};\\ 0, & \text{other case.}\end{cases}\tag{16}$$

On each boundary element, we establish a coordinate system shown as Fig. 1: Suppose that the source point is Q, and the field point P situated on the boundary element $\Gamma_i=\overline{P_iP_{i+1}}$, $\boldsymbol{r}=\overrightarrow{PQ}$. The bases consist of two vectors ($\boldsymbol{n}$, $\boldsymbol{m}$); $\boldsymbol{n}$ is the unitary normal of Γ_i and $\boldsymbol{m}$ is the unitary vector

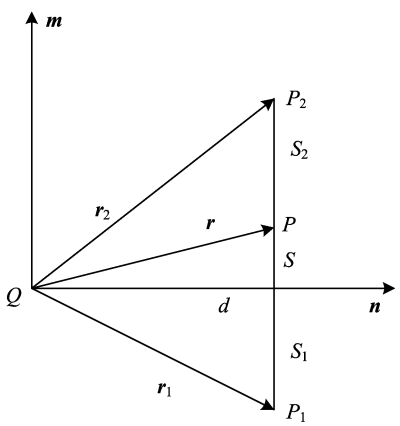

Fig. 1　The local coordinates

along the direction of Γ. The distances from P, P_1 and P_2 to the normal $\boldsymbol{n}$ are denoted by s, s_1 and s_2 respectively. d is the distance from Q to Γ_i. We choose s as the integral variable, then

$$\psi^{(i)} = \frac{s^{(i)} - s_1^{(i)}}{l_i},\ \psi^{(i+1)} = \frac{s_2^{(i+1)} - s^{(i+1)}}{l_{i+1}}. \quad (17)$$

The letter with up-script (i) denotes the local coordinate on the ith element, and l represents the length of the element.

Substitute Eq. (15) into Eq. (14), and let $\mu=\Psi_j (j=1,2,\cdots,N)$, we got the linear system

$$\sum_{i=1}^{N} a_{ij}\varphi_i = b_j \quad (j=1,2,\cdots,N), \quad (18)$$

where

$$b_j = \int_{\Gamma_j \cup \Gamma_{j+1}} g(x)\Psi_j \mathrm{d}s_x$$
$$(j=1,2,\cdots,N;\Gamma_{N+1}=\Gamma_1); \quad (19)$$
$$a_{ij} = \frac{1}{2\pi}\int_{\Gamma_j \cup \Gamma_{j+1}} \overrightarrow{\mathrm{rot}}_\Gamma \Psi_j(x) \cdot \int_{\Gamma_i \cup \Gamma_{i+1}} \ln|x-y| \overrightarrow{\mathrm{rot}}_\Gamma \Psi_i(y)\mathrm{d}s_y \mathrm{d}s_x$$
$$(i,j=1,2,\cdots,N). \quad (20)$$

Further more, the entry a_{ij} of the corresponding matrix can be rewritten as the combination of four integrals,

$$a_{ij} = \frac{1}{2\pi}\int_{\Gamma_j}\int_{\Gamma_i} \ln|x-y| \overrightarrow{\mathrm{rot}}_\Gamma \psi^{(i)}(y) \cdot \overrightarrow{\mathrm{rot}}_\Gamma \psi^{(j)}(x)\mathrm{d}s_y \mathrm{d}s_x + \frac{1}{2\pi}\int_{\Gamma_j}\int_{\Gamma_{i+1}} \ln|x-y| \overrightarrow{\mathrm{rot}}_\Gamma \psi^{(i+1)}(y) \cdot \overrightarrow{\mathrm{rot}}_\Gamma \psi^{(j)}(x)\mathrm{d}s_y \mathrm{d}s_x + \frac{1}{2\pi}\int_{\Gamma_{j+1}}\int_{\Gamma_i} \ln|x-y| \overrightarrow{\mathrm{rot}}_\Gamma \psi^{(i)}(y) \cdot \overrightarrow{\mathrm{rot}}_\Gamma \psi^{(j+1)}(x)\mathrm{d}s_y \mathrm{d}s_x + \frac{1}{2\pi}\int_{\Gamma_{j+1}}\int_{\Gamma_{i+1}} \ln|x-y| \overrightarrow{\mathrm{rot}}_\Gamma \psi^{(i+1)}(y) \cdot \overrightarrow{\mathrm{rot}}_\Gamma \psi^{(j+1)}(x)\mathrm{d}s_y \mathrm{d}s_x$$
$$(i=1,2,\cdots,N;j=1,2,\cdots,N). \quad (21)$$

3 Calculation of the boundary rotation

Suppose that the coordinates of P_1, P_2, P, Q are $P_1:(x_1,y_1), P_2:(x_2,y_2), P:(x,y), Q:(x',y')$, $l=|P_2-P_1|$, we can calculate

$$\boldsymbol{m} = \left(\frac{x_2-x_1}{l}, \frac{y_2-y_1}{l}\right), \quad (22)$$

$$\boldsymbol{n} = \left(\frac{y_2-y_1}{l}, \frac{x_2-x_1}{l}\right), \quad (23)$$

