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EOQ Problem Well-Posedness: an Alternative Approach Establishing Sufficient Conditions

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Abstract

An existence-uniqueness theorem is proved about a minimum cost order for a class of inventory MAB models, leading to a sufficient conditions. They consist of doing a check of some improper integrals, and form the article's main theoretical contribution to the subject. The reason of this article consists of the alternative approach to [5] of proving the existence/uniqueness of the Economic Order Quantity for a generalized Wilson model by means of the integral of the inverse function.

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1 Background and motivation

When the first T Fords were rolling off the assembly line (1913), manufacturers were already appraising the financial benefits of inventory management determining the minimum cost answers. As a matter of fact, long before JIT (Just In Time), TQM (Total Quality Management), TOC (Theory Of Constraints), and MRP (Manufacturing Resources Planning), companies were using these same (then unnamed) concepts in managing both their production and inventory. Economic Order Quantity (EOQ) is a set of models defining the optimal quantity to order that minimizes the total variable costs required to both: order and hold inventory. Inventory models for calculating optimal order quantities and reorder points have been in existence long before the arrival of the computer: in fact the first of them was originally developed by Harris, [3] (1913), though Wilson, [6] (1934), is credited for his early in-depth analysis of the subject. Basic underlying assumptions:

1. the demand for the item is known, and deterministic;

2. no lead time (between order and arrivals) is taken into account;

3. the receipt of the order occurs in a single instant and immediately after ordering it;

4. quantity discounts are not calculated as part of the model;

5. the ordering cost A is a constant.

Several extensions can be made to the EOQ model: the items deterministic demand can change with amount itself or with time; the model can include backordering costs and multiple items. Should they undergo deterioration, perishability can be modelled either constant or variable with the items stock level. Additionally, the economic *order* quantity, and the order interval can be evaluated from the EOQ so that the EPQ, namely the optimal *production* quantity, can be determined in a similar fashion. The above determinism is not the only possible approach to the problem: there is the probabilistic one, but we will keep out of it.

The idea exposed in this paper, even if born first, comes to the press, following an article already published, [5] (2008), but where the authors, through a different approach, generalized its analytic base. Nevertheless its original formulation keeps its interest as a readable contribution as a primer in the authors' research line on EOQ.

2 The MAB models with fixed specific costs

Throughout all this paper $q(t)$ means the instantaneous stock level of our inventory at time t, which is ruled by a blowdown dynamics:

$$\begin{cases} \dot{q}(t) = -f(q(t)) \\ q(0) = Q > 0 \end{cases} \tag{2.2.1}$$

where $f : [0, \infty[\to \mathbb{R}$ is a continuous and positive function, so that the solution to (2.2.1) meets $q(t) \leq Q$ for each $t \geq 0$. Due to the autonomous (2.2.1) nature, we solve by quadratures: if

$$F(q) := \int_q^Q \frac{1}{f(u)}\,\mathrm{d}u = t \tag{2.2.2}$$

then $q(t) = F^{-1}(t)$ solves (2.2.1). We will treat models compliant with the assumptions of the previous section, with monotonic blowdown ($\dot{q} < 0$ for each t) and then named MAB (Monotonic Autonomous Blowdown), whose version with delivery $A > 0$ and holding $h > 0$ time-invariant *specific* costs will be analyzed here. We define *reordering time* generated by Q the real value $T(Q) > 0$ capable of getting zero the solution of (2.2.1):

$$T(Q) = F(0) = \int_0^Q \frac{1}{f(u)}\,\mathrm{d}u.$$

Minding A and h meanings, then the *total* cost for (delivering + holding) a whichever $Q > 0$ amount of goods will be:

$$C(Q) = \frac{A}{T(Q)} + \frac{h}{T(Q)} \int_0^{T(Q)} q(t)\,\mathrm{d}t. \tag{2.2.3}$$

The early Wilson model [6] is found again when the instantaneous rate $\dot{q}$ of stock depletion is assumed to have a constant magnitude: $f(q) = \delta > 0$. The further ones due to Goh, [2] (1994), and to Giri and Chaudhuri, [1] (1998), will correspond to $f(q) = \delta q^{\beta}$ with $0 < \beta < 1$ and to $f(q) = \vartheta\, q + \delta\, q^{\beta}$ with $\delta > 0$ and $0 < \vartheta,\, \beta < 1$.

