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3D Nearest-Nodes Finite Element Method for Solid Continuum Analysis

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Abstract

In this paper, a so-called Nearest-Nodes Finite Element Method (NN-FEM) is proposed for the analysis of a 3D solid continuum. In the method, finite elements are mainly used for numerical integration. For each integration point, a set of element nodes that are the nearest to the point are selected for constructing shape functions, some of them are from neighbouring elements. NN-FEM is an extension of the conventional Finite Element Method, which provides a more flexible way in constructing shape functions. Numerical tests demonstrated that the proposed NN-FEM has a competitive convergence rate compared with the conventional FEM and the meshless method.

Keywords: Nearest-Nodes, Shape Function, Polynomial Interpolation

1 Introduction

The Finite Element Method (FEM), since its appearance around the middle of last century [1, 2, 3], has been a great success and the prevalent numerical tool for solving scientific and engineering problems. The FEM has a lots of merits compared with other numerical methods. Compared with the newly emerged meshless or meshfree methods, one big advantage of the FEM is that it does not need extra time to construct shape functions, as the shape functions for a specific element type are pre-defined. Therefore, the FEM is computationally more efficient. The dilemma is that finite element meshes consisting of only simplexes (triangle or tetrahedron) are easy to generate, but low order elements based on simplexes have a slow convergence. On the other hand, higher order elements with edge or interior nodes have a faster convergence, but the corresponding meshes are hard to handle, especially if mesh adaptation

is involved. Element distortion has been another major obstacle for solving challenging problems such as metal forging. Aiming at resolving the above issues of the conventional FEM, a new category of methods, collectively called meshless or meshfree methods, e. g. [4, 5] and the references therein, have been developed in recent years. One advantage of meshless methods is their flexibility in constructing shape functions. Nevertheless, it is in the price of non-trivial extra computational time spent in looking for neighbor nodes. The amount of extra computational time is even not bounded within a linear order of total nodal number.

In this paper, a so-called Nearest-Nodes Finite Element Method is proposed with an attempt to combine the advantages from both the conventional FEM and meshless methods. In the method, finite elements are mainly used for numerical integration. For each integration point, a set of element nodes that are the nearest to the point are selected for constructing shape functions, some of them are from neighbouring elements. The structure of the paper is outlined in the following. In Section 2, governing equations and variational formulation for a 3D solid continuum are briefly reviewed; A general strategy for constructing shape functions in the proposed NN-FEM is described in Section 3; Results of numerical investigations are reported in Section 4, followed by concluding remarks in Section 5.

2 Governing Equations and Variational Formulation for 3D Solid Continuum

Consider a body of elastic solid continuum that occupies a 3D domain Ω bounded by a closed surface Γ. The equilibrium of the body under static body force and surface traction is governed by

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0} \qquad \text{in} \quad \Omega \tag{1}$$

and satisfies the following boundary conditions

$$\begin{aligned} \boldsymbol{\sigma} \cdot \boldsymbol{n} &= \hat{\boldsymbol{\sigma}} \qquad \text{on} \quad \Gamma_\sigma \\ \boldsymbol{u} &= \hat{\boldsymbol{u}} \qquad \text{on} \quad \Gamma_{\mathrm{u}} \end{aligned} \tag{2}$$

In Eqs. (1) and (2), $\boldsymbol{\sigma}$ is the stress tensor; $\boldsymbol{b}$ the body force; $\boldsymbol{n}$ the outward normal of boundary Γ_σ; $\hat{\boldsymbol{\sigma}}$ is the surface traction and $\hat{\boldsymbol{u}}$ is the prescribed displacement.

Or, equivalently, the equilibrium equations in Eq. (1) and the boundary conditions in Eq. (2) can be formulated as a variational statement, i. e., the principle of minimum potential energy,

$$\delta E = 0 \tag{3}$$

where E is the total energy stored in the continuum body, which in turn includes the strain energy (Π) and the external potential energy (V),

$$E = \Pi - V = \frac{1}{2}\int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}\, d\Omega - \int_{\Omega} \boldsymbol{u}^T \boldsymbol{b}\, d\Omega - \int_{\Gamma_\sigma} \boldsymbol{u}^T \hat{\boldsymbol{\sigma}}\, d\Gamma \tag{4}$$

In the expression, $\boldsymbol{\varepsilon}$ is the strain tensor; operator ':' indicates a scalar product of two tensors. The stress and the strain tensor are related by the material tensor, $\boldsymbol{D}$, via

$$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\varepsilon} \tag{5}$$

Now let $\{\Omega_i\ (i = 1, 2, \cdots, N)\}$ be a partition of Ω, representing N finite elements; Ω_i is the i-th element; $\Omega_i \cap \Omega_j = \emptyset$ (if $i \neq j$) and $\Omega = \cup_{i=1}^{N} \Omega_i$. The total energy is calculated by summing up contributions from all the elements,

$$E = \frac{1}{2}\sum_{i=1}^{N}\int_{\Omega_i} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}\, d\Omega_i - \sum_{i=1}^{N}\int_{\Omega_i} \boldsymbol{u}^T \boldsymbol{b}\, d\Omega_i - \sum_{i=1}^{N}\int_{\Gamma_\sigma} \boldsymbol{u}^T \hat{\boldsymbol{\sigma}}\, d\Gamma_i \tag{6}$$

The third integration on the right side of Eq. (6) is conducted only for those elements that have an element face or element edge classified on boundary Γ_σ.

By introducing shape functions $\phi_k(\boldsymbol{x})$ $(i = 1, 2, \cdots, n)$, the displacements can be approximated as

$$\boldsymbol{u} \approx \sum_{k=1}^{n} \phi_k(\boldsymbol{x})\bar{\boldsymbol{u}}_k \tag{7}$$

with element nodal displacements $\bar{\boldsymbol{u}}_k$ $(i = 1, 2, \cdots, n)$. Correspondingly, the stresses, the strains and the total energy can be expressed by nodal displacements. By applying the principle of minimum potential energy, the finite element equations can be obtained.

3 Strategy for Constructing Shape Functions

Unlike in the conventional FEM, where shape functions are constructed once for a whole element by using nodes belonging to that element, a different strategy is adopted in the proposed NN-FEM. Shape functions are constructed for each quadrature point, using a set of nodes that are the nearest to the quadrature point. Some of those nodes may not belong to the element where the concerned quadrature point is located. A typical scenario is illustrated in Fig.1 by using a 2D mesh. Where, element stiffness matrix for the element with a 'x', which is an integration or quadrature point, is being calculated. For the quadrature point denoted, a number of nearest nodes, marked as solid circles in the figure, are selected for constructing shape functions. Among those nodes, not all the nodes belong to the shaded element. The number of

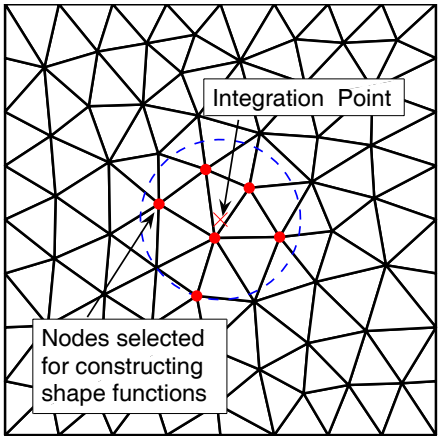


Figure 1: Nearest-Nodes FEM

the selected nodes is determined by the desired order of the resulting shape functions.

For quadrature points in the same element, different nodes may be used in constructing their shape functions, therefore, the procedure of assembling global stiffness matrix is slightly different. In the classical FEM, the assembly is actually done in two steps. First, contributions from all quadrature points over an element are put together into an element stiffness matrix; and then the element stiffness matrix is assembled into the global stiffness matrix. In the NN-FEM, contributions from numerical quadrature points are directly put into the global stiffness matrix.

After selecting n nodes that are the nearest to a quadrature point, any technique used in meshless methods for constructing shape functions can be adopted in NN-FEM. Here, a polynomial interpolation method [6, 7, 8] is used for its simplicity.

Consider the approximation of a generic function $f(\boldsymbol{x})$. With the n nearest nodes, the function value at $\boldsymbol{x}$ can be approximated as

$$\tilde{f}(\boldsymbol{x}) = \sum_{i=1}^{n} a_i p_i(\boldsymbol{x}) = \boldsymbol{p}^T(\boldsymbol{x}) \cdot \boldsymbol{a} \tag{8}$$

where $\boldsymbol{p}(\boldsymbol{x}) = \{p_1(\boldsymbol{x}),\ p_2(\boldsymbol{x}),\ \cdots,\ p_n(\boldsymbol{x})\}$ is a base vector consisting of monomials; $\boldsymbol{a} = \{a_1,\ a_2,\ \cdots,\ a_n\}$ is a coefficient vector. The monomials included in the base vector should satisfy the completeness and symmetry requirement [9].

The coefficient vector $\boldsymbol{a}$ is determined by enforcing the approximation in Equation (8) at the n selected nodes, i. e.

$$\boldsymbol{p}^T(\boldsymbol{x}_i) \cdot \boldsymbol{a} = f_i \qquad (i = 1, 2, \cdots, n) \tag{9}$$

The above equations can be collected in a matrix form

$$\boldsymbol{P}\boldsymbol{a} = \bar{\boldsymbol{f}} \tag{10}$$

where $\bar{\boldsymbol{f}} = \{f_1,\ f_2,\ \cdots,\ f_n\}$ is a vector consisting of function values at the n nodes. $\boldsymbol{P}$ is the Vandermonde matrix with dimensions $n \times n$. The expressions of entries in matrix $\boldsymbol{P}$ depend on problem dimension and polynomial order. For example, for a three-dimensional two-order polynomial, matrix $\boldsymbol{P}$ has the following form

$$\boldsymbol{P} = \begin{bmatrix} 1 & x_1 & y_1 & z_1 & x_1^2 & y_1^2 & z_1^2 & x_1y_1 & y_1z_1 & z_1x_1 \\ 1 & x_2 & y_2 & z_2 & x_2^2 & y_2^2 & z_2^2 & x_2y_2 & y_2z_2 & z_2x_2 \\ & & \cdots & & \cdots & & \cdots & & \cdots & \\ 1 & x_{10} & y_{10} & z_{10} & x_{10}^2 & y_{10}^2 & z_{10}^2 & x_{10}y_{10} & y_{10}z_{10} & z_{10}x_{10} \end{bmatrix} \tag{11}$$

If it is assumed that $\boldsymbol{P}$ is non-singular and its inverse exists, the coefficient vector $\boldsymbol{a}$ is determined by

$$\boldsymbol{a} = \boldsymbol{P}^{-1}\bar{\boldsymbol{f}} \tag{12}$$

and the approximation is now expressed as

$$\tilde{f}(\boldsymbol{x}) = \boldsymbol{\phi}^T(\boldsymbol{x})\bar{\boldsymbol{f}} \tag{13}$$

where

$$\boldsymbol{\phi}^T(\boldsymbol{x}) = \boldsymbol{p}^T(\boldsymbol{x})\boldsymbol{P}^{-1} \tag{14}$$

are shape functions or local polynomials.

Approximate first-derivatives at point $\boldsymbol{x}$ are calculated as

$$\frac{\partial \tilde{f}}{\partial \alpha} = \boldsymbol{\phi}_{,\alpha}^T \bar{\boldsymbol{f}}, \qquad (\alpha = x, y, z) \tag{15}$$

with

$$\boldsymbol{\phi}_{,\alpha} = \frac{\partial \boldsymbol{p}^T(\boldsymbol{x})}{\partial \alpha}\boldsymbol{P}^{-1}, \qquad (\alpha = x, y, z) \tag{16}$$

Any higher order derivatives can be calculated in a similar way, as long as sufficient high order monomials are included in the base vector in Eq. (8). To guarantee that the Vandermonde matrix $\boldsymbol{P}$ is non-singular, in filling $\boldsymbol{P}$ matrix with nearest nodes, the so-called non-singularity criterion is applied. The criterion is based on the following theorem.

Theorem 1 (Rank in terms of determinants)[10]
An $m \times n$ matrix $\boldsymbol{A} = [a_{jk}]$ has rank r $(r \geq 1)$ if and only if $\boldsymbol{A}$ has an $r \times r$

submatrix with nonzero determinant, whereas the determinant of every square submatrix with $r+1$ or more rows that $\boldsymbol{A}$ has (or does not have!) is zero. In particular, if $\boldsymbol{A}$ is square of $n \times n$, it has rank n if and only if $\det(\boldsymbol{A}) \neq 0$.

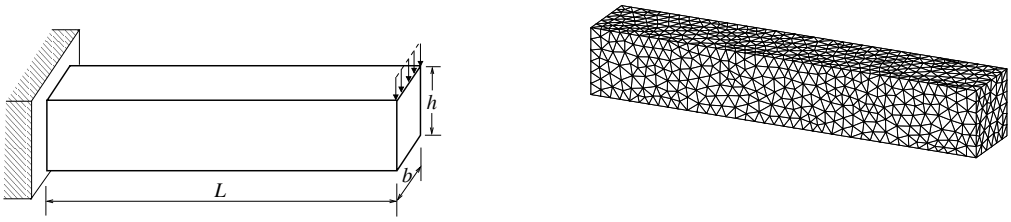
Based on Theorem 1, for constructing a d-dimensional local polynomial, the non-singularity criterion is stated as: in filling $\boldsymbol{P}$ matrix with nearest nodes, for each added new row, say, now the i-th row is added, $i > d$, the determinant of master submatrix, $\boldsymbol{M}_P^i$, is examined, i. e.

$$\text{if} \quad \left|\det(\boldsymbol{M}_P^i)\right| \begin{cases} \geq \delta, & i\text{-th row is kept.} \\ < \delta, & i\text{-th row is discarded.} \end{cases} \tag{17}$$

where δ is a small positive real number, $\delta = 10^{-10} \sim 10^{-20}$.

