

**Exponential Self Similar Solutions Technique for
Instability Phenomenon Arising in Double Phase Flow
through Porous Medium with Capillary Pressure**

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Abstract

The theoretical studies involved development of the differential equations which describe the instability of water by an oil-saturated rock. The basic assumption underlying the present investigation is that the oil and water form two immiscible liquid phases and the latter represents preferentially wetting phase. A formal solution for Saturation of injected water is calculated by Exponential Self Similar Solutions Technique for Nonlinear differential equation of instability phenomena.

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1. Introduction

It is well known that when a fluid contained in a porous medium is displaced by another of lesser viscosity, instead of regular displacement of whole front, perturbation may occur which shoot through the porous medium at relatively great speed. These perturbations are called fingers. Most of the earlier authors have completely neglected the capillary pressure. It was Mehta [3] who included capillary pressure in the analysis of fingers. This problem has great importance for oil production in petroleum technology.

The present chapter deals with phenomenon of instability (fingering) in double immiscible phase flow through homogeneous porous media (viz. oil and water) with mean capillary pressure viz. the assumption made is that the individual pressure of the two flowing phases may be replaced by their common mean pressure and the behavior of the fingers is determined by a statistical treatment. The mathematical formulation yields to a nonlinear differential equation. Displacement of oil from porous medium by an external force which gives rise to a pressure gradient is known as forced instability. Instability can occur in co-current and counter-current flow modes. Liu [2] Putra and Schechter [6]; Tang and Firoozabadi, [8] investigated the effects of injection rate, initial water saturation and gravity on water injection in slightly water-wet fractured porous media.

In this paper we have obtained an approximate solution of the nonlinear differential system governing instability phenomena arising in the flow of two immiscible fluid flows through homogeneous porous media with the effect of capillary pressure by Exponential Self Similar Solutions Technique.

2. Statement of the Problem

We consider here that there is a uniform water injection into oil saturated porous media of homogeneous physical characteristics such that the injecting water flows through the oil formation and gives rise to perturbation (fingers). This furnishes a well developed fingers flow. Since the entire oil at the initial boundary $x = 0$ (x being measured in the direction of displacement), is displaced through a small distance due to water injection. Therefore, it is further assumed that complete saturation exists at the initial boundary.

Our particular interest in the present paper is to obtain an analytical expression for the cross sectional area occupied by fingers. For the mathematical formulation, we consider that Darcy's law is valid for the investigated flow system and assumed further that the macroscopic behavior of fingers is governed by a statistical treatment. In the statistical treatment of finger only the average behavior of the two fluids involved is taken into consideration. It was shown by Scheidegger and Johnson [7], that this treatment of motion with the introduction of the concept of fictitious relative permeability

become formally identical to the Buckley Leverett description of two immiscible fluid flow in porous media .The saturation of water (S_w) is then defined as the average cross sectional area occupied by water at level x, i.e. $S_w(x, t)$. Thus the saturation of displacing fluid in porous medium represents the average cross-sectional area occupied by fingers.



Fig.1.Instability phenomena

3. Mathematical Formulation and Solution of the Problem

The filtration velocity of injected liquid (water) (v_w) and native liquid (oil) (v_o) Scheidegger [7] may be written as (by Darcy’s law)

$$v_o = -\left(\frac{k_o}{\mu_o} K\right) \frac{\partial p_o}{\partial x} \tag{1}$$

$$v_w = -\left(\frac{k_w}{\mu_w} K\right) \frac{\partial p_w}{\partial x} \tag{2}$$

Where ‘K’ is the permeability of homogenous medium, ‘ k_o ’ and ‘ k_w ’ are relative permabilities of native liquid(oil) and injected liquid (water), which are function of ‘ S_o ’, ‘ S_w ’. (‘ S_w ’ and ‘ S_o ’ are saturation of water and oil) respectively and ‘ p_w ’ and ‘ p_o ’ denote the pressure of water and oil, while ‘ μ_w ’, ‘ μ_o ’ are constant kinematic viscosities of the phases.

The equation of continuity (phase densities are regarded as constant) are

$$\phi \frac{\partial S_w}{\partial t} + \frac{\partial v_w}{\partial x} = 0 \tag{3}$$

$$\phi \frac{\partial S_o}{\partial t} + \frac{\partial v_o}{\partial x} = 0 \tag{4}$$

Where ‘ ϕ ’ is the porosity of the medium.From the definition of capillary pressure (p_c) Scheidegger [7] as the pressure discontinuity between the flowing phase yields

$$p_c = p_o - p_w \tag{5}$$

From the definition of phase saturation, it is evident that $S_o + S_w = 1$ (6)

The equation of motion for saturation can be obtained by substituting the value of v_w and v_o from (1) and (2) in equation (3) and (4) respectively, we get

$$\phi \frac{\partial S_w}{\partial t} = \frac{\partial}{\partial x} \left[\left(\frac{k_w}{\mu_w} K \right) \frac{\partial p_w}{\partial x} \right] \quad (7)$$

$$\phi \frac{\partial S_o}{\partial t} = \frac{\partial}{\partial x} \left[\left(\frac{k_o}{\mu_o} K \right) \frac{\partial p_o}{\partial x} \right] \quad (8)$$

Eliminating $\frac{\partial p_w}{\partial x}$ from equation (5) and (7) we have $\phi \frac{\partial S_w}{\partial t} = \frac{\partial}{\partial x} \left[\frac{k_w}{\mu_w} K \left(\frac{\partial p_o}{\partial x} - \frac{\partial p_c}{\partial x} \right) \right]$ (9)

Combining equation (8), (9) and using (6) we have

$$\frac{\partial}{\partial x} \left[\left(\frac{k_w}{\mu_w} K + \frac{k_o}{\mu_o} K \right) \frac{\partial p_o}{\partial x} - \frac{k_w}{\mu_w} K \frac{\partial p_c}{\partial x} \right] = 0 \quad (10)$$

On integrating (10) with respect to x we get

$$\left[\left(\frac{k_w}{\mu_w} K + \frac{k_o}{\mu_o} K \right) \frac{\partial p_o}{\partial x} - \frac{k_w}{\mu_w} K \frac{\partial p_c}{\partial x} \right] = C \quad (11)$$

Where 'C' is the constant of integration.

From (11) $\frac{\partial p_o}{\partial x} = \frac{C}{\left[\frac{k_w}{\mu_w} K + \frac{k_o}{\mu_o} K \right]} + \frac{1}{1 + \left(\frac{k_o \mu_w}{\mu_o k_w} \right)} \frac{\partial p_c}{\partial x}$ (12)

By putting the value of $\frac{\partial p_o}{\partial x}$ in (9) it gives $\phi \frac{\partial S_w}{\partial t} + \frac{\partial}{\partial x} \left[\frac{\frac{k_o}{\mu_o} K}{\left(1 + \frac{k_o \mu_w}{\mu_o k_w} \right)} \frac{\partial p_c}{\partial x} - \frac{C}{\left(1 + \frac{k_o \mu_w}{\mu_o k_w} \right)} \right] = 0$ (13)

The value of the pressure of native liquid (p_o) can be written as

$$p_o = \bar{p} + \frac{1}{2} (p_c) \quad \text{Where } \bar{p} = \frac{p_o + p_w}{2} \quad \text{where } \bar{p} \text{ the Mean pressure and is constant.}$$

Now $\frac{\partial p_o}{\partial x} = \frac{1}{2} \left(\frac{\partial p_c}{\partial x} \right)$, substituting the value of $\left(\frac{\partial p_o}{\partial x} \right)$ in (11) it gives

$$C = \frac{1}{2} \left(\frac{\partial p_c}{\partial x} \right) K \left(\frac{k_o}{\mu_o} - \frac{k_w}{\mu_w} \right) \quad (14)$$

On substituting the value of C we obtain in equation (13)

$$\phi \frac{\partial S_w}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x} \left[K \frac{k_w}{\mu_w} \frac{dp_c}{dS_w} \frac{\partial S_w}{\partial x} \right] = 0 \quad (15)$$

This equation (15) is a non-linear partial differential equation, which describes the Instability phenomenon of two immiscible fluids flow through homogeneous porous cylindrical medium with impervious bounding surfaces on three sides.

It is well known that fictitious relative permeability is the function of displacing fluid saturation. Then at this stage for definiteness of the mathematical analysis, we assume standard forms of Scheidegger and Johnson [7] for the analytical relationship between the relative permeability, phase saturation and capillary pressure phase saturation Mehta [3] as

$k_w = S_w$, $p_c = -\beta S_w$ (β is constant) (-ve sign shows the direction of saturation of water opposite to capillary pressure) Mehta [3]

Substituting this value p_c in equation (15)

$$\frac{\partial S_w}{\partial t} - \frac{k\beta}{2\mu_w\phi} \frac{\partial}{\partial x} \left[S_w \frac{\partial S_w}{\partial x} \right] = 0 \tag{16}$$

Changing equation (16) in to dimensionless form by substituting

$$X = \frac{x}{L} , \quad T = \left(\frac{k\beta}{2\phi\mu_w L^2} \right) t \quad \text{We get} \quad \frac{\partial S_w}{\partial T} - \frac{\partial}{\partial X} \left[S_w \left(\frac{\partial S_w}{\partial X} \right) \right] = 0 \tag{17}$$

with suitable boundary condition $S_w(X,0) = f(X) = e^{-X}$ (18)

Because S_w will decrease linearly as X increase which is significant with phenomenon of instability, Mehta [3]

This is the desired non-linear partial differential equation describing the instability phenomenon with capillary pressure.

4. Exponential Self Similar Solutions Technique

An exponential self similar solution is a solution of the form

$$S_w(X,T) = e^{\alpha T} V(\xi) , \quad \xi = X e^{\beta T} \tag{19}$$

An exponential self similar solution exists if the equation under consideration is invariant under the transformation

$$T = \bar{T} + \ln C , \quad X = C^k \bar{X} , \quad S_w = C^m \bar{S}_w \quad \text{where } C > 0 \text{ is an arbitrary constant,} \tag{20}$$

For some k and m . Transformation (20) is a combination of a shift in T and scaling in X and S_w . We observe that these transformations contain an arbitrary constant C as a parameter.

The above existence criterion is checked, If a pair of k and m in (20) has been found such that the equation remains the same, then there exists an exponential self similar solution with the new variables having the form (19), where $\alpha = m$ and $\beta = -k$ (21)

These relations follow from the condition that the scaling transformation (20) must preserve the form of the variables of (19):

$$S_w = e^{\alpha T} V(\xi), \xi = X e^{\beta T} \quad \Rightarrow \quad \bar{S}_w = e^{\alpha T} V(\bar{\xi}), \bar{\xi} = \bar{X} e^{\beta T}$$

Consider Equation (17) admits an exponential self similar solution. Substituting (20) in to

$$(17) \text{ yields } C^m \frac{\partial \bar{S}_w}{\partial T} = C^{2m-2k} \frac{\partial}{\partial X} \left(\bar{S}_w \frac{\partial \bar{S}_w}{\partial X} \right)$$

Equating the exponent of C, we obtain one linear equation. $m = 2m - 2k$. Hence, we have $k = \frac{m}{2}$, where m is arbitrary. Further, using formulas (19) and (21) and taking

(without loss of generality) $m=2$, which is equivalent to scaling of time T, we find new variables: Polynon [5] $S_w = e^{2T} V(\xi)$, $\xi = X e^{-T}$ (22)

Inserting these into (17), we obtain an ordinary differential equation for the function

$$V(\xi) \text{ as } \frac{d}{d\xi} \left(V \frac{dV}{d\xi} \right) + \xi \frac{dV}{d\xi} - 2V = 0 \quad (23)$$

On Substituting $V(\xi) = \xi^2 U(z)$, $z = \log \xi$ and $U'(z) = P$ in (23) takes the form $UPP'(U) + P^2 + (7U + 1)P + 6U^2 = 0$ (24)

This is Abel's equation of second kind whose solution can be obtained by considering the substitution $UP = \log \theta(z)$, Murphy [4] (25)

Substituting (25) into (24), we get

$$\frac{1}{\theta} \frac{d\theta}{dz} + (7U + 1)P + 6U^2 = 0 \quad (26)$$

Further supposing $\log \theta(z) = M(z)$, then equation (26) can take the form

$$\frac{dM}{dz} + \left(7 + \frac{1}{U}\right)M(z) + 6U^2 = 0 \quad (27)$$

The solution of equation (27) is

$$Me^{\int \left(\frac{7}{\xi} + \frac{1}{V\xi}\right) d\xi} = -\int \frac{6V^2}{\xi^5} e^{\int \left(\frac{7}{\xi} + \frac{1}{V\xi}\right) d\xi} d\xi + C \quad (\text{Where } C \text{ is constant.}) \quad (28)$$

At $t = 0$, $\xi = X$ and $V = e^{-X}$ for $0 \leq X \leq 1$, Equation (28) gives

$$C = Me^{\int_0^1 \frac{1}{X} (7+e^X) dX} + \int_0^1 \frac{6e^{-2X}}{X^5} e^{\int_0^1 \frac{1}{X} (7+e^X) dX} dX$$

Then equation (28) takes the form

$$M \left(e^{\int \left(\frac{7}{\xi} + \frac{1}{V\xi}\right) d\xi} - e^{\int_0^1 \frac{1}{X} (7+e^X) dX} \right) = -\int \frac{6V^2}{\xi^5} e^{\int \left(\frac{7}{\xi} + \frac{1}{V\xi}\right) d\xi} d\xi + \int_0^1 \frac{6e^{-2X}}{X^5} e^{\int_0^1 \frac{1}{X} (7+e^X) dX} dX \quad (29)$$

Equation (29) gives the formal solution in terms of transcendental function.

Conclusion

In this paper we have obtained the analytical solution of the nonlinear differential equation of Instability phenomena by using Exponential Self Similar Solutions technique and the possibility for deriving an expression for the wetting phase saturation in approximate form has been discussed. The analytical solution (29) represents solution of instability phenomena in terms of nonlinear equation (17) with initial condition (18). This solution (29) gives the formal solution in terms of transcendental function satisfies initial condition (18).

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