Output-feedback Stabilization for Stochastic High-order Nonlinear Systems with a Ratio of Odd Integers Power

LIU Liang[1,2] DUAN Na[1,2] XIE Xue-Jun[1]

Abstract This paper investigates the problem of output-feedback control for a class of stochastic high-order nonlinear systems with a ratio of odd integers power. By extending the adding a power integrator technique, introducing a new rescaling transformation, and choosing an appropriate Lyapunov function, an output-feedback controller is constructed to render the closed-loop system globally asymptotically stable in probability and the output can be regulated to the origin almost surely. Furthermore, we address the problem of inverse optimal stabilization in probability. A simulation example is provided to show the effectiveness of the design.

Key words Stochastic high-order nonlinear systems, a ratio of odd integers power, output-feedback control, inverse optimal stabilization

DOI 10.3724/SP.J.1004.2010.00858

Consider the following stochastic high-order nonlinear systems described by

$$\begin{aligned}
\mathrm{d}\eta_1 &= \eta_2^r \mathrm{d}t + \boldsymbol{\psi}_1(\eta_1)^{\mathrm{T}} \mathrm{d}\boldsymbol{\omega} \\
\mathrm{d}\eta_2 &= \eta_3^r \mathrm{d}t + \boldsymbol{\psi}_2(\eta_1, \eta_2)^{\mathrm{T}} \mathrm{d}\boldsymbol{\omega} \\
&\vdots \\
\mathrm{d}\eta_n &= v^r \mathrm{d}t + \boldsymbol{\psi}_n(\eta_1, \cdots, \eta_n)^{\mathrm{T}} \mathrm{d}\boldsymbol{\omega} \\
y &= \eta_1
\end{aligned} \tag{1}$$

where $\boldsymbol{\eta} = (\eta_1, \cdots, \eta_n)^{\mathrm{T}} \in \mathbf{R}^n$, $v \in \mathbf{R}$, and $y \in \mathbf{R}$ are the system state, input, and output, respectively. $\eta_2, \cdots, \eta_n$ are unmeasurable. $r \in \mathbf{R}^* = \{s \in \mathbf{R} : s = q/p \geq 1$ for any positive odd integers $p, q\}$. $\boldsymbol{\omega}$ is an m-dimensional standard Wiener process defined on a probability space $(\Omega, \mathcal{F}, P)$ with Ω being a sample space, $\mathcal{F}$ being a filtration, and P being a probability measure. The mappings $\boldsymbol{\psi}_i : \mathbf{R}^i \to \mathbf{R}^m$, $i = 1, \cdots, n$, are assumed to be at least $\mathcal{C}^1$ functions with $\boldsymbol{\psi}_i(0, \cdots, 0) = \mathbf{0}$.

When $r = 1$, system (1) reduces to the well-known normal form, whose design of globally asymptotically stable (GAS) output-feedback controller design was firstly given by Deng and Krstić in [1]. Since then, by adopting different approaches, much research work has been focused on the output-feedback for more general stochastic nonlinear systems under various structures or growth conditions, e.g, [2−4] and references therein.

For the case of r being positive odd integer and $r > 1$, similar to its deterministic counterpart in [5] and the related papers, some interesting features of (1) are that the Jacobian linearization of the system is neither controllable nor feedback linearizable, so the existing design tools are hardly applicable to (1). Reference [6] firstly considered this class of systems. Subsequently, [7−11] further addressed the different control problems for more general stochastic high-order nonlinear systems with different structures. All the results are achieved under the assumption that the power of stochastic nonlinear system is positive odd integer. While for more general systems in which system′s power is only a ratio of odd integers (i.e., $r \in \mathbf{R}^*$), to the best of authors′ knowledge, the problem of output-feedback stabilization has not yet been considered.

