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On a Nonlinear Problem Intervening in Relativistic Quantum Mechanics Perturbed by a Factor of Amortizement

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Abstract

In this work, we study the existence and the uniqueness and the regularity of the solution of a problem governed by a nonlinear equation intervening in relativistic quantum mechanics perturbed by an amortizement factor

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1. Notations

Let Ω is an open bounded domain of IR^n, $\Sigma = \Omega \times \left]0,T\right[$, where T is a finite positive real number. For each real functions f and g, we pose: $\left(f,g\right) = \int_\Omega f(x)g(x)dx$, and $\left|f\right|^2 = (f,f)$. We denote by $\left\|f\right\|_p$ the norm of the function f in $L^p(\Omega)$, $\quad 1 \le p \le \infty$.

2. Position of the Problem

Let $\alpha \in \left]0,+\infty\right[$, suppose that f is given in $L^2(\Sigma)$.
Our problem is: Find u such that: $u \in L^\infty(0,T;H_0^1(\Omega) \cap L^p(\Omega))$ (2.1)

and $$\frac{\partial u}{\partial t} \in L^{\infty}(0,T;L^2(\Omega)) \quad (2.2)$$

And u is solution of the following problem (P)

$$(P)\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \varepsilon \dfrac{\partial u}{\partial t} - \Delta u + |u|^{\alpha} u = f \quad in \quad \Sigma & (2.3) \\ u = 0 \quad on \quad \Gamma \times]0,T[& (2.4) \\ u(x,0) = u_0(x); \quad x = (x_1, x_2, \ldots.., x_n) \in \Omega & (2.5) \\ \dfrac{\partial u}{\partial t}(x,0) = u_1(x); \quad x = (x_1, x_2, \ldots.., x_n) \in \Omega & (2.6) \end{cases}$$

The equation (2.3) models a phenomenon that occurs in relativistic quantum mechanics perturbed by $\varepsilon \frac{\partial u}{\partial t}$; $\varepsilon > 0$ called amortizement factor for this phenomenon.

We will use the two following lemmas and the following corollary showed in [3].

Lemma 2.1-. *Let be $f \in L^p 0,T;X)$ and $\frac{\partial f}{\partial t} \in L^p 0,T;X)$ where X is a Banach space and $1 \le p \le \infty$. Then $f : [0,T] \to X$ is continious almost every where.*

Lemma 2.2. *Let be θ a bounded open of $IR_x^n \times IR_t$ and g_μ , g two functions of $L^q(\theta)$; $1 < q < \infty$ such that $\|g_\mu\|_{L^p(\theta)} \le C$ and $g_\mu \to g$ almost every where in θ. Then $g_\mu \to g$ in $L^q(\Omega)$ weakly.*

Corollary 2.1. *There exists a sequence $\{w_i\}_1^{\infty} \in H_0^1(\Omega) \cap L^p(\Omega)$ such that for any $m \ge 0$, the vectors $w_1, w_2, \ldots\ldots, w_m$ are linearly independent and any the subspace spanned by this vectors is dense in $H_0^1(\Omega) \cap L^p(\Omega)$*

Theorem 2.1.- *Let be given f, u_0, u_1 such that:*

$$f \in L^2(\Sigma) \quad (2.7)$$

$$u_0 \in H_0^1(\Omega) \cap L^p(\Omega) \qquad p = \alpha + 2 \quad (2.8)$$

$$u_1 \in L^2(\Omega) \quad (2.9)$$

Then the problem (P) admits a solution.

Proof.- The proof of this theoerm 2.1, is done in three steps.

Step 1.- Approximation.-

We pose $u' = \frac{\partial u}{\partial t}$, $u'' = \frac{\partial^2 u}{\partial t^2}$, and $a(u,v) = \sum_{i=1}^{n} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_j} dx$

We use the base introduced into the above corollary, and we define an approximate solution $u_m = u_m(t)$ of the problem (P) under the form :

$u_m(t) = \sum_{i=1}^{m} g_{im}(t) w_i$ such that for $j = 1,2,\ldots,m$, we obtain:

$$\left(u''_m(t), w_j\right) + \varepsilon\left(u'_m(t), w_j\right) + a\left(u_m(t), w_j\right) + \left(\left|u_m(t)\right|^\rho u_m(t), w_j\right) = \left(f(t), w_j\right) \qquad (2.10)$$

with :

$$u_m(0) = u_{0m} \qquad u_{0m} = \sum_{i=1}^{m} \alpha_{im} w_i \to u_0 \text{ in } H_0^1(\Omega) \cap L^p(\Omega) \text{ when } m \to \infty \qquad (2.11)$$

$$u'_m(0) = u_{1m} \qquad u_{1m} = \sum_{i=1}^{m} \beta_{im} w_i \to u_1 \text{ in } L^2(\Omega) \text{ when } m \to \infty \qquad (2.12)$$