$\boldsymbol{r}_1 = (x_1-x', y_1-y')$, $\boldsymbol{r}_2 = (x_2-x', y_2-y')$. We have

$$s_1 = \boldsymbol{r}_1 \cdot \boldsymbol{m},\ s_2 = \boldsymbol{r}_2 \cdot \boldsymbol{m},\ s = \boldsymbol{r} \cdot \boldsymbol{m},$$
$$d = \boldsymbol{r}_1 \cdot \boldsymbol{n} = \boldsymbol{r}_2 \cdot \boldsymbol{n}.$$

Therefore,

$$\frac{s-s_1}{l} = \frac{(\boldsymbol{r}-\boldsymbol{r}_1)\cdot\boldsymbol{m}}{|\boldsymbol{r}_2-\boldsymbol{r}_1|} = \frac{(x-x_1)(x_2-x_1)+(y-y_1)(y_2-y_1)}{l^2}. \quad (24)$$

Let $\boldsymbol{i}, \boldsymbol{j}$ denoted the coordinate axis on the plan and $\boldsymbol{k}=\boldsymbol{i}\times\boldsymbol{j}$,

$$\mathrm{grad}\left(\frac{s-s_1}{l}\right) = \frac{x_2-x_1}{l^2}\boldsymbol{i} + \frac{y_2-y_1}{l^2}\boldsymbol{j}. \quad (25)$$

According to the definition (7),

$$\overrightarrow{\mathrm{rot}}_\Gamma\left(\frac{s-s_1}{l}\right) = \mathrm{grad}\left(\frac{s-s_1}{l}\right)\times\boldsymbol{n} = -\frac{1}{l}\boldsymbol{k}. \quad (26)$$

Similarly

$$\overrightarrow{\mathrm{rot}}_\Gamma\left(\frac{s_2-s}{l}\right) = \mathrm{grad}\left(\frac{s_2-s}{l}\right)\times\boldsymbol{n} = \frac{1}{l}\boldsymbol{k}. \quad (27)$$

Finally, the coefficients of the matrix can be calculated in a simple formula

$$a_{ij} = \frac{1}{2\pi l_i l_j}\int_{\Gamma_j}\int_{\Gamma_i} \ln|x-y| \mathrm{d}s_y \mathrm{d}s_x - \frac{1}{2\pi l_{i+1} l_j}\int_{\Gamma_j}\int_{\Gamma_{i+1}} \ln|x-y| \mathrm{d}s_y \mathrm{d}s_x - \frac{1}{2\pi l_i l_{j+1}}\int_{\Gamma_{j+1}}\int_{\Gamma_i} \ln|x-y| \mathrm{d}s_y \mathrm{d}s_x + \frac{1}{2\pi l_{i+1} l_{j+1}}\int_{\Gamma_{j+1}}\int_{\Gamma_{i+1}} \ln|x-y| \mathrm{d}s_y \mathrm{d}s_x$$
$$(i=1,2,\cdots,N;j=1,2,\cdots,N). \quad (28)$$

Now that the integrations are simplified, the first integral can be calculated by an analytical formula, and the second integral is calculated by Gaussian quadrature.

The linear system (18) gives the values φ_i $(i=1,2,\cdots,N)$ on boundary nodes, and the approximate solution of the problem (1) will be given by a discretized form of Eq. (2),

$$u_h = \frac{1}{2\pi}\sum_{i=1}^{N}\left(\int_\Gamma \Psi_i(y)\frac{\overrightarrow{(y-x)}\cdot\boldsymbol{n}_y}{|y-x|^2}\mathrm{d}s_y\right)\varphi_i + c =$$

$$\frac{1}{2\pi}\sum_{i=1}^{N}\Big(\int_{\Gamma_i}\psi^{(i)}\frac{\overrightarrow{(y-x)}\cdot\boldsymbol{n}_y}{|y-x|^2}\mathrm{d}s_y+\int_{\Gamma_{i+1}}\psi^{(i+1)}\frac{\overrightarrow{(y-x)}\cdot\boldsymbol{n}_y}{|y-x|^2}\mathrm{d}s_y\Big)\varphi_i+c. \tag{29}$$

We need some fixed values to determine the constant c, when $x\in\Omega$, but for the exterior problem, the solution is unique.

4 Numerical results

In order to test the efficiency of the proposed numerical approach, we did some numerical tests on problems with analytical solutions known. Uniform mesh refinement is used.

Example 4.1 $\Omega=\{(x,y)\,|\,x^2+y^2\geqslant 1\}$.

$$u=\frac{1}{r^2}\cos 2\theta=\frac{x^2-y^2}{(x^2+y^2)^2}.$$

The boundary condition is

$$\left.\frac{\partial u}{\partial \boldsymbol{n}}\right|_{\Gamma}=2\cos 2\theta=2x^2-2y^2.$$

The results for this case are shown in Tab. 1. We can see that the relationship of approximation error $E(u)=|u-u_h|$ with the element length $h(h=\frac{2\pi}{N})$ can be described by expression like $E(u)=0(h^2)$. The convergence rates are displayed in Fig. 2.