In this paper, extending the adding a power integrator technique, introducing a new rescaling transformation, and choosing an appropriate Lyapunov function, we develop a systematic design algorithm that achieves a smooth output-feedback controller, which ensures that the equilibrium at the origin of the closed-loop system is globally asymptotically stable (GAS) in probability and the output can be regulated to the origin almost surely. Based on the result, we further address the problem of inverse optimal stabilization in probability.

Notations. The following notations are to be used throughout the paper. $\mathbf{R}_+$ denotes the set of all nonnegative real numbers, and $\mathbf{R}^n$ denotes the real n-dimensional space. $\bar{\boldsymbol{x}}_i = (x_1, \cdots, x_i)^{\mathrm{T}}$, $\bar{\boldsymbol{x}}_n = \boldsymbol{x}$. For a given vector or matrix X, X^{T} denotes its transpose, $\mathrm{tr}\{X\}$ denotes its trace when X is square, and $|\boldsymbol{X}|$ is the Euclidean norm of a vector $\boldsymbol{X}$. C^i denotes the set of all functions with continuous i-th partial derivatives. $\mathcal{K}$ denotes the set of all functions: $\mathbf{R}_+ \to \mathbf{R}_+$, which are continuous, strictly increasing and vanishing at zero; $\mathcal{K}_\infty$ denotes the set of all functions which are of class $\mathcal{K}$ and unbounded; $\mathcal{KL}$ denotes the set of all functions $\beta(s,t)$: $\mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$, which are of $\mathcal{K}$ for each fixed t and decrease to zero as $t \to \infty$ for each fixed s. For a class $\mathcal{K}_\infty$ function γ whose derivative exists and is also a class $\mathcal{K}_\infty$ function, ℓ_γ denotes the transform $\ell_\gamma(s) = s(\dot{\gamma})^{-1}(s) - \gamma((\dot{\gamma})^{-1}(s))$, where $(\dot{\gamma})^{-1}(s)$ stands for the inverse function of $\frac{\mathrm{d}\gamma(s)}{\mathrm{d}s}$ for any variable s, $L_{\boldsymbol{f}} V(\boldsymbol{x}) = \frac{\partial V}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x})$.

1 Mathematics preliminaries

Consider the following stochastic nonlinear system

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x})\mathrm{d}t + g(\boldsymbol{x})^{\mathrm{T}}\mathrm{d}\boldsymbol{\omega}, \quad \boldsymbol{x}(\mathbf{0}) = \boldsymbol{x}_0 \in \mathbf{R}^n \tag{2}$$

where $\boldsymbol{x} \in \mathbf{R}^n$ is the state of the system and $\boldsymbol{\omega}$ is an m-dimensional standard Wiener process defined on the complete probability space $(\Omega, \mathcal{F}, P)$. The Borel measurable functions $\boldsymbol{f} : \mathbf{R}^n \to \mathbf{R}^n$ and $g^{\mathrm{T}} : \mathbf{R}^n \to \mathbf{R}^{n \times m}$ are locally Lipschitz in $\boldsymbol{x} \in \mathbf{R}^n$.

The following conclusions will be used throughout the paper.

Definition 1[12]**.** For any given $V(\boldsymbol{x}) \in \mathcal{C}^2$ associated with stochastic nonlinear system (2), the differential operator $\mathcal{L}$ is defined as

$$\mathcal{L}V(\boldsymbol{x}) = \frac{\partial V}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}) + \frac{1}{2}\mathrm{tr}\left\{ g(\boldsymbol{x}) \frac{\partial^2 V}{\partial \boldsymbol{x}^2} g(\boldsymbol{x})^{\mathrm{T}} \right\} \tag{3}$$

Manuscript received April 13, 2009; accepted September 7, 2009

Supported by National Natural Science Foundation of China (60774010, 10971256), Natural Science Foundation of Jiangsu Province (BK2009083), Program for Fundamental Research of Natural Sciences in Universities of Jiangsu Province (07KJB510114), and Shandong Provincial Natural Science Foundation of China (ZR2009GM008, ZR2009AL014)

