

# Explicit Analytic Solution for Stagnation Flow and Heat Transfer on a Moving Plate

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## Abstract

In this article, we use an efficient analytical method called homotopy analysis method (HAM) to derive an approximate solution of stagnation flow on a moving solid boundary. Actually, we solved the Navier-Stokes and energy equations by the HAM. Unlike the perturbation method, the HAM does not require the addition of a small physical parameter to the differential equation. It is applicable to strongly and weakly nonlinear problems. Moreover, the HAM involves an auxiliary parameter,  $\hbar$  which renders the convergence parameter of series solutions controllable, and increases the convergence, and increases the convergence significantly. This article depicts that the HAM is an efficient and powerful method for solving nonlinear differential equations.

**Keywords:** Nonlinear differential equations; stagnation flow; Homotopy analysis method (HAM), Homotopy perturbation method (HPM).

## 1 Introduction

Modeling of natural phenomena in most sciences yields nonlinear differential

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equations the exact solutions of which are usually rare. Therefore, analytical methods are strongly needed. For instance, one analytical method, called perturbation, involves creating a small physical parameter in the problem, however, finding this parameter is impossible in most cases [1, 2]. Generally speaking, one simple solution for controlling convergence and increasing it does not exist in all analytical methods.

In 1992, Liao [3] presented homotopy analysis method (HAM) based on fundamental concept of homotopy in topology [4-9]. In this method, we do not need to apply the small parameter and unlike all other analytic techniques, the HAM provides us with a simple way to adjust and control the convergence region of approximate series solutions. HAM has been successfully applied to solve many types of nonlinear problems [10-14].

In this work, the basic idea of HAM is described, and then we apply it to the stagnation flow equations. Stagnation flow on a moving solid boundary is basic in many convection-cooling processes. Stagnation flow towards a moving plate has been considered by Root [16], Wang [17], Libby [15] and extended by Weidman and Mahalingam [18]. These sources applied the conventional no slip condition on the solid boundary.

## 2 Basic idea of HAM

Let us consider the following differential equation

$$N[u(\tau)] = 0, \quad (1)$$

where  $N$  is a nonlinear operator,  $\tau$  denotes independent variable,  $u(\tau)$  is an unknown function that is the solution of the equation. We define the function

$$\begin{aligned} \phi(\tau; p) &= u_0(\tau), \\ p &\rightarrow 0 \end{aligned} \quad (2)$$

where,  $p \in [0, 1]$  and  $u_0(\tau)$  is the initial guess which satisfies the initial or boundary condition and is

$$\lim_{p \rightarrow 1} \phi(\tau; p) = u(\tau). \quad (3)$$

By means of generalizing the traditional homotopy method, Liao [3] constructs the so-called *zero-order deformation* equation

$$(1-p) L[\phi(\tau; p) - u_0(\tau)] = p \hbar H(\tau) N[\phi(\tau; p)], \quad (4)$$

where  $\hbar$  is the auxiliary parameter which increases the results convergence,  $H(\tau) \neq 0$  is an auxiliary function and  $L$  is an auxiliary linear operator,  $p$  increases from 0 to 1, the solution  $\phi(\tau; p)$  changes between the initial guess  $u_0(\tau; p)$  and

solution  $u(\tau)$ . Expanding  $\phi(\tau; p)$  in Taylor series with respect to  $p$ , we have

$$\phi(\tau; p) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau) p^m, \tag{5}$$

where

$$u_m(\tau) = \frac{1}{m!} \left. \frac{\partial^m \phi(\tau; p)}{\partial p^m} \right|_{p=0}, \tag{6}$$

if the auxiliary linear operator, the initial guess, the auxiliary parameter  $\hbar$ , and the auxiliary function are so properly chosen, the series (5) converges at  $p = 1$ , and then we have

$$u(\tau) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau), \tag{7}$$

which must be one of the solutions of the original nonlinear equation, as proved by Liao [7]. It is clear that if the auxiliary parameter is  $\hbar = -1$  and auxiliary function is determined to be  $H(\tau) = 1$ , Eq. (4) will be

$$(1-p) L[\phi(\tau; p) - u_0(\tau)] + p N[\phi(\tau; p)] = 0, \tag{8}$$

this statement is commonly used in HPM procedure. Indeed, in HPM we solve the nonlinear differential equation by separating every Taylor expansion term.

