

Central Limit Theorem and Moderate Deviations Principle for Dependent Risks

Li-min Liu

College of Mathematics and Information Science, Henan Normal University
Xinxiang City, Henan Province, 453007, China

Dan-dan Pu

College of Mathematics and Information Science, Henan Normal University
Xinxiang City, Henan Province, 453007, China
pudandan@126.com

Abstract

In the present paper, we consider the models for large dependent risk portfolios, especially for credit risk models. Furthermore, the central limit theorem and moderate deviations principle of models for large risk portfolios are obtained by analyzing its mathematical structure.

Keywords: Dependence structures; Copulas; Central limit theorems; Moderate deviations principle; Mixing distributions

1 Introduction

In an influential paper, Marshall and Olkin [7] introduced the multivariate mixture models, which are a special family of copulas. Multivariate mixture models depend on a latent factor Θ (the frailty parameter). This frailty parameter plays the role of common economic conditions (the so-called systematic risks). Recently, Schönbucher [10] and McNeil et al. [8] have further studied the models, and Maier and Wüthrich [11] obtained the large deviation and law of large numbers for dependent risks by analyzing the mathematical structure of the multivariate mixture models.

In [11], these stochastic behaviors were studied with the help of limiting behaviors. In particular, they proved that if the latent parameter Θ has a bounded support, then they obtained two different regimes of behavior, namely there is a phase transition point: below this phase transition point they obtained law of large numbers or central limit theorem behavior; above this phase

transition point they obtained large deviation behavior. This means that, for values of portfolio losses above the phase transition point, they observed an exponential decay of the corresponding probabilities. The authors studied multivariate mixture models, which obtained two different regimes: below the phase transition point they model systematic risks that cause joint defaults of several risks; above the phase transition point the risks behave like independent random variables and they observed typical diversification effects for independent risks.

The definition of a multivariate mixture model of the dependence structure proceeds via Laplace transforms (see Marshall and Olkin [7, formula (2.6)]): let F be the common marginal distribution function of $X_i, i = 1, \dots, n$, and M_Θ a univariate distribution with $M_\Theta(0) = 0$. Denote ψ by the Laplace transform of M_Θ . Then we assume that random vector X_1, \dots, X_n has the following distribution for $(x_1, \dots, x_n) \in R^n$:

$$G(x_1, \dots, x_n) = \int_0^\infty \prod_{i=1}^n \exp\{-\psi^{-1}(F(X_i))\}^\theta dM_\Theta(\theta) \quad (1)$$

For technical reasons we assume that the marginals X_i are non-negative, i.e., we can restrict ourselves to $X_i > 0, i = 1, \dots, n$.

In this paper, motivated by their works, we prove the central limit theorem and moderate deviation for the aggregated risk portfolio by depending on the choice of the distribution of Θ .

2 Central limit theorem

We introduce the following notation for the conditional mean and of the conditional variance X_1 , given Θ :

$$A(\Theta) = E[X_1|\Theta] = \int_0^\infty 1 - \exp\{-\Theta \cdot \psi^{-1}(F(x))\} dx, \quad (2)$$

$$\begin{aligned} B(\Theta) &= \text{Var}[X_1|\Theta] = B_1(\Theta) - [A(\Theta)]^2 \\ &=: 2 \int_0^\infty x(1 - \exp\{-\Theta \cdot \psi^{-1}(F(x))\}) dx \\ &\quad - \left(\int_0^\infty 1 - \exp\{-\Theta \cdot \psi^{-1}(F(x))\} dx \right)^2. \end{aligned} \quad (3)$$

Theorem 2.1. *Assume that the multivariate distribution of X_1, \dots, X_n ($n \geq 1$) is given by (1) and that the marginal distribution F has a finite second moment. Then, for all $x \in R$, we have*

$$\lim_{n \rightarrow \infty} P \left(\frac{\sum_{i=1}^n (X_i - A(\Theta))}{\sqrt{nB(\Theta)}} \leq x \mid \Theta \right) \xrightarrow{\mathcal{D}} \Phi(x), \quad M_\Theta - a.s. \quad (4)$$

where $\Phi(\cdot)$ denotes the distribution function of standard normal random variable.