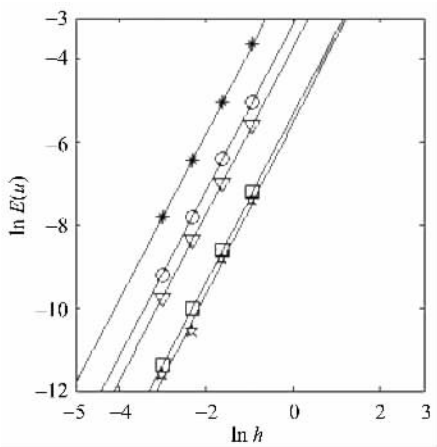

Fig. 2 $\ln E(u)\approx 2\ln h+\ln\alpha$ (α is a positive constant)

Example 4.2 Ω is a square domain with side length 8. $u(x,y)=xy$. Subjected to the boundary condition $g=\pm y$ at $x=\pm 4$; $g=\pm x$ at $y=\pm 4$.

The results for this case are shown in Tab. 2. The convergence rate is also about $O(h^2)$, which agrees with the theoretical convergence for Galerkin boundary element method (12). Fig. 3 illustrates the visual simulated results.

Tab. 1 The comparison of the solutions with the numbers of boundary elements increasing

r	approximate solution u_h				analytical solution
	$N=16$	$N=32$	$N=64$	$N=128$	
1.5	0.418 084 7	0.437 849 3	0.442 814 2	0.444 039 6	0.444 444 4
3.0	0.104 447 9	0.109 463 8	0.110 703 6	0.111 010 0	0.111 111 1
5.0	0.037 600 95	0.039 406 17	0.039 853 26	0.039 963 57	0.040 000 00
7.0	0.019 184 58	0.020 105 87	0.020 333 31	0.020 389 58	0.020 408 16
9.0	0.011 605 04	0.012 162 25	0.012 300 39	0.012 334 44	0.012 345 68
12.0	0.006 527 783	0.006 841 213	0.006 918 968	0.006 938 119	0.006 944 444

【Note】 N is numbers of the elements, r is the distance to the centre of the circle

Tab. 2 The comparison of the solutions with the numbers of boundary elements increasing

x	y	approximate solution u_h				analytical solution
		$N=16$	$N=32$	$N=64$	$N=128$	
3.0	3.0	0.901 894 7E+1	0.899 411 8E+1	0.899 980 5E+1	0.899 980 5E+1	9.0
1.0	1.0	0.998 112 1E+0	0.999 757 2E+0	0.100 014 1E+1	0.100 000 1E+1	1.0
0.5	0.2	0.998 773 5E−1	0.999 745 4E−1	0.100 005 2E+0	0.100 000 4E+0	0.1
−0.4	−0.6	0.239 612 8E+0	0.239 941 1E+0	0.240 002 3E+0	0.240 000 5E+0	0.24
−2.0	−2.0	0.398 890 5E+1	0.399 895 0E+1	0.399 997 8E+1	0.399 999 2E+1	4.0
3.0	−2.0	−0.598 808E+1	−0.599 804 1E+1	−0.599 986E+1	−0.599 998E+1	−6.0
−1.5	1.0	−0.149 689E+1	−0.149 963 8E+1	−0.150 004E+1	−0.150 000E+1	−1.5

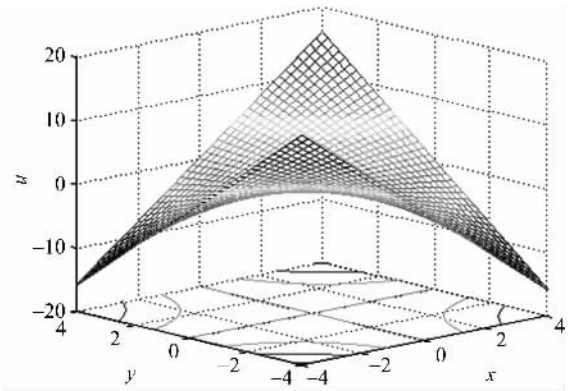

Fig. 3 Visual simulated result

5 Conclusion

This paper presents an approach that overcomes the barrier of hyper singular integration, encountered while applying the boundary element method, when a double layer solution for the Neumann problem is used. We introduce a Galerkin variational formulation in order to implement the integration by parts. The difference from other integrations by parts is the employment of the boundary rotation instead of the tangential derivative. While linear boundary elements are used in two-dimensional cases, the boundary rotation can be discretized into a constant vector on each element, and the hyper singular integrals turn into weak ones, so that the integrations can be performed smoothly. Numerical tests show good convergence behavior.

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