Now we define the vector of  $\vec{u}_m$  as follows

$$\vec{u}_m = \{\vec{u}_1, \vec{u}_2, \vec{u}_3, \dots, \vec{u}_n\}.$$

According to the definition Eq. (6), the governing equation and the corresponding initial condition of  $u_m(\tau)$  can be deduced from *zero-order deformation* Eq. (4). Differentiating Eq. (4) for  $m$ -times with respect to the embedding parameter  $p$  and setting  $p = 0$  and finally dividing by  $m!$ , we will have the so-called *mth order deformation* equation in the following from

$$L[u_m(\tau) - x_m u_{m-1}(\tau)] = \hbar H(\tau) R_m(\vec{u}_{m-1}), \tag{9}$$

where

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} N[\phi(\tau; p)]}{\partial p^{m-1}} \right|_{p=0}, \tag{10}$$

and

$$x_m = \begin{cases} 0 & m \leq 1, \\ 1 & m > 1. \end{cases} \tag{11}$$

So by applying inverse linear operator to both sides of the linear equation, Eq. (9), we can easily solve the equation and compute the generation constant by applying the initial or boundary condition.

## 4 Applications

Now we consider the two-dimensional stagnation flow. Fig. 1 shows a two-dimensional stagnation flow in the  $x$ - $z$  plane impinging on a plate at  $z=0$  moving with velocity  $U$  in the  $x$  direction and velocity  $V$  in the  $y$  direction. The flow far from the plate is given by the potential flow

$$u = ax, \quad v = 0, \quad w = -az, \quad p = p_0 - \rho a^2(x^2 + z^2)/2, \quad (12)$$

where  $u, v, w$  are velocity components in the Cartesian  $x, y, z$  directions,  $a$  is the strength of the stagnation flow,  $\rho$  is the density,  $p$  is the pressure and  $p_0$  is the stagnation pressure. For viscous flow, set

$$u = axf'(\eta) + Ug(\eta), \quad v = Vh(\eta), \quad w = -\sqrt{av}f(\eta), \quad (13)$$

$$p = p_0 - \rho(a^2x^2/2 + w^2/2 - vw_z), \quad (14)$$

where  $\eta \equiv \sqrt{a/\nu}z$  and  $\nu$  is the kinematic viscosity. The subscript  $z$  denotes differentiation with respect to  $z$ . The three-dimensional Navier- Stokes equations then reduce to the similarity ordinary differential equations [19].

$$f''' + ff'' - (f')^2 + 1 = 0, \quad (15)$$

$$g'' + fg' - f'g = 0, \quad (16)$$

$$h'' + fh' = 0. \quad (17)$$

On the plate, Navier's condition gives

$$u - U = N\rho\nu u_z, \quad v - V = N\rho\nu v_z, \quad (18)$$

Where  $N$  is a slip constant. The no slip condition is recovered when  $N=zero$ . For (no slip) the boundary conditions:

$$f(0) = 0, \quad f'(\infty) = 1, \quad f'(0) = 0, \quad (19)$$

$$g(\infty) = 0, \quad g(0) = 1, \quad (20)$$

$$h(\infty) = 0, \quad h(0) = 1, \quad (21)$$

Let the temperature far from the plate be  $T_\infty$  and temperature on the plate be  $T_0$ . Set

$$\theta(\eta) = \frac{T - T_\infty}{T_0 - T_\infty}. \tag{22}$$

The energy equation becomes

$$\theta'' + Pf\theta' = 0, \tag{23}$$

where  $P$  is the Prandtl number. A temperature slip condition similar to Navier's condition is

$$T - T_0 = ST_z, \tag{24}$$

where  $S$  is a proportionality constant. Eq. (24) can be written as

$$\theta(0) = 1 + \beta\theta'(0), \tag{25}$$

where  $\beta \equiv S\sqrt{\frac{a}{\nu}}$  is the thermal slip factor. At infinity the condition is

$$\theta(\infty) = 0. \tag{26}$$

Now we solve the problem for no slip condition.

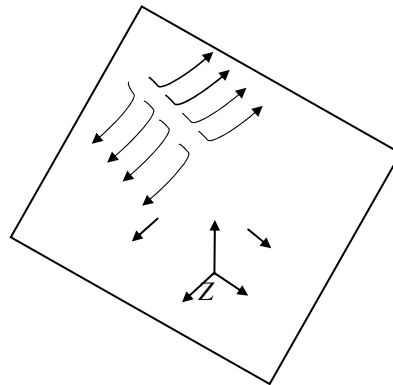


Fig.1. Two-dimensional stagnation flow.