Proof We first prove that $B(\Theta)$ is finite, M_Θ -a.s. The proof is based on the ideas of Maiery and Wüthrich in [13]. For $n = 1$, observe that

$$G(x_1) = \int_0^\infty \exp\{-\psi^{-1}(F(x_1))\}^\theta dM_\Theta(\theta) = \psi\{-\psi^{-1}(F(x_1))\} = F(x_1) \quad (5)$$

which implies that, for $X_1 \sim F$ (using our assumptions of finite second moments),

$$E[X_1^2] = \int_0^\infty E[X_1^2|\Theta = \theta]dM_\Theta(\theta) = \int_0^\infty B_1(\theta)dM_\Theta(\theta) < \infty \quad (6)$$

with, conditionally, given $\Theta = \theta$, $X_1 \sim \exp\{-\theta \cdot \psi^{-1}(F(\cdot))\}$. This immediately shows that $B_1(\Theta)$ is finite M_Θ -a.s., which yields that $B(\Theta)$ is also finite M_Θ -a.s. by the expression (3)

From (1), we see that, conditionally, given Θ , the components of X_1, \dots, X_n ($n \geq 1$) are i.i.d. with mean $A(\Theta)$ and variance $B(\Theta)$. Henceforth, using the central limit theorem, we obtain conditional distribution of X_1 , given Θ , for $n \rightarrow \infty$:

$$P\left(\frac{\sum_{i=1}^n(X_i - A(\Theta))}{\sqrt{nB(\Theta)}} \leq x|\Theta\right) \xrightarrow{\mathcal{D}} \Phi(x), \quad M_\Theta - a.s. \quad (7)$$

Corollary 2.2. *Under the assumptions of Theorem 2.1, we have the following asymptotic estimate: for any $x \in R$,*

$$P\left(\frac{\sum_{i=1}^n X_i}{\sqrt{n}} \leq x\right) \approx \int \Phi\left(\frac{x - \sqrt{n}A(\theta)}{\sqrt{B(\theta)}}\right) dM_\Theta(\theta).$$

Proof From Theorem 2.1, it is easy to check that for all sufficiently large n , we have

$$\begin{aligned} P\left(\frac{\sum_{i=1}^n X_i}{\sqrt{n}} \leq x\right) &= E\left(P\left(\frac{\sum_{i=1}^n X_i}{\sqrt{n}} \leq x\right) \middle| \Theta\right) \\ &= E\left(P\left(\frac{\sum_{i=1}^n (X_i - A(\Theta))}{\sqrt{nB(\Theta)}} \leq \frac{\sqrt{n}x - nA(\Theta)}{\sqrt{nB(\Theta)}}\right) \middle| \Theta\right) \\ &\approx E\left(\Phi\left(\frac{x - \sqrt{n}A(\theta)}{\sqrt{B(\theta)}}\right) \middle| \Theta\right) = \int \Phi\left(\frac{x - \sqrt{n}A(\theta)}{\sqrt{B(\theta)}}\right) dM_\Theta(\theta). \end{aligned} \quad (8)$$

3 Moderate deviations principle

In the last section we have proved the central limit theorem, which give a asymptotic distribution when n is sufficiently large. Furthermore, in this section, we will obtain the moderate deviations principle for the above model, which show the exponential convergence rate of $\sum_{i=1}^n X_i/n$. Assume that the marginal of X_1 satisfies the following exponential integrable condition: there exists a positive constant $\delta > 0$, such that

$$E \exp \{ \delta |X_1| \} < \infty. \quad (9)$$

In addition, we define the average default ratio as follows

$$Z_n = \frac{1}{\sqrt{nb_n}} \sum_{i=1}^n (X_i - A(\Theta)) \quad (10)$$

where (b_n) is the moderate deviation scale, i.e., it is a sequence of positive numbers satisfying $b_n \rightarrow \infty$, $\frac{\sqrt{n}}{b_n} \rightarrow \infty$.

Theorem 3.1. *Suppose that the condition (9) holds, then we have the moderate deviations principle for Z_n , i.e., for any $r > 0$,*

$$\lim_{n \rightarrow \infty} \frac{1}{b_n^2} \log P(Z_n \geq r) = -\frac{r^2}{2E(B(\Theta))}.$$

Proof For any $\lambda \in R$, let us consider the following logarithmic moment generating function of Z_n ,

$$\Lambda_n(\lambda) := \log E \exp(\lambda Z_n). \quad (11)$$

By the Gärtner-Ellis Theorem (cf. [1]), in order to obtain the desired result, it is enough to get the following claim

$$\lim_{n \rightarrow \infty} \frac{1}{b_n^2} \Lambda_n(\lambda b_n^2) = \frac{\lambda^2 E(B(\Theta))}{2}. \quad (12)$$

It is easy to see that

$$\begin{aligned} \frac{1}{b_n^2} \Lambda_n(\lambda b_n^2) &= \frac{1}{b_n^2} \log E \exp \left(\frac{\lambda b_n}{\sqrt{n}} \sum_{i=1}^n (X_i - A(\Theta)) \right) \\ &= \frac{1}{b_n^2} \log E \left\{ E \left[\exp \left(\frac{\lambda b_n}{\sqrt{n}} \sum_{i=1}^n (X_i - A(\Theta)) \right) \middle| \Theta \right] \right\} \\ &= \frac{1}{b_n^2} \log E \left\{ \prod_{i=1}^n E \left[\exp \left(\frac{\lambda b_n}{\sqrt{n}} (X_i - A(\Theta)) \right) \middle| \Theta \right] \right\} \\ &= \frac{n}{b_n^2} \log E \left\{ E \left[\exp \left(\frac{\lambda b_n}{\sqrt{n}} (X_1 - A(\Theta)) \right) \middle| \Theta \right] \right\}. \end{aligned}$$

By the condition (9) and Taylor's formula, we have

$$\lim_{n \rightarrow \infty} \frac{1}{b_n^2} \Lambda_n(\lambda b_n^2) = \frac{\lambda^2 E(\text{Var}(X_1|\Theta))}{2},$$

which is the claim (12).

Corollary 3.2. *Assume that the conditions in Theorem 3.1 hold, then with $I(r)$ defined as $I(r) = \frac{r^2}{2E(B(\Theta))}$, we have for any closed $F \subset R$,*

$$\limsup_{n \rightarrow \infty} \frac{1}{b_n^2} \log P(Z_n \in F) \leq - \inf_{r \in F} I(r)$$

and for any open $G \subset R$,

$$\limsup_{n \rightarrow \infty} \frac{1}{b_n^2} \log P(Z_n \in G) \geq - \inf_{r \in G} I(r).$$

References

- [1] A. Dembo and O. Zeitouni, *Large Deviations Techniques and Applications*, 2nd edn. Springer, New York, 1998.
- [2] A. Joe, *Multivariate Models and Dependence Concepts*, Chapman and Hall, London, 1997.
- [3] A. Juri and M. V. Wüthrich, Tail dependence from a distributional point of view, *Extremes*, **6** (2004), 213-246.
- [4] A.W. Marshall and I. Olkin, Families of multivariate distributions, *Journal of the American Statistical Association*, **83** (198) 834-841.
- [5] D. Li, On default correlation: a copula function approach, *Journal of Fixed Income*, **9**(2001), 43-54.
- [6] I. H. Dinwoodie and S. L. Zabell, Large deviations for exchangeable random vectors, *The Annals of Probability*, **3** (1992), 1147-1166.
- [7] J. Galambos, *The Asymptotic Theory of Extreme Order Statistics*, 2nd ed, Krieger, Malabar, 1987.
- [8] M. Denuit, J. Dhaene, M. Goovaerts and R. Kaas, *Actuarial Theory for Dependent Risks*, Wiley, New York, 2005.
- [9] N. Whelan, Sampling from Archimedean copulas, *Quantitative finance*, **4** (2004), 339-352.
- [10] P. J. Schönbucher, Taken to the limit: simple and not-so-simple loan loss distributions, *Wilmott magazine*, **1** (2003), 63-72.
- [11] R. Maiery and M. V. Wüthrich, Law of large numbers and large deviations for dependent risks, *Quantitative finance*, **2** (2009), 207-215.

Received: November, 2009