3 The existence of a minimum cost

The well known MAB models have not been studied in their general features so far. What we mean to do is what has not been done yet, namely to establish, on the above assumptions, some conditions sufficient to ensure that a cost function like (2.2.3) attains a minimum, and eventually its unicity. Such a research arises adequately motivated because the minimum computation requires a certain transcendental equation to be solved numerically. Then it makes a little sense to go on without proofs of existence and unicity of this aim. Nevertheless such a research is neglected at all in literature which faces single particular problems without making sure of their well-posedness.

Let us start with the cost function (2.2.3). It will be recalled that for integrating an inverse function, say g^{-1}, one can also operate on the direct one, see Key (1994):

$$\int_{g(b)}^{g(a)} g^{-1}(y)\,\mathrm{d}y = ag(a) - bg(b) + \int_a^b g(x)\,\mathrm{d}x$$

for $g : [a, b] \to \mathbb{R}$ strictly decreasing function. We will prove the following:

Lemma 3.1. *Assume that both functions:*

$$\frac{1}{f(u)}, \quad \frac{u}{f(u)}.$$

are integrable at the origin. Then the cost function (2.2.3) *relevant to a Q-amount of goods can be expressed by:*

$$C(Q) = \frac{A + h\displaystyle\int_0^Q \frac{u}{f(u)}\,\mathrm{d}u}{\displaystyle\int_0^Q \frac{1}{f(u)}\,\mathrm{d}u}. \tag{3.3.1}$$

Proof. (3.3.1) stems applying to (2.2.3) the aforementioned formula for inverse integration:

$$C(Q) = \frac{A}{T(Q)} + \frac{h}{T(Q)} \int_{F(Q)}^{F(0)} F^{-1}(t)\,\mathrm{d}t = \frac{A}{T(Q)} + \frac{h}{T(Q)} \int_0^Q F(q)\,\mathrm{d}q.$$

Then, minding (2.2.2), through the double integrals reduction, we get:

$$C(Q) = \frac{A}{T(Q)} + \frac{h}{T(Q)} \int_0^Q \left(\int_q^Q \frac{\mathrm{d}u}{f(u)} \right) \mathrm{d}q = \frac{A}{T(Q)} + \frac{h}{T(Q)} \int_0^Q \left(\int_0^u \frac{dq}{f(u)} \right) \mathrm{d}u$$

and then (3.3.1). □

Theorem 3.2. *Assume that:*

$$\int_0^\infty \frac{\mathrm{d}u}{f(u)} = \infty, \tag{3.3.2}$$

or:

$$\int_0^\infty \frac{\mathrm{d}u}{f(u)} \in \mathbb{R}, \quad \int_0^\infty \frac{u}{f(u)}\,\mathrm{d}u = \infty, \tag{3.3.3}$$

or:

$$\int_0^\infty \frac{\mathrm{d}u}{f(u)} \in \mathbb{R}, \quad \int_0^\infty \frac{u}{f(u)}\,\mathrm{d}u \in \mathbb{R}. \tag{3.3.4}$$

In any of such three occurrences the cost function (3.3.1) *attains a minimum for exactly one* $Q^* > 0$ *value.*

Proof. We can immediately check that:

$$\lim_{Q \to 0^+} C(Q) = \infty.$$

Such a first divergence has its theoretical explanation: for a batch of zero consistency the reordering time will be zero, and then each of sub-costs being distributed on a zero time span will give specific infinite costs. Furthermore, if we assume (3.3.2), then the cost function will go to infinite if $Q \to \infty$:

$$\lim_{Q\to\infty} C(Q) = \lim_{Q\to\infty} \frac{\dfrac{h\,Q}{f(Q)}}{\dfrac{1}{f(Q)}} = \infty$$

according to the De l'Hospital rule. This second divergence complies with the fact that a cost function gets unlimited growing, if the batch consistency does so. The double divergence and the continuity of $C(Q)$ as well, imply that $C(Q)$ is bounded from below and then it shall have somewhere at least a critical point. Then a minimum does exist and, in addition, it has to be unique. In fact, the first derivative of $C(Q)$ becomes zero if and only if Q solves the equation:

$$h\,Q \int_0^Q \frac{\mathrm{d}u}{f(u)} - \left\{ A + h \int_0^Q \frac{u}{f(u)}\,\mathrm{d}u \right\} = 0. \tag{3.3.5}$$

Notice that the function

$$Q \to h\,Q \int_0^Q \frac{\mathrm{d}u}{f(u)} - \left\{ A + h \int_0^Q \frac{u}{f(u)}\,\mathrm{d}u \right\} := \mathcal{N}(Q)$$

is a difference of two increasing functions, so that the critical point is unique.

The way to prove our statement is similar if we assume (3.3.3). Finally, if relation (3.3.4) holds, notice that $\mathcal{N}(0) = -A < 0$ and in our specific case we have:

$$\lim_{Q\to\infty} \mathcal{N}(Q) = \infty$$

then only one critical point exists capable of minimizing $C(Q)$. The existence and unicity of the minimum cost is then, in our assumptions, completely proved.

□

The reader can check that for Wilson model, [6], corresponding to $f(u) = \delta$, (3.3.1) gives back the classic optimum condition $hQ^2 - 2\delta A = 0$. For the Goh model, [2], the optimum condition gives $hQ^{2-\beta} - A\,(\beta^2 - 3\beta + 2)\,\delta = 0$. Finally, in Giri-Chaudhuri model, [1], the detection of the optimum batch means that:

$$\frac{hQ}{(1-\beta)\vartheta} \ln\left(1 + \frac{\vartheta}{\delta} Q^{1-\beta}\right) = A + \frac{hQ}{\vartheta}\left[1 - {}_2\mathrm{F}_1\left(\begin{matrix} 1, 1/(1-\beta) \\ (2-\beta)/(1-\beta) \end{matrix} \,\middle|\, -\frac{\vartheta}{\delta} Q^{1-\beta} \right)\right]$$

to be solved to Q. Anyway the above formula involving the Gauss hypergeometric function ${}_2\mathrm{F}_1$is not present in the article [1]; it is founded upon the integral identities:

$$\int_0^Q \frac{\mathrm{d}u}{\vartheta\, u + \delta\, u^\beta} = \frac{1}{\vartheta(1-\beta)} \ln\left(1 + \frac{\vartheta}{\beta} Q^{1-\beta}\right),$$
$$\int_0^Q \frac{u\,\mathrm{d}u}{\vartheta\, u + \delta\, u^\beta} = \frac{1}{\vartheta}\left[Q - Q\,{}_2\mathrm{F}_1\left(\begin{array}{c} 1, 1/(1-\beta) \\ (2-\beta)/(1-\beta) \end{array} \middle| -\frac{\vartheta}{\delta} Q^{1-\beta}\right)\right].$$

We limit here to recall that ${}_2F_1$ is the Gauss hypergeometric x -power series, $|x| < 1$:

$${}_2\mathrm{F}_1\left(\begin{array}{c} a, b \\ c \end{array} \middle| x\right) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n} \frac{x^n}{n!},$$

where $(a)_k$ is a Pochhammer symbol: $(a)_k = a(a+1)\cdots(a+k-1)$. We used the integral representation theorem for ${}_2\mathrm{F}_1$:

$${}_2\mathrm{F}_1\left(\begin{array}{c} a, b \\ c \end{array} \middle| x\right) = \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)} \int_0^1 \frac{t^{a-1}(1-t)^{c-b-1}}{(1-xt)^b}\,\mathrm{d}t,$$

$\mathrm{Re}\, c > \mathrm{Re}\, a > 0$, $|x| < 1$. The IRT provides the way for extending the region where the (complex) hypergeometric function is defined, namely for its analytical continuation to the (almost) whole complex plane excluding the half line $]1, \infty[$.

4 Conclusions

An existence uniqueness theorem 3.2 is proved about a minimum cost batch for a class of inventory MAB models, leading to a set of, completely new, sufficient conditions. They require to check some improper integrals, and form the article's main theoretical effort. In [5] an analogous existence/uniqueness theorem has been established following a fully different approach and relevant applications have been provided.

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