1. School of Electrical Engineering and Automation, Xuzhou Normal University, Xuzhou 221116, P. R. China 2. Institute of Automation, Qufu Normal University, Qufu 273165, P. R. China

Definition 2[12]**.** For the stochastic nonlinear system (2) with $\boldsymbol{f}(\boldsymbol{0})=\boldsymbol{0}$ and $g(\boldsymbol{0})=0$, the equilibrium $\boldsymbol{x}(t)=\boldsymbol{0}$ of (2) is GAS in probability if for any $\varepsilon>0$, there exists a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ such that $P\{|\boldsymbol{x}(t)|<\beta(|\boldsymbol{x}_0|,t)\}\geq 1-\varepsilon$ for any $t\geq 0$ and $\boldsymbol{x}_0\in\mathbf{R}^n\setminus\{\boldsymbol{0}\}$.

Lemma 1[12]**.** Consider the stochastic nonlinear system (2). If there exist a $\mathcal{C}^2$ function $V(\boldsymbol{x})$, class $\mathcal{K}_\infty$ functions α_1 and α_2, constants $c_1>0$, $c_2\geq 0$, and a nonnegative function $W(\boldsymbol{x})$ such that $\alpha_1(|\boldsymbol{x}|)\leq V(\boldsymbol{x})\leq\alpha_2(|\boldsymbol{x}|)$ and $\mathcal{L}V\leq -c_1W(\boldsymbol{x})+c_2$, then

1) For (2), there exists an almost surely unique solution on $[0,\infty)$;

2) When $c_2=0$, $\boldsymbol{f}(\boldsymbol{0})=\boldsymbol{0}$, $g(\boldsymbol{0})=0$, and $W(\boldsymbol{x})=\alpha_3(|\boldsymbol{x}|)$, where $\alpha_3(\cdot)$ is a class $\mathcal{K}$ function, the equilibrium $\boldsymbol{x}=\boldsymbol{0}$ is GAS in probability, and $P\{\lim_{t\to\infty}|\boldsymbol{x}(t)|=0\}=1$.

Lemma 2[13]**.** For $x\in\mathbf{R}$, $y\in\mathbf{R}$, and $p\geq 1$ is a constant, we have $|x+y|^p\leq 2^{p-1}|x^p+y^p|$, if $p\in\mathbf{R}^*$, then $|x-y|^p\leq 2^{p-1}|x^p-y^p|$.

Lemma 3[14]**.** Let c, d be positive constants. For any positive number $\gamma>0$, then $|x|^c|y|^d\leq\frac{c}{c+d}\gamma|x|^{c+d}+\frac{d}{c+d}\gamma^{-\frac{c}{d}}|y|^{c+d}$.

Lemma 4[11]**.** If $x_1,\cdots,x_n,p$ are positive real numbers, then $(x_1+\cdots+x_n)^p\leq\max\{n^{p-1},1\}(x_1^p+\cdots+x_n^p)$.

Lemma 5[13]**.** Let $p\in\mathbf{R}^*$ and x, y be real-valued functions. For a constant $c>0$, one has $|x^p-y^p|\leq p|x-y|(x^{p-1}+y^{p-1})\leq c|x-y||(x-y)^{p-1}+y^{p-1}|$.

Lemma 6. If x, y are any real numbers and $p\in\mathbf{R}^*$, then $-(x-y)(x^p-y^p)\leq-\frac{1}{2^{p-1}}(x-y)^{p+1}$.

Proof. Since $p\in\mathbf{R}^*$, from Lemma 2, one has

$$|x-y|^p\leq 2^{p-1}|x^p-y^p| \tag{4}$$

Multiplying inequality (4) on both sides by $|x-y|$, we obtain

$$|x-y|^{p+1}\leq 2^{p-1}|x-y||x^p-y^p| \tag{5}$$

We consider two cases.

