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Numerical Solution of nth-Order Integro-Differential Equations with Trigonometric Wavelet

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Abstract

Integro-differential equations are appeared in many engineering and physic fields. We select fredholm integro-differential equation and basis gained from trigonometric wavelet scaling function. In this procedure, we use collocation method as a projection method to convert integral equation to the system of linear equation. It seems that due to the nature of trigonometric wavelet, the use of this wavelets, it makes little error in contrast to the use of other wavelets. Finally some numerical examples indicate the accuracy of this method.

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1 Introduction

Sometimes in the science and other branches of mathematics such as mathematic modeling or ordinary differential equations, the result of the work leads to the solution of integro-differential equation, the importance of this method leads us to the solution of this equations. In the present study we attempt to solve a large class of this equations. We consider the following integro-differential equation of second kind:

$$\begin{cases} f^{(n)}(x) = g(x) + \int_0^{2\pi} k(x,t) f(t)\, dt \\ \alpha_i = f^{(n-i)}(0)\,; i = 1, 2, ..., n \end{cases} \tag{1}$$

Where k(x,t) and g(x) are known functions,and f(x) is an unknown function.In this paper for solving this equation firstly,we define some scaling basis function and use them to approximate the unknown function.Basis function in this work are scaling function which will be defined in the next section.

2 Trigonometric Scaling and Wavelet Function

Forℓ $\in$N, the Dirichlet kernel $D_\ell \in T_\ell$ and the conjugate Dirichlet kernel $\tilde{D}_\ell \in T_\ell$ are defined as

$$D_\ell(X) = \frac{1}{2} + \sum_{K=1}^{\ell} Coskx = \begin{cases} \frac{\sin(\ell+1/2)x}{2\sin(x/2)} & x \notin 2\pi\mathbf{Z} \\ \ell + 1/2 & x \in 2\pi\mathbf{Z} \end{cases}$$

and

$$\tilde{D}_\ell(x) = \sum_{k=1}^{\ell} \sin(kx) = \begin{cases} \frac{\cos(x/2)-\cos(\ell+1/2)x}{2\sin(x/2)} & x \notin 2\pi\mathbf{Z} \\ 0 & x \in 2\pi\mathbf{Z} \end{cases}$$

Where T_ℓ denotes the linear space of trigonometric polynomial of degree ℓ.The inner product $\langle .,. \rangle$ of two functions f and g in $L^2_{2\pi}$ is defined as usual,by $\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}dx$. For any j$\in N$, consider the following two functions:

$$\phi^0_{j,0} = \frac{1}{2^{2j+1}} \sum_{\ell=0}^{2^{j+1}-1} D_\ell(x)$$

$$\phi^1_{j,0}(x) = \frac{1}{2^{2j+1}}(\tilde{D}_{2^{j+1}-1}(x) + \frac{1}{2}\sin(2^{j+1}x))$$

let $x_{j,n} = \frac{n\pi}{2^j}$,for n=0,1,. . . ,$2^{j+1} - 1$; define $\phi^0_{j,n} = \phi^0_{j,0}(x - x_{j,n})$,and $\phi^1_{j,n}(x) = \phi^1_{j,n}(x - x_{j,n})$, for any j$\in N$ and n=0,1,. . . , $2^{j+1} - 1$. The following interpolatory properties hold for each k=0,1,. . . ,$2^{j+1} - 1$:

$$\phi^0_{j,n}(x_{j,k}) = \delta_{k,n}\ , \quad \left(\phi^0_{j,n}(x_{j,k})\right)' = 0\ , \quad \phi^1_{j,n}(x_{j,k}) = 0\ , \quad \left(\phi^1_{j,n}(x_{j,k})\right)' = \delta_{k,n}$$

Where

$$\delta_{k,n} = \begin{cases} 1 & k = n \\ 0 & k \neq n \end{cases}$$

Given j$\in N$, the space V_j is defined by $V_j = span\left\{\phi^0_{j,n}, \phi^1_{j,n}, n = 0, 1, ..., 2^{j+1} - 1\right\}$ It is easy to see that the spaces V_j ,j=0,1,. . . ,J-1,form a sequence of nested subspaces of $L^2_{2\pi}$ with:

$$L^2_{2\pi} = \overline{\bigcup_{j=-1}^{+\infty} V_j}\ ,\ \bigcap_{j=-1}^{+\infty} V_j = \{0\}\ , V_j \subset V_{j+1}$$

Where $L^2_{2\pi}$ is a set of square-integrable 2π- periodic functions and $V_{-1} = \{0\}$. There for $\{V_j\}^{+\infty}_{j=-1}$ forms a hermite-type MRA of $L^2_{2\pi}$. The wavelet spaces are defined by $W_j = span\left\{\psi^0_{j,n}, \psi^1_{j,n}, n = 0, 1, ..., 2^{j+1} - 1\right\}$,where:

$$\psi^0_{j,0} = \frac{1}{2^{j+1}}\cos(2^{j+1}x) + \frac{1}{3 \times 2^{2j+1}} \sum_{\ell=2^{j+1}+1}^{2^{j+2}-1} (3 \times 2^{j+1} - \ell)\cos(\ell x)$$

$$\psi^1_{j,0} = \frac{1}{2^{2j+3}}\sin(2^{j+2}x) + \frac{1}{3 \times 2^{2j+1}} \sum_{\ell=2^{j+1}+1}^{2^{j+2}-1} \sin(\ell x)$$

For any j$\in N$ and n=0,1,. . . ,$2^{j+1} - 1$, define $\psi^0_{j,n}(x) = \psi^0_{j,0}(x - x_{j,n})$, and $\psi^1_{j,n}(x) = \psi^1_{j,0}(x - x_{j,n})$. It can be verified that $V_j \oplus W_j = V_{j+1}$and $L^2_{2\pi} = V_0 \oplus (\oplus^{+\infty}_{j=0} W_j)$.
For J$\in$N , it is obtained basis set $\left\{\phi^0_{J,n}, \phi^1_{J,n}, 0 \leq k \leq 2^{J+1} - 1\right\}$;for approximation space $V_J \subset L^2_{2\pi}$.For any function $f \in L^2_{2\pi}$, based on the dual function $\tilde{\phi}^p(x)$ and $\tilde{\psi}^p(x)$ in [2],p=0,1. we define the orthogonal projection operator P_j as $P_j f(x) = \sum_{k=0}^{2^{j+1}-1} \tilde{f}^0_{j,k}\phi^0_{j,k}(x) + \sum_{k=0}^{2^{j+1}-1} \tilde{f}^1_{j,k}\phi^1_{j,k}(x)$,where $P_j f \in V_j$,and $\tilde{f}^p_{j,k} = \left\langle f, \tilde{\phi}^p_{j,k}\right\rangle$ is the scaling coefficient,p=0,1.It follows that $P_J f(x) = \sum_{k=0}^{2^{J+1}-1}(\tilde{f}^0_{J,k}\phi^0_{J,k}(x) + \tilde{f}^1_{J,k}\phi^1_{J,k}(x)) = \sum_{j=-1}^{J-1}\sum_{k=0}^{2^{j+1}-1}(\hat{f}^0_{j,k}\psi^0_{j,k}(x) + \hat{f}^1_{j,k}\psi^1_{j,k}(x))$,where $\hat{f}^p_{j,k} = \left\langle f, \tilde{\psi}^p_{j,k}\right\rangle$is the trigonometric wavelet coefficient of the function f(x) and p=0,1.
Remark. Define the orthogonal complement projection Q_j to satisfy $Q_j = P_{j+1} - P_j$ and $Q_j f = \sum_{k=0}^{2^{j+1}-1}(\hat{f}^0_{j,k}\psi^0_{j,k}(x) + \hat{f}^1_{j,k}\psi^1_{j,k}(x))$, thus, for any f(x)$\in L^2_{2\pi}$ and J$\in N$, it follows that $f - P_J f(x) = \sum_{j=J}^{+\infty}(P_{j+1} - P_j)f = \sum_{j=J}^{+\infty}(Q_j f)$, and when $J \to +\infty$ we have $P_J f(x) \to f(x)$.

3 Collocation Method

In this section we use the wavelet basis functions which are introduced in the previous section to approximate the unknown function in the Eq (1). we consider $f^{(n)}(x)$ continuous, so we have:

$$f^{(n)}(x) = \sum_{k=0}^{2^{J+1}-1} c_{J,k}\phi_{J,k}^{0}(x) + \sum_{k=0}^{2^{J+1}-1} d_{J,k}\phi_{J,k}^{1}(x). \tag{2}$$

with integration n times as $\int_0^x$ from sides (2) and using the following formula

$$\underbrace{\int_0^x ... \int_0^x}_{n} A(t) = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} A(t)\, dt$$

we have:

$$f(x) = \sum_{k=0}^{2^{J+1}-1} c_{J,k}\int_0^x \frac{(x-t)^{n-1}}{(n-1)!}\phi_{J.k}^{0}(t)\, dt + \sum_{k=0}^{2^{J+1}-1} d_{J,k}\int_0^x \frac{(x-t)^{n-1}}{(n-1)!}\phi_{J.k}^{1}(t)\, dt +$$

$\sum_{i=0}^{n-1} \frac{x^i}{i!}\alpha_{(n-i)}$.
by substituting $f(t), f^{(n)}(x)$ in Eq(1) and some changes, we have:

$$\sum_{k=0}^{2^{J+1}-1} c_{J,k}\left(\phi_{J,k}^{0}(x) - \int_0^{2\pi} k(x,t)\left(\int_0^t \frac{(t-r)^{n-1}}{(n-1)!}\phi_{J,k}^{0}(r)\, dr\right) dt\right) +$$

$$\sum_{k=0}^{2^{J+1}-1} d_{J,k}\left(\phi_{J,k}^{1}(x) - \int_0^{2\pi} k(x,t)\left(\int_0^t \frac{(t-r)^{n-1}}{(n-1)!}\phi_{J,k}^{1}(r)\, dr\right) dt\right) =$$

$$g(x) + \sum_{i=0}^{n-1} \frac{\alpha_{(n-i)}}{i!}\int_0^{2\pi} k(x,t)\, t^i dt$$

by regard to the above relation we can define residual function as

$$R_n(x) = \sum_{k=0}^{2^{J+1}-1} c_{J,k}\left(\phi_{J,k}^{0}(x) - \int_0^{2\pi} k(x,t)\left(\int_0^t \frac{(t-r)^{n-1}}{(n-1)!}\phi_{J,k}^{0}(r)\, dr\right) dt\right) +$$

$$\sum_{k=0}^{2^{J+1}-1} d_{J,k}\left(\phi_{J,k}^{1}(x) - \int_0^{2\pi} k(x,t)\left(\int_0^t \frac{(t-r)^{n-1}}{(n-1)!}\phi_{J,k}^{1}(r)\, dr\right) dt\right) -$$

$$g(x) - \sum_{i=0}^{n-1} \frac{\alpha_{(n-i)}}{i!}\int_0^{2\pi} k(x,t)\, t^i dt$$

Now for determining unknown coefficients $c_{J,k}$ and $d_{J,k}$, we have some alternatives. In this paper, we choose collocation method which is defined as follows

In collocation method we select some collocation points then let residual equation in this point equal to zero. With choose collocation point as

$$x_i = b + \frac{i\,(b-a)}{2^{J+2}} = \frac{i\pi}{2^{J+1}}; i = 0, 1, ..., 2^{J+2} - 1$$

we have

$$R_n(x) = \sum_{k=0}^{2^{J+1}-1} c_{J,k} \left(\phi^0_{J,k}(x_i) - \int_0^{2\pi} k(x_i, t) \left(\int_0^t \frac{(t-r)^{n-1}}{(n-1)!} \phi^0_{J,k}(r)\, dr \right) dt \right) +$$

$$\sum_{k=0}^{2^{J+1}-1} d_{J,k} \left(\phi^1_{J,k}(x_i) - \int_0^{2\pi} k(x_i, t) \left(\int_0^t \frac{(t-r)^{n-1}}{(n-1)!} \phi^1_{J,k}(r)\, dr \right) dt \right) -$$

$$g(x_i) - \sum_{i=0}^{n-1} \frac{\alpha_{(n-i)}}{i!} \int_0^{2\pi} k(x_i, t)\, t^i dt \begin{cases} k = 0, 1, ..., 2^{J+1} - 1 \\ i = 0, 1, ..., 2^{J+2} - 1 \end{cases}$$

Thus, we have system of linear equation $A_J X = b_J$ where $A_J = \begin{bmatrix} A^1 & A^2 \\ A^3 & A^4 \end{bmatrix}$, $A^p = (a^p_{ik})_{2^{J+1} \times 2^{J+1}}$, p=1,2,3,4. $X = \begin{bmatrix} c_{J,k} \\ d_{J,k} \end{bmatrix}$.$b_J = \begin{bmatrix} b^1 \\ b^2 \end{bmatrix}$.and:

$$a^1_{ik} = \phi^0_{J,k}(x_i) - \int_0^{2\pi} k(x_i, t) \left(\int_0^t \frac{(t-r)^{n-1}}{(n-1)!} \phi^0_{J,k}(r)\, dr \right) dt; \begin{cases} i = 0, 1, ..., 2^{J+1} - 1 \\ k = 0, 1, ..., 2^{J+1} - 1 \end{cases}$$

$$a^2_{ik} = \phi^1_{J,k}(x_i) - \int_0^{2\pi} k(x_i, t) \left(\int_0^t \frac{(t-r)^{n-1}}{(n-1)!} \phi^1_{J,k}(r)\, dr \right) dt; \begin{cases} i = 0, 1, ..., 2^{J+1} - 1 \\ k = 2^{J+1}, ..., 2^{J+2} - 1 \end{cases}$$

$$a^3_{ik} = \phi^0_{J,k}(x_i) - \int_0^{2\pi} k(x_i, t) \left(\int_0^t \frac{(t-r)^{n-1}}{(n-1)!} \phi^0_{J,k}(r)\, dr \right) dt; \begin{cases} i = 2^{J+1}, ..., 2^{J+2} - 1 \\ k = 0, 1, ..., 2^{J+1} - 1 \end{cases}$$

$$a^4_{ik} = \phi^1_{J,k}(x_i) - \int_0^{2\pi} k(x_i, t) \left(\int_0^t \frac{(t-r)^{n-1}}{(n-1)!} \phi^1_{J,k}(r)\, dr \right) dt; \begin{cases} i = 2^{J+1}, ..., 2^{J+2} - 1 \\ k = 2^{J+1}, ..., 2^{J+2} - 1 \end{cases}$$

$$b^1_i = g(x_i) + \sum_{i=0}^{n-1} \frac{\alpha_{(n-i)}}{i!} \int_0^{2\pi} k(x_i, t)\, t^i dt; i = 0, 1, ..., 2^{J+1} - 1$$

$$b^2_i = g(x_i) + \sum_{i=0}^{n-1} \frac{\alpha_{(n-i)}}{i!} \int_0^{2\pi} k(x_i, t)\, t^i dt; i = 2^{J+1}, ..., 2^{J+2} - 1$$

4 Numerical Example

In this section we present some numerical examples to illustrate the stated method in this paper.

Example 1. In this example we solve equation

$$\begin{cases} f^{(1)}(x) = xCosx + Sinx - x + \int_0^{\frac{\pi}{2}} xf(t)\,dt \\ f(0) = 0 \end{cases}.$$

Where the exact solution is f(x)=xSinx , and results are shown in Fig 1 and 2.

Example 2. In this example we solve equation:

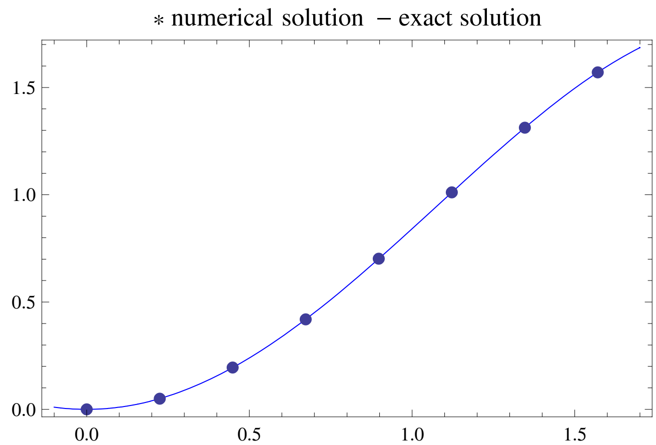

Figure 1: Result for J=1

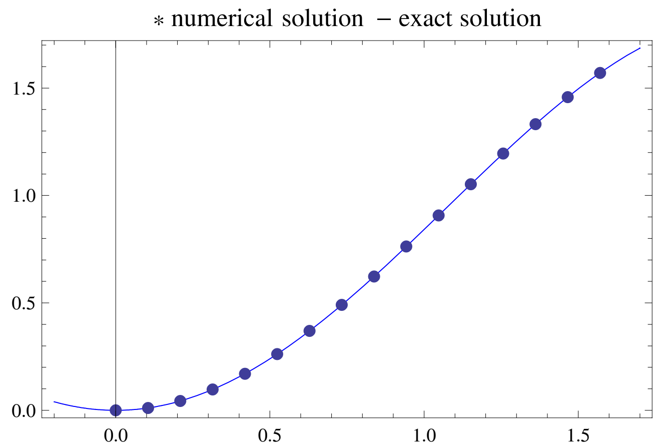

Figure 2: Result for J=2

$$\begin{cases} f^{(2)}(x) = -Sinx + x - \int_0^{\frac{\pi}{2}} xtf(t)\,dt \\ f(0) = 0, f^{(1)}(0) = 1 \end{cases}.$$

Where the exact solution is f(x)=Sinx, and Results are shown in Fig 3,4.

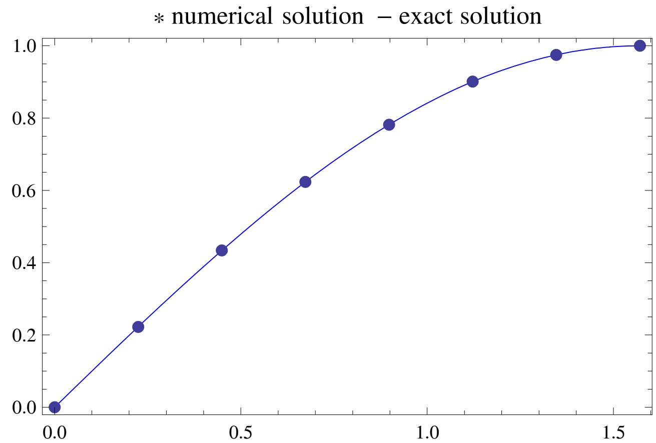

Figure 3: Result for J=1

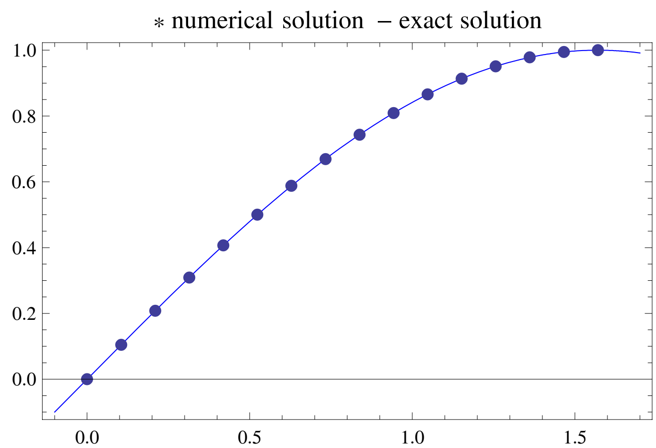

Figure 4: Result for J=2

5 Conclusion

In this paper , we reduce the integral equation to a linear system of equation using collocation method with trigonometric wavelet basis. figure(1),(2),(3), and (4), show numerical solution convergence to exact solution.
This shows the efficiency trigonometric wavelet in the solution of integral equations. we can easily generalize this numerical method to the solution of other equation like

$$\begin{cases} f^{(n)}(x) = g(x) + \int_0^{2\pi} k(x,t) f^{(m)}(t)\,dt; n > m \\ \alpha_i = f^{(n-i)}(0); i = 1, 2, ..., n \end{cases}$$

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