This is a nonlinear ordinary differential system. If one takes into account conditions (2.11) and (2.12), this system admits a solution on $[0, t_m]$, where $t_m \le T$. The following priori estimates will show that $t_m = T$

Step 2 Priori estimates

By multiplying (2.10) by $g'_{jm}(t)$ and sum over j, we obtain:

$$\left(u''_m(t), u'_m(t)\right) + a(u_m(t), u'_m(t)) + \varepsilon\left\|u'_m(t)\right\|_2^2 + \left(\left|u_m(t)\right|^\alpha u_m(t), u'_m(t)\right) = \left(f(t), u'_m(t)\right)$$

According to $\alpha = p - 2$ (see (2.8)), the above expression can be written as:

$$\frac{1}{2}\frac{d}{dt}\left[\left\|u'_m(t)\right\|_2^2 + a(u_m(t), u_m(t)) + \frac{1}{p}\left(\int_\Omega \left|u_m(x,t)\right|^p dx\right)\right] + \varepsilon\left\|u'_m(t)\right\|_2^2 = \left(f(t), u'_m(t)\right) \quad (2.13)$$

We pose $\|v\|^2 = a(v,v)$ this is a norm in $H_0^1(\Omega)$ equivalent to the usual norm in $H^1(\Omega)$ [1]. Along this work the norms without index are those defined in $H_0^1(\Omega)$ by the above definition. From (2.13), we deduce that :

$$\frac{1}{2}\left[\left\|u'_m(t)\right\|_2^2 + \left\|u_m(t)\right\|^2)\right] + \frac{1}{p}\left\|u_m(t\right\|_p^p \le \frac{1}{2}\left(\left\|u_{1m}\right\|_2^2 + \left\|u_{0m}\right\|^2\right) + \frac{1}{p}\left\|u_m(0)\right\|_p^p + \varepsilon\int_0^t \left\|u'_m(\sigma)\right\|_2^2 d\sigma$$
$$+ \int_0^t \left\|f(\sigma)\right\|_2 \left\|u'_m(\sigma)\right\|_2 d\sigma \qquad (2.14)$$

From (2.11) and (2.12), we deduce that: There exists a constante $C > 0$ such that:

$$\frac{1}{2}\left(\left\|u_{1m}\right\|_2^2 + \left\|u_{0m}\right\|_2^2\right) + \frac{1}{p}\left\|u_m(0)\right\|_p^p \le C$$

This implies that:

$$\frac{1}{2}\left[\left|u'_m(t)\right|^2 + \left\|u_m(t)\right\|^2)\right] + \frac{1}{p}\left\|u_m(t)\right\|_{L^p(\Omega)}^p \le C + \frac{1}{2}\int_0^t \left\|f(\sigma)\right\|_2^2 d\sigma + \frac{1-2\varepsilon}{2}\int_0^t \left\|u'_m(t)\right\|_2^2 d\sigma$$

Since f is given in $L^2(\Sigma)$, we deduce that: There exists a constante $K > 0$ such that: $\int_0^t \left|f(\sigma)\right|^2 \le K$. From the above inequality, we deduce that: There is a constante $K_3 > 0$ independent to m such that: $\left\|u'_m(t)\right\| + \left\|u'_m(t)\right\|_{L^p(\Omega)} \le K_3$

Therefore $t_m = T$, moreover when $m \to \infty$, u_m remains in a bounded of $L^\infty(0,T; H_0^1(\Omega) \cap L^p(\Omega))$ and u'_m in a bounded of $L^\infty(0,T; L^2(\Omega))$

Step 3 Taking the limit

According to the Danford Pettis theorem cf. [5], $L^{\infty}(0,T;H_0^1(\Omega)\cap L^p(\Omega))$ respectively $L^{\infty}(0,T;L^2(\Omega))$ is the dual space of $L^1(0,T;H^{-1}(\Omega)+L^q(\Omega))$ respectively $L^1(0,T;L^2(\Omega))$ where $\left(\frac{1}{p}+\frac{1}{q}=1\right)$

Consequently one can extract from $\{u_m\}$ the subsequence $\{u_\mu\}$ such that:

$u_\mu \to u$ in $L^{\infty}(0,T;H_0^1(\Omega)\cap L^p(\Omega))$ weak star (2.18)

and $u'_\mu \to u'$ dans $L^{\infty}(0,T;L^2(\Omega))$ weak star

According to (1.23), we deduce that $\{u_m\}$ is bounded in $L^2(0,T;H_0^1(\Omega))$ and $\{u'_m\}$ is bounded in $L^2(0,T;L^2(\Omega))$ then $\{u_m\}$ remains in a bounded of $H^1(\Sigma)$.