We choose the initial approximation.

$$f_0(\eta) = 0/616295\eta^2, \tag{27}$$

$$g_0(\eta) = 1 - 0/8113\eta, \tag{28}$$

$$h_0(\eta) = 1 - 0/57047\eta, \tag{29}$$

$$\theta_0(\eta) = \eta^2, \quad (30)$$

and the linear operator for equation (15)

$$L[\phi_1(\eta; p)] = \frac{\partial^3 \phi_1(\eta; p)}{\partial \eta^3}, \quad (31)$$

and the linear operator for equation (16)

$$L[\phi_2(\eta; p)] = \frac{\partial^2 \phi_2(\eta; p)}{\partial \eta^2}, \quad (32)$$

and the linear operator for equation (17)

$$L[\phi_3(\eta; p)] = \frac{\partial^2 \phi_3(\eta; p)}{\partial \eta^2}, \quad (33)$$

and the linear operator for equation (23)

$$L[\phi_4(\eta; p)] = \frac{\partial^2 \phi_4(\eta; p)}{\partial \eta^2}, \quad (34)$$

we change equations (15), (16), (17) and (23) to nonlinear form

$$N_1[\phi_1(\eta; p)] = \frac{\partial^3 \phi_1(\eta; p)}{\partial \eta^3} + \phi_1(\eta; p) \frac{\partial^2 \phi_1(\eta; p)}{\partial \eta^2} - \left( \frac{\partial \phi_1(\eta; p)}{\partial \eta} \right)^2 + 1, \quad (35)$$

$$N_2[\phi_1(\eta; p), \phi_2(\eta; p)] = \frac{\partial^2 \phi_2(\eta; p)}{\partial \eta^2} + \phi_1(\eta; p) \frac{\partial \phi_2(\eta; p)}{\partial \eta} - \frac{\partial \phi_1(\eta; p)}{\partial \eta} \phi_2(\eta; p), \quad (36)$$

$$N_3[\phi_1(\eta; p), \phi_3(\eta; p)] = \frac{\partial^2 \phi_3(\eta; p)}{\partial \eta^2} + \phi_1(\eta; p) \frac{\partial \phi_3(\eta; p)}{\partial \eta}, \quad (37)$$

$$N_4[\phi_1(\eta; p), \phi_4(\eta; p)] = \frac{\partial^2 \phi_4(\eta; p)}{\partial \eta^2} + P \phi_1(\eta; p) \frac{\partial \phi_4(\eta; p)}{\partial \eta}, \quad (38)$$

assuming  $H(\tau) = 1$ , we use above definition to construct the *zero-order* deformation equations.

$$(1-p)L[\phi_1(\eta; p) - f_0(\eta)] = p\hbar N_1[\phi_1(\eta; p)], \quad (39)$$

$$(1-p)L[\phi_2(\eta; p) - g_0(\eta)] = p\hbar N_2[\phi_1(\eta; p), \phi_2(\eta; p)], \quad (40)$$

$$(1-p)L[\phi_3(\eta; p) - h_0(\eta)] = p\hbar N_3[\phi_1(\eta; p), \phi_3(\eta; p)], \quad (41)$$

$$(1-p)L[\phi_4(\eta; p) - \theta_0(\eta)] = p\hbar N_4[\phi_1(\eta; p), \phi_4(\eta; p)], \quad (42)$$

Obviously, when  $p=0$  and  $p=1$ ,

$$\phi_1(\eta;0) = f_0(\eta), \quad \phi_1(\eta;1) = f(\eta), \quad (43)$$

$$\phi_2(\eta;0) = g_0(\eta), \quad \phi_2(\eta;1) = f(\eta), \quad (44)$$

$$\phi_3(\eta;0) = h_0(\eta), \quad \phi_3(\eta;1) = h(\eta), \quad (45)$$

$$\phi_4(\eta;0) = \theta_0(\eta), \quad \phi_4(\eta;1) = \theta(\eta), \quad (46)$$

Differentiating the *zero-order* deformation equations (39), (40), (41) and (42)  $m$ -times with respect to  $p$ .

$$L[f_m - x_m f_{m-1}] = \hbar R_{1m}(\bar{f}_{m-1}, \bar{g}_{m-1}, \bar{h}_{m-1}, \bar{\theta}_{m-1}), \quad (47)$$

$$L[g_m - x_m g_{m-1}] = \hbar R_{2m}(\bar{f}_{m-1}, \bar{g}_{m-1}, \bar{h}_{m-1}, \bar{\theta}_{m-1}), \quad (48)$$

$$L[h_m - x_m h_{m-1}] = \hbar R_{3m}(\bar{f}_{m-1}, \bar{g}_{m-1}, \bar{h}_{m-1}, \bar{\theta}_{m-1}), \quad (49)$$

$$L[\theta_m - x_m \theta_{m-1}] = \hbar R_{4m}(\bar{f}_{m-1}, \bar{g}_{m-1}, \bar{h}_{m-1}, \bar{\theta}_{m-1}), \quad (50)$$

where

$$R_{1m}(\bar{f}_{m-1}, \bar{g}_{m-1}, \bar{h}_{m-1}, \bar{\theta}_{m-1}) = \frac{\partial^3 f_{m-1}(\eta)}{\partial \eta^3} + \sum_{n=0}^{m-1} [f_n(\eta) \frac{\partial^2 f_{m-1-n}(\eta)}{\partial \eta^2} - \frac{\partial f_n(\eta)}{\partial \eta} \frac{\partial f_{m-1-n}(\eta)}{\partial \eta}] + (1-x_m), \quad (51)$$

$$R_{2m}(\bar{f}_{m-1}, \bar{g}_{m-1}, \bar{h}_{m-1}, \bar{\theta}_{m-1}) = \frac{\partial^2 g_{m-1}(\eta)}{\partial \eta^2} + \sum_{n=0}^{m-1} [f_n(\eta) \frac{\partial g_{m-1-n}(\eta)}{\partial \eta} - \frac{\partial f_n(\eta)}{\partial \eta} g_{m-1-n}(\eta)], \quad (52)$$

$$R_{3m}(\bar{f}_{m-1}, \bar{g}_{m-1}, \bar{h}_{m-1}, \bar{\theta}_{m-1}) = \frac{\partial^2 h_{m-1}(\eta)}{\partial \eta^2} + \sum_{n=0}^{m-1} f_n(\eta) \frac{\partial h_{m-1-n}(\eta)}{\partial \eta}, \quad (53)$$

$$R_{4m}(\bar{f}_{m-1}, \bar{g}_{m-1}, \bar{h}_{m-1}, \bar{\theta}_{m-1}) = \frac{\partial^2 \theta_{m-1}(\eta)}{\partial \eta^2} + P \sum_{n=0}^{m-1} f_n(\eta) \frac{\partial \theta_{m-1-n}(\eta)}{\partial \eta}, \quad (54)$$

$$\text{and } x_m = \begin{cases} 0 & m \geq 1, \\ 1 & m > 1. \end{cases} \quad (55)$$

From (27) to (30) and (51) to (54), we now successively obtain the  $f(\eta)$ ,  $g(\eta)$ ,  $h(\eta)$  and  $\theta(\eta)$ . The equations (47), (48), (49) and (50) are linear and thus can be easily solved, especially by means of symbolic computation software such as Mathematica, Maple, MATLAB and so on. We used 25 terms in evaluating the approximate solution.

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{25} f_m(\eta), \quad (56)$$

$$g(\eta) = g_0(\eta) + \sum_{m=1}^{25} g_m(\eta), \quad (57)$$

$$h(\eta) = h_0(\eta) + \sum_{m=1}^{25} h_m(\eta), \tag{58}$$

$$\theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{25} \theta_m(\eta), \tag{59}$$

Note that this series contains the auxiliary parameter  $\hbar$ , which influence its convergence region and rate. We should therefore focus on the choice of  $\hbar$  by plotting of  $\hbar$ -curve. Fig.2 shows the  $\hbar$ -curve of  $\frac{\partial^3 f(0)}{\partial \eta^3}$ , Fig.3, shows the  $\hbar$ -curve of  $\frac{\partial^3 g(0)}{\partial \eta^3}$  and Fig.4 shows the  $\hbar$ -curve of  $\frac{\partial^4 h(0)}{\partial \eta^4}$  and Fig.5 shows the  $\hbar$ -curve of  $\frac{\partial \theta(0)}{\partial \eta}$ .

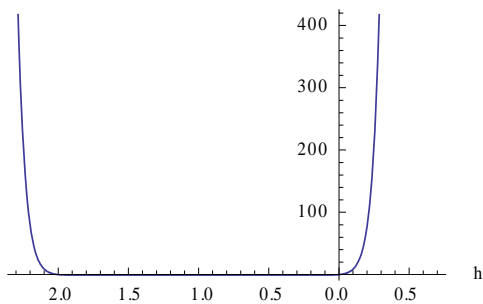


Fig.2. The 25th-order approximation of  $f'''(0)$  versus  $\hbar$

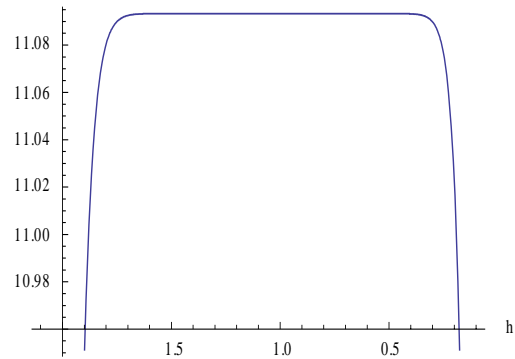


Fig.3. The 25th-order approximation of  $g'''(0)$  versus  $\hbar$

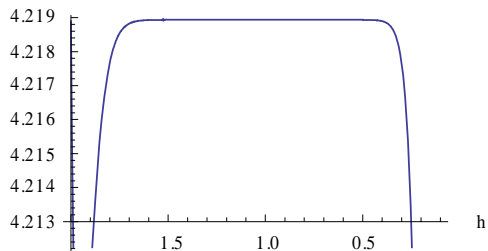


Fig.4. The 25th-order approximation of  $h^{iv}(0)$  versus  $\hbar$

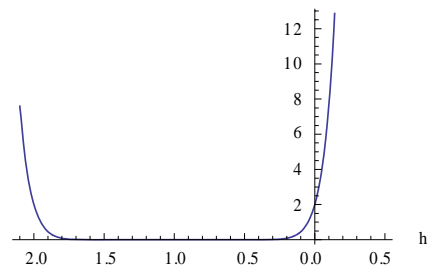


Fig.5. The 25th-order approximation of  $\theta'(0)$  versus  $\hbar$



We should select optimal  $\hbar$  from the region in which the diagram is quite horizontal. Horizontal region is the optimal  $\hbar$  region. Regarding figure (2) optimal  $\hbar$  equals zero, regarding figure (3) optimal  $\hbar$  equals -1, regarding figure (4) optimal  $\hbar$  equals -1 regarding figure (5) optimal  $\hbar$  equals -1. In this article we have obtained the values of  $f, g, h$  by applying HAM remarkable method as well as by numerical method and you will see the consequences of these methods in figures No,6,7 and 8. These three diagrams apparently show that quite analytic method of HAM is so close to numerical solution with great exactness which is a token of its high accuracy.

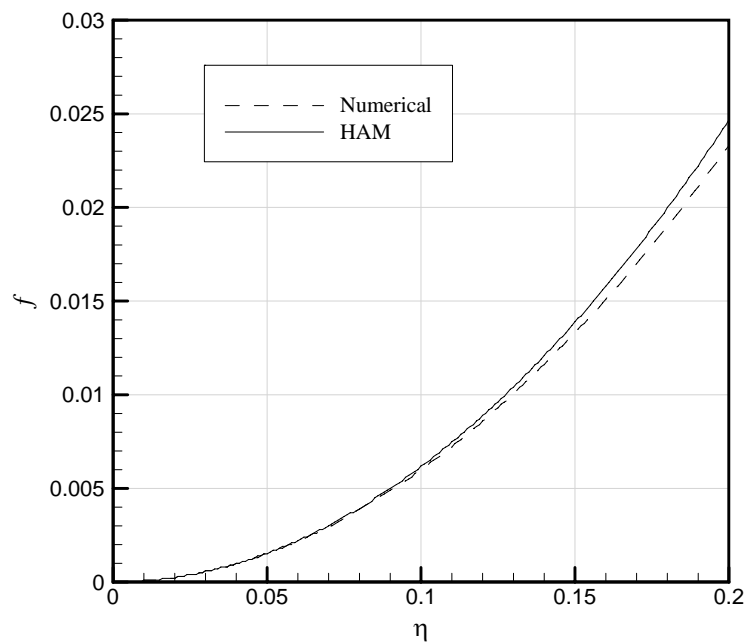


Fig.6. Comparison of numerical results with HAM of  $f(\eta)$

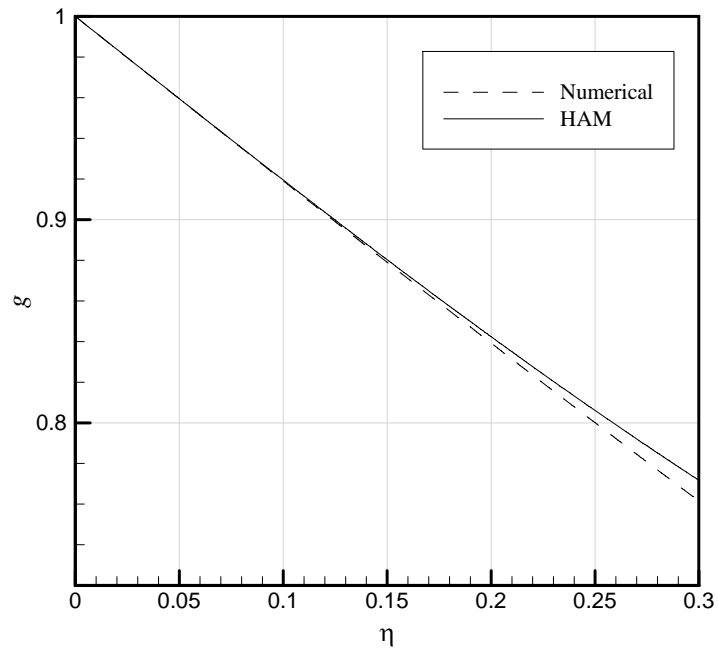
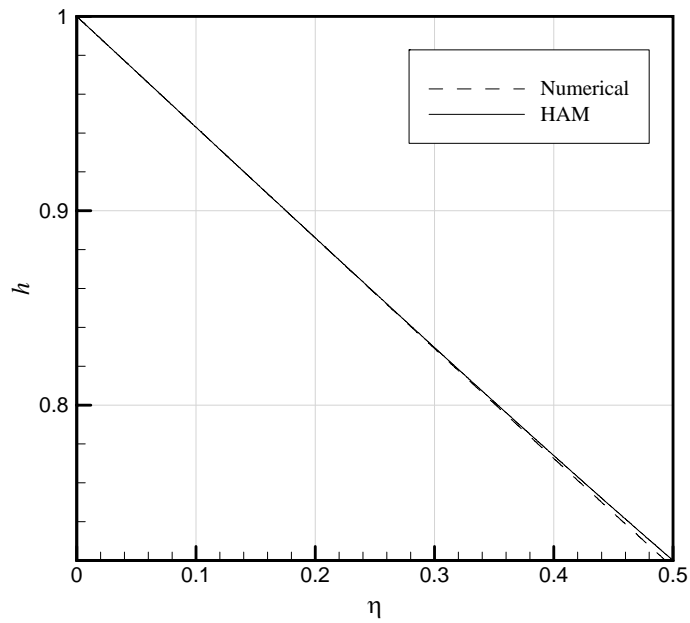
Fig.7. Comparison of numerical results with HAM of  $g(\eta)$ Fig.8. Comparison of numerical results with HAM of  $h(\eta)$

Table 1

The values of  $\theta'(0)$ . for no slip condition. (  $P = 7$  )

$\beta$	0	0.5	1	5
<i>numerical</i>	1.21176	0.752941	0.523529	0.170588
<i>HAM</i>	1.21175	0.752941	0.523528	0.170585

We consider four different situation of the no slip energy equation (  $\beta = 0, \beta = 0.5, \beta = 1$  and  $\beta = 5$  ) which led us to identical  $h$ -curve for different values of  $\beta$ . Then we obtain  $\theta'(0)$  for Prandtl number seven. Prandtl number 7 represents liquids. In Table1, we can see that the HAM solution is very close to the numerical solution.

#### 4 Conclusions

In this paper, we utilized the powerful method of homotopy analysis to obtain the stagnation flow equations. We achieved a very good approximation with the numerical solution of the considered problem. In addition, this technique is algorithmic and it is easy to implementation by symbolic computation software, such as Maple and Mathematica. Different from all other analytic techniques, it provides us with a simple way to adjust and control the convergence region of approximate series solutions. Unlike perturbation methods, the HAM does not need any small parameter. It shows that the HAM is a very efficient method. We sincerely hope this method can be applied in a wider range.

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