1) When $x-y>0$, (5) becomes $(x-y)^{p+1}\leq 2^{p-1}(x-y)|x^p-y^p|$. It is easy to prove that for any $x\in\mathbf{R},p\in\mathbf{R}^*,f(x)=x^p$ is a increasing function, hence $x^p\geq y^p$, from which $(x-y)^{p+1}\leq 2^{p-1}(x-y)(x^p-y^p)$ holds.

2) By the symmetry property, it is similar to deduce that the inequality $(x-y)^{p+1}\leq 2^{p-1}(x-y)(x^p-y^p)$ also holds for $x-y<0$. Combining 1) with 2) leads to the inequality. □

Consider the following stochastic nonlinear system

$$\mathrm{d}\boldsymbol{x}=\hat{\boldsymbol{f}}(\boldsymbol{x})\mathrm{d}t+\hat{g}_1(\boldsymbol{x})\mathrm{d}\boldsymbol{\omega}+\hat{\boldsymbol{g}}_2(\boldsymbol{x})u^r\mathrm{d}t,\quad \boldsymbol{x}_0\in\mathbf{R}^n \tag{6}$$

where $\boldsymbol{x}$ and $\boldsymbol{\omega}$ have the same definitions as those in (2), $\hat{\boldsymbol{f}}:\mathbf{R}^n\to\mathbf{R}^n$, $\hat{g}_1:\mathbf{R}^n\to\mathbf{R}^{n\times m}$, and $\hat{\boldsymbol{g}}_2:\mathbf{R}^n\to\mathbf{R}^n$ are some Borel measurable functions, and u is the input. We give the result on the problem of inverse optimal stabilization in probability.

Lemma 7[7]**.** Consider the control law

$$u=\alpha(\boldsymbol{x})=-\left[R(\boldsymbol{x})^{-1}(L_{\hat{\boldsymbol{g}}_2}V)^{\mathrm{T}}\frac{\ell_\gamma(|(L_{\hat{\boldsymbol{g}}_2}V)R(\boldsymbol{x})^{-\frac{1}{2}}|)}{|(L_{\hat{\boldsymbol{g}}_2}V)R(\boldsymbol{x})^{-\frac{1}{2}}|^2}\right]^{\frac{1}{r}} \tag{7}$$

where $V(\boldsymbol{x})$ is a Lyapunov function candidate, $\gamma(\cdot)$ is a class $\mathcal{K}_\infty$ function whose derivative exists and is also a class $\mathcal{K}_\infty$ function, and $R(\boldsymbol{x})=R(\boldsymbol{x})^{\mathrm{T}}>0$ is a matrix-valued function. If the control law (7) achieves GAS in probability for (6) with respect to $V(\boldsymbol{x})$, then the control law

$$u^*=\alpha^*(\boldsymbol{x})=-\left[\frac{\beta}{2}R(\boldsymbol{x})^{-1}(L_{\hat{\boldsymbol{g}}_2}V)^{\mathrm{T}}\frac{(\dot{\gamma})^{-1}(|(L_{\hat{\boldsymbol{g}}_2}V)R(\boldsymbol{x})^{-\frac{1}{2}}|)}{|(L_{\hat{\boldsymbol{g}}_2}V)R(\boldsymbol{x})^{-\frac{1}{2}}|}\right]^{\frac{1}{r}} \tag{8}$$

solves the problem of inverse optimal stabilization in probability for (6) by minimizing the cost function

$$J(u)=\mathrm{E}\left\{\int_0^\infty\left[l(\boldsymbol{x})+\beta^2\gamma\left(\frac{2}{\beta}|R(\boldsymbol{x})^{\frac{1}{2}}u^r|\right)\right]\mathrm{d}\tau\right\} \tag{9}$$

where $l(\boldsymbol{x})=2\beta[\ell_\gamma(|(L_{\hat{\boldsymbol{g}}_2}V)R(\boldsymbol{x})^{-\frac{1}{2}}|)-L_{\hat{\boldsymbol{f}}}V-\frac{1}{2}\mathrm{tr}\{\hat{g}_1(\boldsymbol{x})^{\mathrm{T}}\frac{\partial^2V(\boldsymbol{x})}{\partial\boldsymbol{x}^2}\hat{g}_1(\boldsymbol{x})\}]+\beta(\beta-2)\ell_\gamma(|(L_{\hat{\boldsymbol{g}}_2}V)R(\boldsymbol{x})^{-\frac{1}{2}}|)$, $\beta\geq 2$.

2 Output-feedback controller design

The objective of this paper is to design an output-feedback controller for system (1) such that the closed-loop system is GAS in probability at the origin, the output can be regulated to the origin almost surely, and the controller is also optimal in probability.

In this paper, we need the following assumption.

Assumption 1. Given r defined in (1), there exists a nonnegative constant a such that

$$|\boldsymbol{\psi}_i(\eta_1,\cdots,\eta_i)|\leq a(|\eta_1|^{\frac{1+r}{2}}+|\eta_2|^{\frac{1+r}{2}}+\cdots+|\eta_i|^{\frac{1+r}{2}})$$

Remark 1. To obtain a linear smooth state-feedback controller and use the certainty equivalence principle in [15] to achieve an implementable controller, Assumption 1 is necessary.

Before giving the design of controller, we first introduce a new rescaling transformation

$$\begin{aligned}&\eta_1=x_1,\quad \eta_i=N^{\frac{1}{r}+\cdots+\frac{1}{r^{i-1}}}x_i,\quad i=2,\cdots,n\\ &v=p_0N^{\frac{1}{r}+\cdots+\frac{1}{r^n}}u\end{aligned} \tag{10}$$

with which (1) can be expressed as

$$\begin{aligned}\mathrm{d}x_1&=Nx_2^r\mathrm{d}t+\boldsymbol{\varphi}_1(x_1)^{\mathrm{T}}\mathrm{d}\boldsymbol{\omega}\\ \mathrm{d}x_2&=Nx_3^r\mathrm{d}t+\boldsymbol{\varphi}_2(\bar{\boldsymbol{x}}_2)^{\mathrm{T}}\mathrm{d}\boldsymbol{\omega}\\ &\ \ \vdots\\ \mathrm{d}x_n&=Np_0^ru^r\mathrm{d}t+\boldsymbol{\varphi}_n(\bar{\boldsymbol{x}}_n)^{\mathrm{T}}\mathrm{d}\boldsymbol{\omega}\\ y&=x_1\end{aligned} \tag{11}$$

where $N\geq 1$ and $p_0>0$ are rescaling factors to be assigned later, $\boldsymbol{\varphi}_1=\boldsymbol{\psi}_1$, $\boldsymbol{\varphi}_i=\frac{\boldsymbol{\psi}_i}{N^{\frac{1}{r}+\cdots+\frac{1}{r^{i-1}}}}$, $i=2,\cdots,n$. Using $N\geq 1$ and Assumption 1, it is easy to deduce that

$$\begin{aligned}|\boldsymbol{\varphi}_1|&=|\boldsymbol{\psi}_1|\leq a|\eta_1|^{\frac{1+r}{2}}=a|x_1|^{\frac{1+r}{2}}\\ |\boldsymbol{\varphi}_i|&\leq aN^{\frac{1}{2}-\frac{1}{2r^{i-1}}}\left(|x_1|^{\frac{1+r}{2}}+\cdots+|x_i|^{\frac{1+r}{2}}\right)\end{aligned} \tag{12}$$

where $i=2,\cdots,n$.

The design procedure of output-feedback controller can be divided into two steps.

2.1 State-feedback design

In this subsection, under the assumption that all the states are available for measurement, one will construct a partial state-feedback linear controller.

Step 1. Introduce $\xi_1 = x_1$ and construct the first Lyapunov function $V_1(\xi_1) = \frac{1}{4}p_1\xi_1^4$, where $p_1 > 0$ is a constant. With the help of (3), (11), (12), and $N \geq 1$, it can be verified that

$$\mathcal{L}V_1(\xi_1) = