Analytical Solution for Solute Transport along and against Time Dependent Source Concentration in Homogeneous Finite Aquifers

Mritunjay Kumar Singh

Department of Applied Mathematics, Indian School of Mines, Dhanbad-826004, Jharkhand, India
drmks29@rediffmail.com

Premlata Singh

Department of Applied Mathematics, Indian School of Mines, Dhanbad-826004, Jharkhand, India
prema.singh@rediffmail.com

Vijay P. Singh

Department of Biological and Agricultural Engineering
Department of Civil & Environmental Engineering
Texas A and M University
321 Scoates Hall, 2117 TAMU, College Station Texas 77843-2117 USA
vsingh@tamu.edu

Abstract

A solute transport equation is analytically solved to predict the contaminant concentration distribution along and against transient groundwater flow in finite aquifer. The horizontal dispersion along and against transient groundwater flow in homogenous and finite aquifer is considered. Initially the aquifer is not supposed to be solute free, i.e., aquifer is not clean. The time-dependent input concentration is considered at the source of an aquifer and at the other end the concentration gradient

is supposed to be zero. The time-dependent forms of velocities are considered in which one form, sinusoidal form, represents the seasonal variation in a year in tropical regions. The Laplace Transform Technique (LTT) is used to obtain analytical solutions which are perhaps most useful for benchmarking numerical codes and solutions.

Keywords: Aquifer, Solute transport, Contaminant concentration, Groundwater flow, Analytical solution.

1. Introduction

Rapid growth of urbanization, industrialization as well as agricultural activities is causing groundwater contamination worldwide, and especially in India. Increasing demand for water to meet environmental, domestic, industrial and agricultural needs has resulted in an unplanned abstraction and exploitation of natural groundwater resources. The utilization of fertilizers, pesticides, disposal of solid wastes, and untreated waste water on land has further deteriorated groundwater quality (USEPA 1989 1990; Anderson and Woessner 1992; Charbeneau 2000; Kebew 2001; Sharma and Ready 2004; Rausch et al. 2005; Thangarajan 2006) [35], [36], [2], [7], [17], [30], [24], [34]. A well-known water contamination problem results from mining due to pumping of mine water and its discharge into an existing drainage system. A huge quantity of water is used in the beneficiation and coal preparation plants, which, when discharged into streams, causes water contamination. For example, the Jharia coal field is one of the best examples for producing coal in Jharkhand and it is directly or indirectly affecting the surface water and subsurface water contamination.

In India, many aquifers are being contaminated by a host of human activities, such as sewage disposal, refuse dumps, pesticide and chemical fertilizer contamination, industrial effluent discharges, and toxic waste disposal (Sharma and Ready 2004; Rausch et al. 2005; Thangarajan 2006) [30], [24], [34]. For managing groundwater resources and rehabilitation of contaminated aquifers, mathematical modeling is a powerful tool. Groundwater transport and its mathematical models were presented by Fried (1975), Javendal and Tsang (1984), and Bear and Verruijt (1987) [12], [16], [6]. The role of mathematical modeling in groundwater resource management was discussed by Rai (2004) [22]. Mathematical modeling of groundwater in mining areas has been discussed by numerous scientists, hydrologists and civil engineers in Advanced Training courses organized by Department of Hydrology, CMRI, Dhanbad, and Sponsored by Department of Science and Technology, New Delhi, and has been presented by Gupta et al. (2005) [14]. In the present work, the contaminant concentration distribution along/against transient groundwater flow in an aquifer is investigated.

There is a wide variety of applications of dispersion in finite, semi-infinite and infinite porous media. Many examples of groundwater system and other flow domains are bounded by infinite regions. Analysis of flow against dispersion in porous media was presented by Al-Niami and Rushton (1977) [1]. The study of flow against dispersion in non-adsorbing porous media was presented by Marino (1978) [21] in which the flows were opposite to dispersion. The flows were assumed to be one-dimensional in a horizontal direction and the average velocities were taken to be constant throughout the flow field. Concentrations varying as exponential functions of time were enforced at the source of the dispersion, while a condition in which the change in concentration is proportional to the flow was applied at the other boundary. Analytical solutions of one dimensional advective-dispersive-reactive solute transport equations under a variety of conditions were presented by Van Genuchten and Alves (1982) [33]. Unsteady flow against dispersion in finite porous media was also explored which was controlled by flow (with transient unidirectional velocity distribution) of a low concentration fluid towards a region of higher concentration (Kumar 1983) [18]. In field applications, upstream spreading of contaminant plumes may be controlled by the flow of fresh water in a direction opposite to the dispersive expansion direction of the plume. In the groundwater literature this type of control is identified as flow against dispersion or contrary flow. Aral and Tang (1992) [3] presented an analytic methods which are used to investigate contrary flow conditions for two-dimensional applications. In particular, special attention is given to the dispersive spread of the contaminant plume in the transverse direction under equilibrium flow against dispersion.

Analytical solutions of the convective-dispersive equation with different initial and boundary conditions were developed by Lindstrom and Boersma (1989); Fry et al. (1993) [20], [13]. Analytical solution of one dimensional time-dependent transport equation was also presented (Basha and Habel 1993) [4]. One-dimensional virus transport in homogeneous porous media with a time dependent distribution coefficient was presented by Chrysikopoulos and Sim (1996) [8]. An analytical solution for solute transport with depth dependent transformation or sorption coefficient was developed by Flury et al. (1998) [11]. Analytical solution/numerical solutions with suitable initial and boundary conditions to predict the concentration distribution of pollutants along transient groundwater flow in semi-infinite aquifers in splitting time domain were studied by Kumar and Kumar (1998) [19]. The water-table variation in response to time varying recharge was explored by Rai and Manglik (1999) [23]. Analytical solution for unsteady transport dispersion of non-conservative pollutant with time dependent periodic waste discharge concentration was presented by Shukla (2002) [31]. Analytical solutions to transient, unsaturated transport of water and contaminants through horizontal porous media were discussed by Sander Braddock (2005) [26]. Analytical solutions for sequentially coupled one-dimensional reactive transport problems were discussed by Srinivasan and Clement (2008) [32]. An analytical solution for the longitudinal dispersion with time dependent source concentration in a semi infinite aquifer was presented by Singh et al. (2008) [28].

Analytical solution for conservative solute transport in one-dimensional homogeneous porous formations with time dependent velocity was also derived by Singh et al. (2009) [29].

In most investigations, groundwater velocity has been considered steady. However, when the groundwater table rises and falls, the velocity of flow in the aquifer may be transient or unsteady. The objective of this study is to derive the spatio-temporal analytical solutions for contaminant concentration along and against transient groundwater flow in a finite aquifer and the aquifer is subject to time-dependent source contamination. These results can be used to benchmark numerical models and perhaps significant for groundwater resource management.

2. Solute Transport Mechanism

Advection (convection) is the movement of solute caused by groundwater flow. The bulk movement of water through the aquifer causes solute transport via advection. Advection is the primary process by which solute moves in the groundwater system. Due to advection, non-reactive solute travel at an average rate equal to the seepage velocity of the fluid. Diffusion is also known as molecular diffusion refers to the movement of contaminants under a chemical concentration gradient (i.e. from an area of greater concentration toward an area of lower concentration). It can occur even when the fluid is not flowing or is flowing in the direction opposite to contaminant movement. Diffusion will cease only if there is no concentration gradient. Hydrodynamic dispersion occurs when two miscible fluids mix each other and also any soluble matter is dissolved in a fluid. Dispersion is essentially macroscopic phenomenon caused by a combination of molecular diffusion and hydrodynamic mixing occurring with laminar flow through porous media. The pattern of point traces as its move downstream from its source tends to a normal distribution both longitudinal and transverse. Furthermore, the longitudinal component is larger than that of the transverse so that the major axis of mixing occurs in the direction of flow. Hydrodynamic dispersion is normally referred to as mixing of miscible fluid. It is a hydraulic mixing process by which waste concentration is attenuated while waste contaminants are being transported downstream. At the macro scale level, contaminant transport is defined by the average groundwater velocity. However, at the microscale level the actual velocity of water may vary from point to point and can be either lower or higher than the average velocity. This difference in micro scale level water velocities arises due to pore size, path length and friction in pores. Because of these differences in velocities mixing occurs along the flow path. This mixing is known as mechanical dispersion or hydrodynamic dispersion (Fried 1975; Sharma and Reddy 2004) [12], [30]. The dispersion sources in uniform groundwater flow were reported by Hunt (1978) [15]. In general, the theory of dispersion in porous media was presented by Scheidegger (1961) [27].

3. Mathematical Formulation

Consider a homogeneous finite aquifer of length *L*. The aquifer may have an initial natural concentration. The aquifer is subjected to contamination due to the time-dependent source concentration. The groundwater flow in the aquifer is transient where the velocity follows either a sinusoidal form or an exponentially decreasing form. The sinusoidal form of velocity may represent the seasonal variation in a year in tropical regions.

In order to mathematically formulate the problem, let *c(x, t)* be the contaminant concentration in the aquifer [ML^{-3}], *u* the groundwater velocity [LT^{-1}], and *D* the dispersion coefficient [L^{2}T^{-1}] at any time *t* [T]. Then the problem can be mathematically formulated as follows:

$$D\frac{\partial^2 c}{\partial x^2} - u\frac{\partial c}{\partial x} = \frac{\partial c}{\partial t} \tag{1}$$

$$u(t) = u_0 V(t) \tag{2}$$

where u_0 is the initial groundwater velocity[LT^{-1}] at distance *x*[L]. Here, two forms of $V(t)$ are considered such as $V(t) = 1 - \sin mt$ and $V(t) = \exp(-mt)$, *mt < 1*, where *m* is the flow resistance coefficient [T^{-1}]. However, linear and exponentially time dependent forms of seepage velocity for purposes of studying salinity problems were derived (Banks and Jerasate 1962) [5]. The dispersion coefficient varies approximately in direct proportion to the seepage velocity for various types of porous media (Ebach and White 1958) [10]. It has also been found that such a relationship established for steady flow is also approximately valid for transient or unsteady flow with sinusoidal varying seepage velocity (Rumer 1962) [25]. Let $D = au$, where *a* is the dispersivity [L] that depends upon the pore geometry. Using equation (2), one gets $D = D_0 V(t)$. Here $D_0 = au_0$ is an initial dispersion coefficient.

4. Initial and Boundary Conditions

Let c_i be the initial contaminant concentration in the aquifer [ML^{-3}] describe the distribution of the concentration at all points of the flow domain at the beginning of investigation, i.e., at time *t =0*. It is assumed that initially, the aquifer is not clean. The time-dependent source concentration $c_0\{1 + \exp(-qt)\}$ at the origin of the aquifer (i.e., *x=0*) is considered where c_0 is the solute concentration [ML^{-3}] and *q* is the parameter like flow resistance coefficient[T^{-1}] and let the concentration gradient at the other end of the aquifer (i.e. *x=L*) be zero. Here the concentration is prescribed for all points of the boundary for the entire period of investigation. The boundary

conditions describe the nature of interaction of the flow system with its surrounding. The initial and boundary conditions can be expressed as follows:

$$c(x,t) = c_i \; ; \; x \geq 0 \,, \; t = 0 \tag{3}$$

$$c(x,t) = c_0 \{1 + \exp(-qt)\}; x = 0, t > 0 \tag{4a}$$

$$\frac{\partial c}{\partial x} = 0; \; t \geq 0, \; x = L \tag{4b}$$

5. Analytical Solution: Along the Flow

Combining equations (1) and (2), we get:

$$D_0 \frac{\partial^2 c}{\partial x^2} - u_0 \frac{\partial c}{\partial x} = \frac{1}{V(t)} \frac{\partial c}{\partial t} \tag{5}$$

Introducing a new time variable (Crank 1975) [9] or transformation as

$$T^* = \int_0^t V(t) dt \tag{6}$$

equation (5) and the corresponding initial and boundary conditions given by equations (3)-(4) become

$$D_0 \frac{\partial^2 c}{\partial x^2} - u_0 \frac{\partial c}{\partial x} = \frac{\partial c}{\partial T^*} \tag{7}$$

$$c(x, T^*) = c_i; T^* = 0, x \geq 0 \tag{8}$$

$$c(x, T^*) = c_0 (2 - qT^*); x = 0, T^* > 0 \tag{9a}$$

$$\frac{\partial c}{\partial x} = 0; T^* \geq 0, x = L \tag{9b}$$

It may be convenient to present equations (7)-(9) in dimensionless form. For that the set of non-dimensional variables can be defined as:

$$X = \frac{x}{L}, C = \frac{c}{c_0}, T = \frac{D_0 T^*}{L^2}, U = \frac{u_0 L}{D_0}, Q = \frac{qL^2}{D_0} \tag{10}$$

Equation (7), along with initial and boundary conditions (8)-(9) in non-dimensional form, can be written as:

$$\frac{\partial^2 C}{\partial X^2} - U \frac{\partial C}{\partial X} = \frac{\partial C}{\partial T} \tag{11}$$

$$C(X,T) = c_i / c_0; X \geq 0, T = 0 \tag{12}$$

$$C(X,T) = 2 - QT \; ; \; X = 0, \; T > 0 \tag{13a}$$

$$\frac{\partial C}{\partial X} = 0; X = 1, \; T \geq 0 \tag{13b}$$

Using the transformation $C(X,T) = K(X,T) \exp(UX/2 - U^2 T/4)$ (14)

in equations (11)-(13) and applying the Laplace transform, one can get the solution of the boundary value problem as:

$$\bar{K}(X,p) = (2 - c_i/c_0)\{\bar{K}_1(X,p) - \bar{K}_2(X,p) - \bar{K}_3(X,p)\}$$
$$- Q\{\bar{K}_4(X,p) - \bar{K}_5(X,p) - \bar{K}_6(X,p)\} + (c_i/c_0)\bar{K}_7(X,p) \tag{15}$$

where

$$\bar{K}_1(X,p) = \left(\frac{1}{p - U^2/4}\right) e^{-\sqrt{pX}} \tag{15a}$$

$$\bar{K}_2(X,p) = \left(\frac{1}{p - U^2/4}\right)\left(1 - \frac{U}{\sqrt{p} + U/2}\right)\left\{e^{-(2+X)\sqrt{p}} - e^{-(2-X)\sqrt{p}}\right\} \tag{15b}$$

$$\bar{K}_3(X,p) = \left(\frac{1}{p - U^2/4}\right)\left(1 - \frac{U}{\sqrt{p} + U/2}\right)^2 e^{-(4-X)\sqrt{p}} \tag{15c}$$

$$\bar{K}_4(X,p) = \left(\frac{1}{p - U^2/4}\right)^2 e^{-\sqrt{pX}} \tag{15d}$$

$$\bar{K}_5(X,p) = \left(\frac{1}{p - U^2/4}\right)^2\left(1 - \frac{U}{\sqrt{p} + U/2}\right)\left\{e^{-(2+X)\sqrt{p}} - e^{-(2-X)\sqrt{p}}\right\} \tag{15e}$$

$$\bar{K}_6(X,p) = \left(\frac{1}{p - U^2/4}\right)^2\left(1 - \frac{U}{\sqrt{p} + U/2}\right)^2 e^{-(4-X)\sqrt{p}} \tag{15f}$$

$$\bar{K}_7(X,p) = \left(\frac{1}{p - U^2/4}\right) e^{-\frac{UX}{2}} \tag{15g}$$

where

$$\bar{K}(X,p) = \int_0^\infty K(X,T) e^{-pT} dT$$

Taking inverse Laplace transform for (15), we get

$$K(X,T) = \left(2 - \frac{c_i}{c_0}\right)\left\{\begin{array}{l} F(X,T) - F(2+X,T) + F(2-X,T) - F(4-X,T) + UG(2+X,T) \\ -UG(2-X,T) + 2UG(4-X,T) - U^2H(4-X,T) \end{array}\right\}$$
$$- Q\left\{\begin{array}{l} I(X,T) - I(2+X,T) + I(2-X,T) - I(4-X,T) + UJ(2+X,T) \\ -UJ(2-X,T) + 2UJ(4-X,T) - U^2L(4-X,T) \end{array}\right\}$$
$$+ \frac{c_i}{c_0}\exp\left(\frac{U^2T}{4} - \frac{UX}{2}\right)$$

$$\tag{16}$$

$$F(X,T)=\frac{1}{2}\left\{\exp\left(\frac{U^{2}T}{4}-\frac{UX}{2}\right)erfc\left(\frac{X}{2\sqrt{T}}-\frac{U\sqrt{T}}{2}\right)+\exp\left(\frac{U^{2}T}{4}+\frac{UX}{2}\right)erfc\left(\frac{X}{2\sqrt{T}}+\frac{U\sqrt{T}}{2}\right)\right\}$$

(16a)

$$G(X,T)=\sqrt{\frac{T}{\pi}}\exp\left(-\frac{X^{2}}{4T}\right)+\frac{1}{2U}\exp\left(\frac{U^{2}T}{4}-\frac{UX}{2}\right)erfc\left(\frac{X}{2\sqrt{T}}-\frac{U\sqrt{T}}{2}\right)$$

$$-\frac{1}{2U}\left(1+UX+U^{2}T\right)\exp\left(\frac{U^{2}T}{4}+\frac{UX}{2}\right)erfc\left(\frac{X}{2\sqrt{T}}+\frac{U\sqrt{T}}{2}\right)$$

(16b)

$$H(X,T)=\frac{1}{2U^{2}}\left[\begin{array}{l}-2U\sqrt{\dfrac{T}{\pi}}\left(1+\dfrac{UX}{2}+\dfrac{U^{2}T}{2}\right)\exp\left(-\dfrac{X^{2}}{4T}\right)\\[2mm]+\left\{-1+UX+2U^{2}T+\dfrac{U^{2}}{2}(X+UT)^{2}\right\}\exp\left(\dfrac{U^{2}T}{4}+\dfrac{UX}{2}\right)erfc\left(\dfrac{X}{2\sqrt{T}}+\dfrac{U\sqrt{T}}{2}\right)\\[2mm]+\exp\left(\dfrac{U^{2}T}{4}-\dfrac{UX}{2}\right)erfc\left(\dfrac{X}{2\sqrt{T}}-\dfrac{U\sqrt{T}}{2}\right)\end{array}\right]$$

(16c)

$$I(X,T)=\frac{1}{2U}\left\{\begin{array}{l}(UT-X)\exp\left(\dfrac{U^{2}T}{4}-\dfrac{UX}{2}\right)erfc\left(\dfrac{X}{2\sqrt{T}}-\dfrac{U\sqrt{T}}{2}\right)\\[2mm]+(UT+X)\exp\left(\dfrac{U^{2}T}{4}+\dfrac{UX}{2}\right)erfc\left(\dfrac{X}{2\sqrt{T}}+\dfrac{U\sqrt{T}}{2}\right)\end{array}\right\}$$

(16d)

$$J(X,T)=\frac{1}{2U^{3}}\left[\begin{array}{l}2U\sqrt{\dfrac{T}{\pi}}\left(1+\dfrac{UX}{2}+\dfrac{U^{2}T}{2}\right)\exp\left(-\dfrac{X^{2}}{4T}\right)\\[2mm]+\left(-1-UX+U^{2}T\right)\exp\left(\dfrac{U^{2}T}{4}-\dfrac{UX}{2}\right)erfc\left(\dfrac{X}{2\sqrt{T}}-\dfrac{U\sqrt{T}}{2}\right)\\[2mm]-\left\{-1+U^{2}T+\dfrac{U^{2}}{2}(X+UT)^{2}\right\}\exp\left(\dfrac{U^{2}T}{4}+\dfrac{UX}{2}\right)erfc\left(\dfrac{X}{2\sqrt{T}}+\dfrac{U\sqrt{T}}{2}\right)\end{array}\right]$$

(16e)

$$L(X,T)=\frac{1}{U^{4}}\left[\begin{array}{l}\dfrac{2U}{3}\sqrt{\dfrac{T}{\pi}}\left\{-3+U^{2}T+\dfrac{U^{2}}{4}(X+UT)^{2}\right\}\exp\left(-\dfrac{X^{2}}{4T}\right)\\[2mm]-\left\{\left(1+UX-\dfrac{U^{2}T}{2}\right)\exp\left(\dfrac{U^{2}T}{4}-\dfrac{UX}{2}\right)erfc\left(\dfrac{X}{2\sqrt{T}}-\dfrac{U\sqrt{T}}{2}\right)\right\}\\[2mm]+\left\{1+\dfrac{U}{2}\left(U^{2}T-1\right)(X+UT)+\dfrac{U^{3}}{6}(X+UT)^{3}\right\}\exp\left(\dfrac{U^{2}T}{4}+\dfrac{UX}{2}\right)erfc\left(\dfrac{X}{2\sqrt{T}}+\dfrac{U\sqrt{T}}{2}\right)\end{array}\right]$$

(16f)

Substituting the value of $K(X,T)$ in equation (14), we get the desired solution as

$$C(X,T) = \left(2 - \frac{c_i}{c_0}\right)\exp\left(\frac{UX}{2} - \frac{U^2T}{4}\right) \left\{ \begin{matrix} F(X,T) - F(2+X,T) + F(2-X,T) \\ -F(4-X,T) + UG(2+X,T) - UG(2-X,T) \\ +2UG(4-X,T) - U^2H(4-X,T) \end{matrix} \right\}$$

$$- Q\exp\left(\frac{UX}{2} - \frac{U^2T}{4}\right) \left\{ \begin{matrix} I(X,T) - I(2+X,T) + I(2-X,T) \\ -I(4-X,T) + UJ(2+X,T) - UJ(2-X,T) \\ +2UJ(4-X,T) - U^2L(4-X,T) \end{matrix} \right\} + \frac{c_i}{c_0}$$

$$(17)$$

6.   Analytical Solution: Against the Flow

The initial and boundary conditions of the same problem against transient groundwater flow in non-dimensional form can be written as:

$$\frac{\partial^2 C}{\partial X^2} - U\frac{\partial C}{\partial X} = \frac{\partial C}{\partial T} \tag{18}$$

$$C(X,T) = c_i / c_0; X \geq 0, T = 0 \tag{19}$$

$$\frac{\partial C}{\partial X} = 0; X = 0, T \geq 0 \tag{20a}$$

$$C(X,T) = 2 - QT \; ; X = 1, T > 0 \tag{20b}$$

Using the same transformation $C(X,T) = K(X,T)\exp\left(UX/2 - U^2T/4\right)$ in equations (18)-(20) and applying the Laplace transform, we get the solution of the boundary value problem as:

$$\bar{K}(X,p) = \left(2 - c_i/c_0\right)\exp(-U/2)\left\{\bar{K}_1(X,p) - \bar{K}_2(X,p) - \bar{K}_3(X,p)\right\}$$
$$- Q\exp(-U/2)\left\{\bar{K}_4(X,p) - \bar{K}_5(X,p) - \bar{K}_6(X,p)\right\} + \left(c_i/c_0\right)\bar{K}_7(X,p) \tag{21}$$

where

$$\bar{K}_1(X,p) = \left(\frac{1}{p - U^2/4}\right)e^{-(1-X)\sqrt{p}} \tag{21a}$$

$$\bar{K}_2(X,p) = \left(\frac{1}{p - U^2/4}\right)\left(1 + \frac{U}{\sqrt{p} - U/2}\right)\left\{e^{-(3-X)\sqrt{p}} - e^{-(1+X)\sqrt{p}}\right\} \tag{21b}$$

$$\bar{K}_3(X,p) = \left(\frac{1}{p - U^2/4}\right)\left(1 + \frac{U}{\sqrt{p} - U/2}\right)^2 e^{-(3+X)\sqrt{p}} \tag{21c}$$

$$\bar{K}_4(X,p)=\left(\frac{1}{p-U^2/4}\right)^2 e^{-(1-X)\sqrt{p}} \tag{21d}$$

$$\bar{K}_5(X,p)=\left(\frac{1}{p-U^2/4}\right)^2\left(1+\frac{U}{\sqrt{p}-U/2}\right)\left\{e^{-(3-X)\sqrt{p}}-e^{-(1+X)\sqrt{p}}\right\} \tag{21e}$$

$$\bar{K}_6(X,p)=\left(\frac{1}{p-U^2/4}\right)^2\left(1+\frac{U}{\sqrt{p}-U/2}\right)^2 e^{-(3+X)\sqrt{p}} \tag{21f}$$

$$\bar{K}_7(X,p)=\left(\frac{1}{p-U^2/4}\right)e^{-\frac{UX}{2}} \tag{21g}$$

where
$$\bar{K}(X,p)=\int_0^\infty K(X,T)e^{-pT}\,dT$$

Taking inverse Laplace transform of equations (21), we get

$$
\begin{aligned}
K(X,T)=&\left(2-\frac{c_i}{c_0}\right)\exp\left(-\frac{U}{2}\right)\left\{\begin{array}{l}F(1+X,T)-F(3+X,T)+F(1-X,T)-F(3-X,T)\\+UG(1+X,T)-UG(3-X,T)-2UG(3+X,T)-U^2H(3+X,T)\end{array}\right\}\\
&-Q\exp\left(-\frac{U}{2}\right)\left\{\begin{array}{l}I(1+X,T)-I(3+X,T)+I(1-X,T)-I(3-X,T)\\+UJ(1+X,T)-2UJ(3+X,T)-UJ(3-X,T)-U^2L(3+X,T)\end{array}\right\}\\
&+\frac{c_i}{c_0}\exp\left(\frac{U^2T}{4}-\frac{UX}{2}\right)
\end{aligned}
\tag{22}
$$

$$F(X,T)=\frac{1}{2}\left\{\exp\left(\frac{U^2T}{4}-\frac{UX}{2}\right)erfc\left(\frac{X}{2\sqrt{T}}-\frac{U\sqrt{T}}{2}\right)+\exp\left(\frac{U^2T}{4}+\frac{UX}{2}\right)erfc\left(\frac{X}{2\sqrt{T}}+\frac{U\sqrt{T}}{2}\right)\right\} \tag{22a}$$

$$
\begin{aligned}
G(X,T)=&\sqrt{\frac{T}{\pi}}\exp\left(-\frac{X^2}{4T}\right)-\frac{1}{2U}\exp\left(\frac{U^2T}{4}+\frac{UX}{2}\right)erfc\left(\frac{X}{2\sqrt{T}}+\frac{U\sqrt{T}}{2}\right)\\
&+\frac{1}{2U}\left(1-UX+U^2T\right)\exp\left(\frac{U^2T}{4}-\frac{UX}{2}\right)erfc\left(\frac{X}{2\sqrt{T}}-\frac{U\sqrt{T}}{2}\right)
\end{aligned}
\tag{22b}
$$

$$H(X,T) = \frac{1}{U}\sqrt{\frac{T}{\pi}}\left(1 - \frac{UX}{2} + \frac{U^2T}{2}\right)\exp\left(-\frac{X^2}{4T}\right) + \frac{1}{2U^2}\exp\left(\frac{U^2T}{4} + \frac{UX}{2}\right)$$

$$erfc\left(\frac{X}{2\sqrt{T}} + \frac{U\sqrt{T}}{2}\right) + \frac{1}{2U^2}\left\{-1 - UX + 2U^2T + \frac{U^2}{2}(X-UT)^2\right\}$$

$$\exp\left(\frac{U^2T}{4} - \frac{UX}{2}\right)erfc\left(\frac{X}{2\sqrt{T}} - \frac{U\sqrt{T}}{2}\right)$$

(22c)

$$I(X,T) = \frac{1}{2U}\left\{\begin{array}{l}(UT - X)\exp\left(\dfrac{U^2T}{4} - \dfrac{UX}{2}\right)erfc\left(\dfrac{X}{2\sqrt{T}} - \dfrac{U\sqrt{T}}{2}\right) \\[3mm] +(UT + X)\exp\left(\dfrac{U^2T}{4} + \dfrac{UX}{2}\right)erfc\left(\dfrac{X}{2\sqrt{T}} + \dfrac{U\sqrt{T}}{2}\right)\end{array}\right\}$$

(22d)

$$J(X,T) = \frac{1}{2U^3}\left[\begin{array}{l}2U\sqrt{\dfrac{T}{\pi}}\left(1 - \dfrac{UX}{2} + \dfrac{U^2T}{2}\right)\exp\left(-\dfrac{X^2}{4T}\right) \\[3mm] -\left(-1 + UX + U^2T\right)\exp\left(\dfrac{U^2T}{4} + \dfrac{UX}{2}\right)erfc\left(\dfrac{X}{2\sqrt{T}} + \dfrac{U\sqrt{T}}{2}\right) \\[3mm] +\left\{-1 + U^2T + \dfrac{U^2}{2}(X-UT)^2\right\}\exp\left(\dfrac{U^2T}{4} - \dfrac{UX}{2}\right)erfc\left(\dfrac{X}{2\sqrt{T}} - \dfrac{U\sqrt{T}}{2}\right)\end{array}\right]$$

(22e)

$$L(X,T) = \frac{1}{U^4}\left[\begin{array}{l}-\dfrac{2U}{3}\sqrt{\dfrac{T}{\pi}}\left\{-3 + U^2T + \dfrac{U^2}{4}(X-UT)^2\right\}\exp\left(-\dfrac{X^2}{4T}\right) \\[3mm] -\left\{\left(1 - UX - \dfrac{U^2T}{2}\right)\exp\left(\dfrac{U^2T}{4} + \dfrac{UX}{2}\right)erfc\left(\dfrac{X}{2\sqrt{T}} + \dfrac{U\sqrt{T}}{2}\right)\right\} \\[3mm] +\left\{1 - \dfrac{U}{2}(U^2T - 1)(X - UT) - \dfrac{U^3}{6}(X - UT)^3\right\}\exp\left(\dfrac{U^2T}{4} - \dfrac{UX}{2}\right)erfc\left(\dfrac{X}{2\sqrt{T}} - \dfrac{U\sqrt{T}}{2}\right)\end{array}\right]$$

(22f)

We now obtain the desired solution as:

$$C(X,T) = \left(2 - \frac{c_i}{c_0}\right)\exp\left(\frac{U(X-1)}{2} - \frac{U^2T}{4}\right)\begin{Bmatrix} F(1+X,T) - F(3+X,T) + F(1-X,T) \\ -F(3-X,T) + UG(1+X,T) - UG(3-X,T) \\ -2UG(3+X,T) - U^2H(3+X,T) \end{Bmatrix} \qquad (23)$$

$$-Q\exp\left(\frac{U(X-1)}{2} - \frac{U^2T}{4}\right)\begin{Bmatrix} I(1+X,T) - I(3+X,T) + I(1-X,T) \\ -I(3-X,T) + UJ(1+X,T) - 2UJ(3+X,T) \\ -UJ(3-X,T) - U^2L(3+X,T) \end{Bmatrix} + \frac{c_i}{c_0}$$

7.   An Example Application and discussion

Banks and Jerasati (1962) [25] derived linear and exponentially time dependent form of seepage velocity for purposes of studying the salinity problem. They considered unsteady flow in porous media and longitudinal disperson. In the present discussion the sinusoidally varying and exponentially decreasing forms of velocities are considered:

$$u(t) = u_0(1 - \sin mt) \qquad (24a)$$

$$u(t) = u_0\exp(-mt), \quad mt < 1 \qquad (24b)$$

where m(/day) is the flow resistance coefficient. In aquifers in tropical regions, groundwater velocity and water level may exhibit seasonally sinusoidal behavior (as noted by Kumar and Kumar (1998; Thangarajan (2006) [19], [34]. In tropical regions in India, groundwater velocity and water level are minimum during the peak of the summer season (the period of greatest pumping), which fall in the month of June, just before rainy season. Maximum values are observed during the peak of winter season around December, after the rainy season (the period of lowest pumping). In these regions, groundwater infiltration is from rainfall and rivers. Groundwater flow depends on aquifer properties, such as porosity, permeability, hydraulic conductivity, etc. For a homogeneous aquifer, its properties are spatially invariant. Graphs for both velocity forms are shown in the Figs. 1(a,b).

For both the expressions, the non-dimensional time variable T may be written as

$$T = \frac{D_0}{mL^2}\{mt - (1 - \cos mt)\} \qquad (25a)$$

$$T = \frac{D_0}{mL^2}\{1 - \exp(-mt)\} \qquad (25b)$$

where mt = 2, 5, 8, 11, 14, 17,…, 38, 41 and 44 are chosen. For m = 0.0165 (/day), equation (24a) yield, t (days) =121, 303, 485, 667, 849, 1030,…, 2303, 2485, and 2667, respectively. For these values of mt, the velocity u, is alternatively minimum and maximum. Hence it represents the groundwater level and velocity minimum

during the month of June and maximum during December just after six months (Approximately 182 days) in one year. The next data of t represents minimum and maximum records during June and December respectively in subsequent years as depicted in Fig. 1(a). Analytical solutions given by equations (17) and (23) are computed for the values of $c_i = 0.01$, $c_0 = 1.0$, $u_0 = 0.005$ km/day, $D_0 = 0.5$ km^2/day, $q = 0.0001$(/day), and $L=100$ km. The concentration values are depicted graphically in the presence of time dependent source of contaminant concentration at mt = 26,29,32,35,38, and 41 which represents minimum and maximum records of groundwater level and velocity during June and December in 5th, 6th and 7th years, respectively. The contaminant concentration distribution behaviour along and against transient groundwater flow of sinusoidally varying velocity are shown in the Fig. (2a,b). It is observed that the contaminant concentration decreases at the source and emerges at a point near the origin. After emergence the tendency of the contaminant concentration is the same reaching towards the minimum or harmless concentration. But the values of the contaminant concentration decrease and increase with time just before and after the emergence, respectively. For example, before emergence the 5th year December concentration is less than 5th year June concentration, while after emergence the trend is just reverse. For the same set of inputs, except for m=0.0002 (/day) as mt < 1(as in case of exponentially decreasing velocity), equation (17) and (23) are also computed for the exponentially decreasing form of velocity and shown in the Fig(3a,b).

8.   Conclusions

A solute transport model is formulated with time dependent source concentration in homogeneous finite aquifer and analytical solution is obtained to predict contaminant concentration along and against transient groundwater flow in a homogeneous finite aquifer. Time dependent forms of velocities, i.e., sinusoidally varying and exponentially decreasing velocity are considered. Analytical solution of the problems may help model numerical solutions and codes. It may be used as a preliminary predictive tool in groundwater management.

References

[1] A. N. S. Al-Niami and K. R. Rushton, Analysis of flow against dispersion in porous media, J. Hydrology, 33(1977), 87-97.
[2] M. P. Anderson and W. W. Woessner Applied groundwater modeling: Simulation of Flow and Advective Trasnport, 381p, Academic Press, London, England (1992).

[3] M. M. Aral and Y Tang, Flow against dispersion in two-dimensional regions, J Hydrol., 140(1992), 261 – 277.

[4] H. A. Basha and F. S. El-Habel, Analytical solution of one dimensional time-dependent transport equation, Water Resources Research, 29(9) (1993), 3209-3214.

[5] R. P. Banks and S. T. Jerasate, Dispersion in unsteady porous media flow, J Hydraulic Div., 88(1962), 1-21.

[6] J. Bear and A. Verruijt, Modeling Groundwater Flow and Pollution, D. Reidel Publ. Comp., Netherlands, (1987).

[7] R. J. Charbeneau, Groundwater Hydraulics and Pollutant Transport, 593 p., Prentice Hall, New Jersey, (2000).

[8] C. V. Chrysikopoulos and Y. Sim, One-dimensional virus transport homogeneous porous media with time dependent distribution coefficient, J. Hydrology, 185 (1996), 199-219.

[9] J. Crank, The Mathematics of Diffusion, Oxford Univ. Press, Oxford(1975).

[10] E. H. Ebach and R. White, Mixing of fluid flowing through beds of packed solids, J Am Inst Chem Eng, 4(1958), 161-164.

[11] M. Flury , Q. J. Wu , L. Wu and L. Xu, Analytical solution for solute transport with depth dependent transformation or sorption coefficient, Water Resources Research, 34(11) (1998), 2931-2937.

[12]J. J. Fried, Groundwater Pollution, Elsevier Scientific Pub. Comp., Amsterdam (1975).

[13]V. A. Fry, J. D. Istok and R. B. Guenther, An analytical solution of the solute transport equation with rate-limited desorption and decay, Water Resource Research, 29(9) (1993), 3201-3208.

[14]P. K. Gupta, A. P. Singh and M. S. Alam editors, Mathematical Modeling of Groundwater in Mining Area, Department of Hydrology, CMRI, Dhanbad(2005).

[15]B. Hunt, Dispersion sources in uniform ground water flow, J Hydraulic Div., 104(1) (1978), 75-85.

[16]I. C. Javendel and C. F. Tsang, Groundwater Transport: Handbook of Mathematics Models, Water Resource Monogrser 10 (AGU Washington) (1984).

[17]A. E. Kebew, Applied Chemical Hydrogeology, 368 p, Prentice Hall, New Jersey(2001).

[18]N. Kumar, Unsteady flow against dispersion in finite porous media, J Hydrol., 63(1983), 345-358.

[19]N. Kumar and M. Kumar, Solute dispersion along unsteady groundwater flow in a semi-infinite aquifer, Hydrol. & Earth System Sci., 2(1) (1998), 93-100.

[20]F. T. Lindstrom and L. Boersma, Analytical solutions for convective dispersive transport in confined aquifers with different initial and boundary conditions, Water Resources Research, 25(2): (1989), 241-256.

[21]M. A. Marino, Flow against dispersion in non-adsorbing porous media, J Hydrol., 37(1-2) (1978), 149-158.

[22] S. N. Rai, Role of Mathematical Modeling in Groundwater Resources Management, NGRI, Hyderabad (2004).

[23] S. N. Rai and A. Manglik, Modelling of water table variation in response to time varying recharge from multiple basin using the linearised Boussinesq equation, J Hydrology, 220 (1999), 141-148.

[24] R. Rausch, W. Schafer, R. Therrien and C. Wagner, Solute Transport Modelling: An Introduction to Models and Solution Strategies, 205 p Gebr. Borntrager Verlagsbuchhandlung Science Publishers, Berlin, Germany (2005).

[25] R. R. Rumer, Longitudinal dispersion in steady and unsteady flow, J Hydraulic Div., 88(HY4), (1962), 147-172.

[26] G. C. Sander and R. D. Braddock, Analytical solutions to the transient, unsaturated transport of water and contaminants through horizontal porous media, Adv. Water Resour. 28(2005), 1102–1111.

[27] A. E. Scheidegger, General theory of dispersion in porous media, J Geophysics Research, 66 (1961), 10.

[28] M. K. Singh, N. K. Mahato and P. Singh, Longitudinal dispersion with time dependent source concentration in semi infinite aquifer, J Earth System Sci., 117 (6) (2008), 945-949.

[29] M. K. Singh, V. P. Singh, P. Singh and D. Shukla, Analytical solution for conservative solute transport in one-dimensional homogeneous porous formations with time dependent velocity, J Engg. Mech., ASCE 135 (9) (2009), 1015-1021.

[30] H. D. Sharma and K.R. Reddy, Geo-environmental Engineering, Wiley, New York (2004).

[31] V. P. Shukla, Analytical solution for unsteady transport dispersion of non-conservative pollutant with time dependent periodic waste discharge concentration, J Hydraulic Engineering, 128(9) (2002), 866-869.

[32] V. Srinivasan and T. P. Clement, Analytical solutions for sequentially coupled one-dimensional reactive transport problems-Part-I: Mathematical Derivations, Advances Water Resources, 31 (2008), 203-218.

[33] M. Th. Van Genucheten and W. J. Alves, Analytical solution of one dimensional convective-dispersion solute transport equation, Tech Bull 1661: (1982), 1-51.

[34] M. Thangarajan, editor Groundwater: Resource Evaluation, Augmnetation, Contamination, Restoration, Modeling and Managament, 362p. Capital Publishing Company, New Delhi, India (2006).

[35] USEPA, Transport and Fate of Contaminants in the Subsurface EPA 625(19), 4-89 Washington, DC (1989).

[36] USEPA, Groundwater and contamination EPA 625(016a):6-90, Office of Research and Development, Washington, DC (1990).
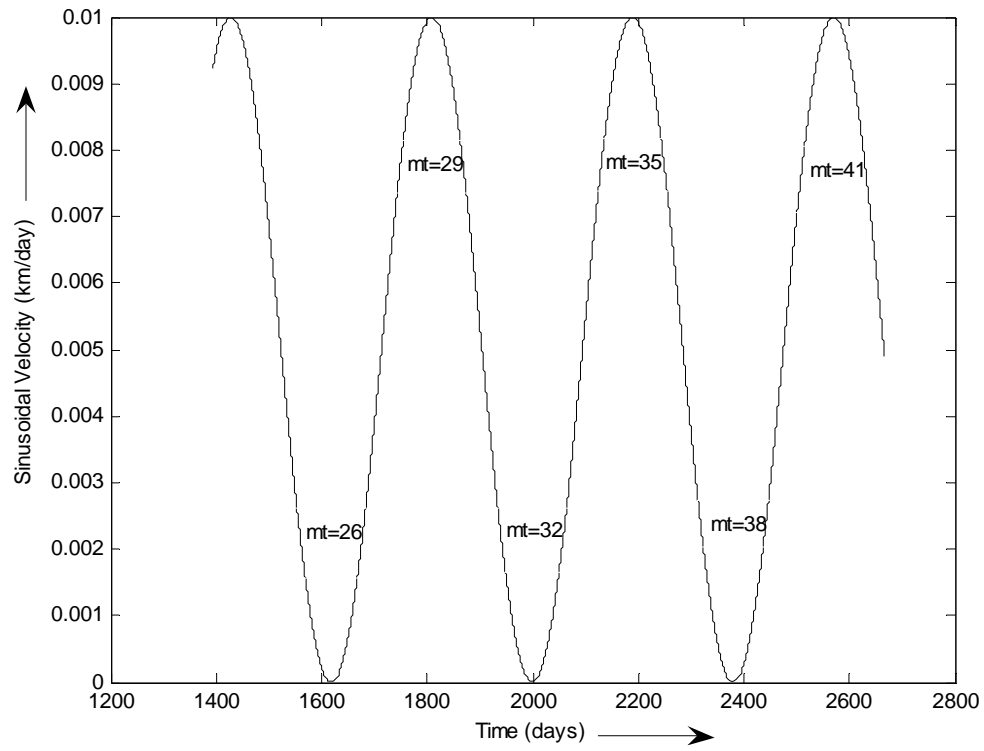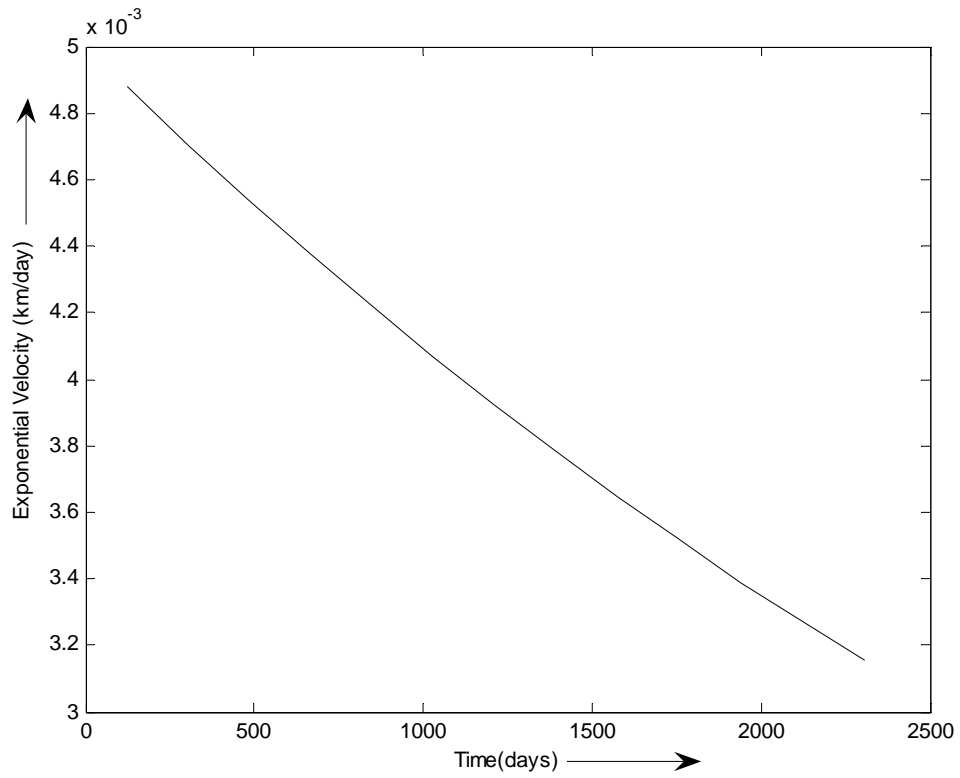
Fig. 1(a) Sinusoidally varying velocity

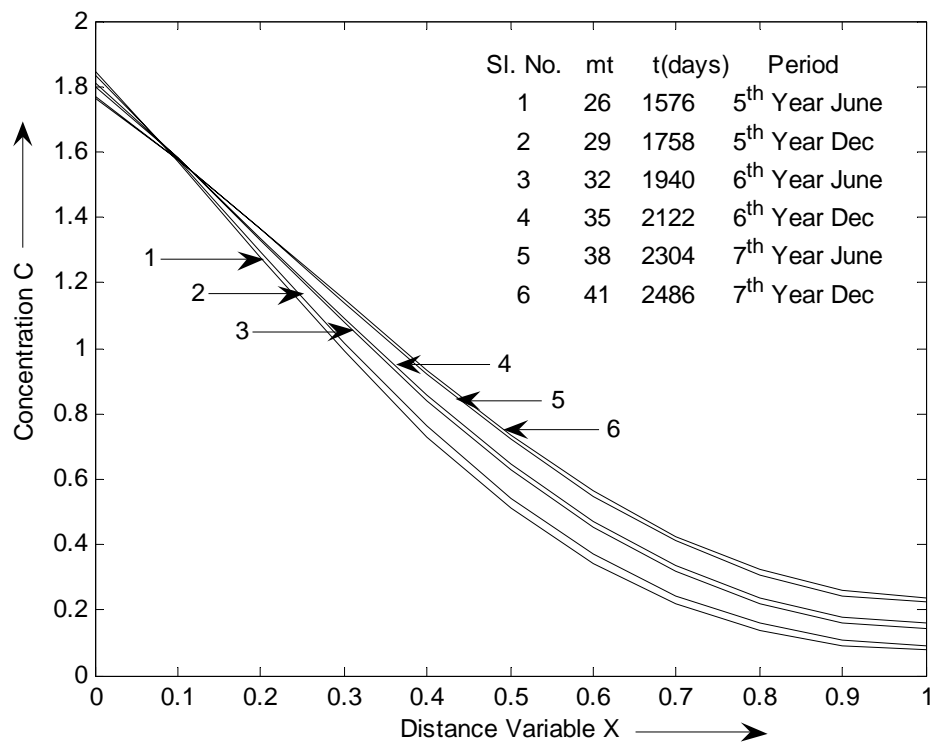Fig. 1(b) Exponentially decreasing velocity

SI. No.	mt	t(days)	Period
1	26	1576	5th Year June
2	29	1758	5th Year Dec
3	32	1940	6th Year June
4	35	2122	6th Year Dec
5	38	2304	7th Year June
6	41	2486	7th Year Dec

Fig. 2(a) Contaminant concentration along unsteady groundwater flow of sinusoidally

varying velocity

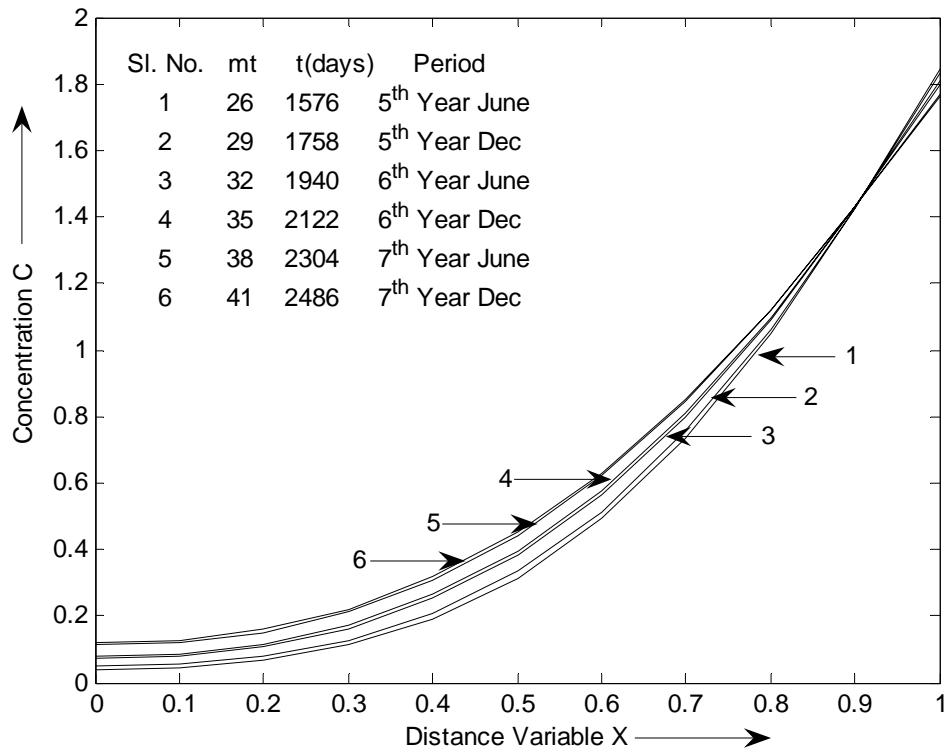| Sl. No.	mt	t(days)	Period
1	26	1576	5th Year June
2	29	1758	5th Year Dec
3	32	1940	6th Year June
4	35	2122	6th Year Dec
5	38	2304	7th Year June
6	41	2486	7th Year Dec

Fig. 2(b) Contaminant concentration against unsteady groundwater flow of
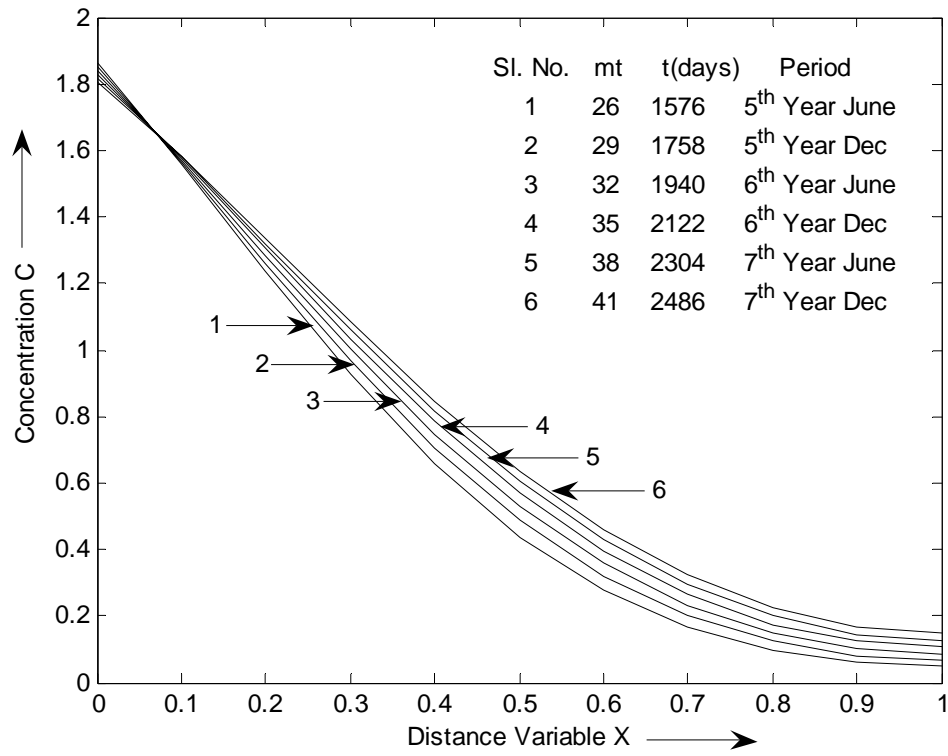
sinusoidally varying velocity

Fig. 3(a) Contaminant concentration along unsteady groundwater flow of
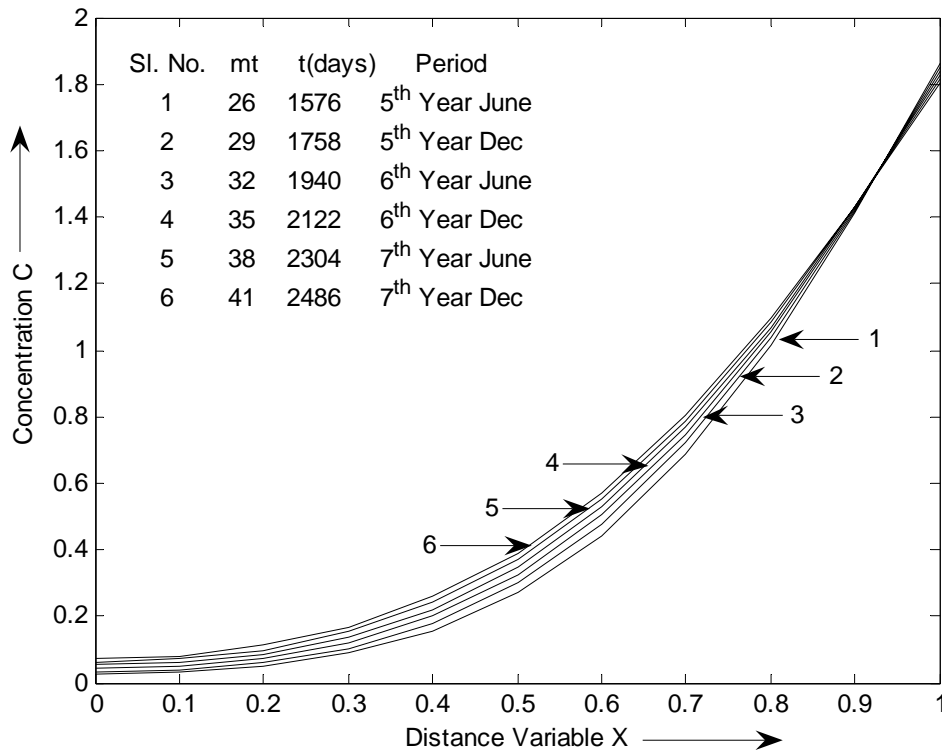
exponentially decreasing velocity

Fig. 3(b) Contaminant concentration against unsteady groundwater flow of

exponentially decreasing velocity