However from the Rellich Kondrachoff compcteness theorem see [2], the injection of $H^1(\Sigma)$ in $L^2(\Sigma)$ is compact. We can then suppose that $u_\mu \to u$ in $L^2(\Sigma)$ strongly and almost everywhere (2.19)

Since the application $v \to |v|^{\alpha} v$ maps $L^p(\Omega)$ on $L^q(\Omega)$ where $\left(\frac{1}{p}+\frac{1}{q}=1\right)$,

then $|u_\mu|^{\alpha} u_\mu$ remains in a bounded domain of $L^{\infty}(0,T;L^q(\Omega))$, thus there exists $w \in L^{\infty}(0,T;L^q(\Omega))$ such that $|u_\mu|^{\alpha} u_\mu \to w$ weak star (2.20)

It remains to show that $w = |u|^{\alpha} u$:

But this results from(2.19) and (2.20) and the lemma 2.2 with $\theta = \Sigma$;

$g_\mu = |u_\mu|^{\alpha} u_\mu$, $q = \frac{\alpha+2}{\alpha+1}$, where $\left(\frac{1}{p}+\frac{1}{q}=1\right)$ and from (2.18), we deduce that:

$g_\mu \to |u|^{\alpha} u = g$ and $g_\mu \to w$ in $L^q(\theta)$; $1 < q < \infty$ weakly and from this we obtain $w = g = |u|^{\alpha} u$. According to the lemma 2.2, we can now take the limit in (2.10) that we use for $m = \mu$. Let *j* is a fixed natural number such that $\mu > j$ then:

$$\left(u''_\mu(t), w_j\right) + \varepsilon\left(u'_\mu(t), w_j\right) + a\left(u_\mu(t), w_j\right) + \left(|u_\mu(t)|^{\alpha} u_\mu(t), w_j\right) = \left(f(t), w_j\right) \qquad (2.21)$$

According to (2.18), we obtain $a(u_\mu, w_j) \to a(u, w_j)$ in $L^{\infty}(0,T)$ weak star and $(u'_\mu, w_j) \to (u', w_j)$ in $L^{\infty}(0,T)$ weak star.

then $(u''_\mu, w_j) = \frac{d}{dt}(u'_\mu, w_j) \to (u'', w_j)$ in $D'(0,T)$ and according (2.20), we have $\left(|u_\mu|^{\alpha} u_\mu, w_j\right) \to \left(|u|^{\alpha} u, w_j\right)$ in $L^{\infty}(0,T)$ weak star

One deduces from (2.21) that :

$$\frac{d^2}{dt^2}\left(u, w_j\right) + \varepsilon \frac{d}{dt}\left(u, w_j\right) + a\left(u, w_j\right) + \left(|u|^{\alpha} u, w_j\right) = \left(f, w_j\right)$$

According to the properties of the density of the base $w_1, w_2, \ldots\ldots, w_m$, see the corollary 2.1, we deduce that:

$$\frac{d^2}{dt^2}(u,v)+\varepsilon\frac{d}{dt}(u,v)+a(u,v)+\left(|u|^{\alpha}u,v\right)=(f,v) \qquad \forall v\in H_0^1(\Omega)\cap L^p(\Omega)$$

then u satisfies (2.3) ,(2.1) and (2.2). It remains to check (2.5) and(2.6). According to (2.11) , (2.12) and the lemma 2.2, we have $u_\mu(0)\to u(0)$ in $L^2(\Omega)$ weakly, but according to (2.11) $u_\mu(0)=u_{0\mu}\to u_0$ in $H_0^1(\Omega)\cap L^p(\Omega)$ then (2.5) is checked. Using the lemma 2.1, we obtain: $\left(u_\mu^{''},w_j\right)\to\left(u^{''},w_j\right)$ in $L^\infty(0,T)$ weak star. So using the lemma 2.1 with $X=IR$, we will have $\left(u_\mu^{'}(0),w_j\right)\to\left(u^{'},w_j\right)_{|t=0}=\left(u^{'}(0),w_j\right)$, from (2.12) we deduce that :

$\left(u^{'}(0),w_j\right)\to\left(u_1,w_j\right)$, on a $\left(u^{'}(0),w_j\right)=\left(u_1,w_j\right)$ $\forall j$ then $\dfrac{\partial u}{\partial t}(0)=u_1$

3. Uniqueness

Theorem 3.1. *Using the hypothesis of the theorem 2.1 with the condition:*

$$\alpha\le\frac{2}{n-2} \quad \textit{if } n\ne 2 \ (\alpha \textit{ is any finite real number if } n=2) \tag{3.1}$$

Then the problem (P) admits an unic solution.

Proof.-

Let u and v are two solutions of the problem (P), then $w=u-v$ checks :

$$w^{''}+\varepsilon w^{'}-\Delta w=|v|^{\alpha}v-|u|^{\alpha}u \tag{3.2}$$

$$w(0)=0;\ \ w^{'}(0)=0$$

$$w\in L^\infty(0,T;H_0^1(\Omega)\cap L^p(\Omega))$$

$$w^{'}\in L^\infty(0,T;L^2(\Omega))$$

by multiplying the two members of (3.2) by $w^{'}$ and integrating over Ω, it becames :

$$\frac{1}{2}\frac{d}{dt}\left[\left\|w^{'}(t)\right\|_2^2+\|w(t)\|^2\right]+\varepsilon\|w(t)\|_2^2=\int_\Omega\left(|v|^{\alpha}v-|u|^{\alpha}u\right)w^{'}dx \tag{3.3}$$

According to the Hölder inequality, the above expression is majored by :

$c\left(\left\||u|^{\alpha}\right\|_n+\left\||v|^{\alpha}\right\|_n\right)\|w(t)\|_q\left\|w^{'}(t)\right\|_2$ where $\dfrac{1}{q}+\dfrac{1}{n}+\dfrac{1}{2}=1$

from (3.1), we deduce that $\alpha n\le q$. According to the prologation theorem of Sobolev [2], we get $H_0^1(\Omega)\subset L^q(\Omega)$ with $\dfrac{1}{q}+\dfrac{1}{n}+\dfrac{1}{2}=1$, $n>2$. Then:

$$\left|\int_\Omega\left(|v|^{\alpha}v-|u|^{\alpha}u\right)w^{'}dx\right|\le c\left(\left\|\,|u|^{\alpha}\right\|_n+\left\|\,|v|^{\alpha}\right\|_n\right)\|w(t)\|\,\left\|w^{'}(t)\right\|_2 \tag{3.4}$$

Since $u,v\in L^\infty(0,T;H_0^1(\Omega))$ we obtain finaly:

$$\left[\left\|w^{'}(t)\right\|_2^2+\|w(t)\|^2\right]\le\frac{c_2}{2}\left\|w^{'}(\sigma)\right\|_2^2+\frac{c_2-2\varepsilon}{2}\int_0^t\|w(\sigma)\|^2d\sigma \qquad \text{Then } \ w=0$$

4 Regularity

Theorem 4.1.- *Let us return to the hypothesis of the theorem 2.1, that we use with the following complementary conditions:*

$$\frac{\partial f}{\partial t} \in L^2(\Sigma) \tag{4.1}$$

$$u_0 \in H_0^1(\Omega) \cap H^2(\Omega) \tag{4.2}$$

$$u_1 \in H_0^1(\Omega) \tag{4.3}$$

$$\alpha \le \frac{2}{n-2} \qquad (\alpha \text{ is a finite real number if } n = 2) \tag{4.4}$$

Then the problem (P) admits an unic solution u checking

$$u \in L^\infty(0,T;H_0^1(\Omega) \cap H^2(\Omega)) \tag{4.5}$$

$$\frac{\partial u}{\partial t} \in L^\infty(0,T;H_0^1(\Omega)) \tag{4.6}$$

$$\frac{\partial^2 u}{\partial t^2} \in L^\infty(0,T;L^2(\Omega)) \tag{4.7}$$

Proof.-
Existence
We will use the base defined in the corollary 2.1 and that introduced int the step1 of the proof of the theorem 2.1, but in this case this base is defined in $H_0^1(\Omega) \cap H^2(\Omega)$.

We suppose that $u_{0m} \to u_0$ in $H_0^1(\Omega) \cap H^2(\Omega)$ and $u_{1m} \to u_1$ in $H_0^1(\Omega)$ (4.8)

And we use the same methode as that used in proff of the theorem 2.1

Uuniqueness.-
We use the same technic as that used in the proof of the theorem3.1
It becames: There is a constante $c > 0$ such that

constante *M > 0* such that: $\frac{1}{2}\frac{d}{dt}\left[\|w'(t)\|_2^2 + \|w(t)\|^2\right] \le M\|w(t)\|\,\|w'(t)\|_2$

Since $\|w(t)\|\,\|w'(t)\|_2 \le \|w(t)\|^2 + \|w'(t)\|_2^2$ and by applying the Gronwal Lemma [5] and [6], we deduce $w = 0$.

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