4 Numerical Investigation

In this section, numerical investigations on the proposed NN-FEM are reported. A cantilever beam under an end shear force shown in Fig. 2(a) was



(a) (b)

Figure 2: A cantilever beam (a) physical model; (b) a mesh

analyzed. The beam has the following geometric and material parameters: length L=6, width $b = 1$, height $h = 1$, elasticity modulus E=1000.0, Poisson's ratio ν=0. The parameters have consistent units. The beam cross-section at the right end is loaded with a uniformly distributed shear force. Convergence was studied by increasing mesh density. One representative mesh is displayed in Fig. 2(b).

With $\nu = 0.$, three-dimensional solutions should be close to the beam solutions. The obtained displacements of the beam cross-section center at the loaded end are normalized by the analytical beam solution and plotted in Fig. 3. A deformed configuration of the beam and the corresponding effective stress distribution are given in Fig. 4. For comparison purpose, results from the FEM and from the Element-Free Galerkin (EFG) method [11] are also displayed. The FEM results were produced by commercial software ANSYS

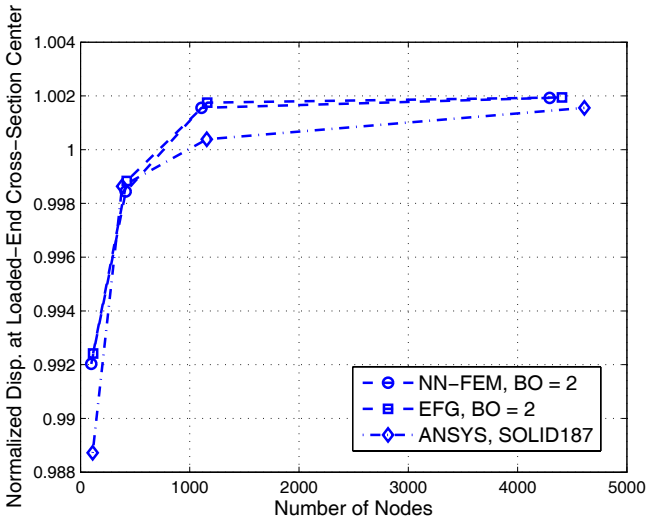
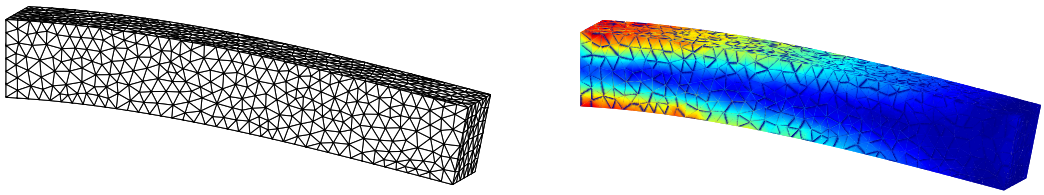


Figure 3: *h*-Convergence. (BO—Base order)



(a) (b)

Figure 4: (a) Deformed configuration; (b) Effective stress distribution

using SOLID187. SOLID187 is a second order tetrahedral element with ten nodes. For the EFG method, the radius of a circular domain of influence was so determined that approximately 10 nodes are covered by the domain of influence, and a complete set of second order monomials were included in the base. A fair base was tried for the comparison, that is, for all the compared methods they have the same order of shape functions, a similar number of element nodes, etc. From the obtained results, it can be observed that compared with the FEM and EFG method, the proposed NN-FEM with a second order base has a competitive convergence rate. For the EFG method, a weight function is usually used in the construction of shape functions. Therefore, if a second-order base is used, the resulting shape functions will have an order higher than two [11]. This is why the EFG method has a slightly higher convergence rate.

It was also found from the investigation that although by increasing the order of local polynomials, NN-FEM can have a gain in convergence rate, but the bandwidth of global stiffness matrix also becomes larger. Therefore more storage and memory are demanded and more solution time is needed. Based on a comprehensive consideration of balance between convergence rate, computational time and memory consumption, second-order local polynomials would be the most efficient for NN-FEM.

5 Concluding Remarks

A 3D Nearest-Nodes Finite Element Method (NN-FEM) is proposed for the analysis of solid continuum. In the method, finite elements are mainly used for numerical integration. For each integration point, a set of element nodes that are the nearest to the point are selected for constructing shape functions, some of them are from neighbouring elements. Numerical tests demonstrated that with a similar number of elements and element nodes, the proposed NN-FEM has a competitive convergence rate compared with the conventional FEM and the meshless method. NN-FEM has several very attractive features. First, higher-order shape functions can be constructed using simplex meshes. The quality of shape functions are solely determined by the locations of element nodes. The order of shape functions can be arbitrarily high, as long as enough element nodes are available. Therefore, p-version of adaptation can be easily implemented in NN-FEM.

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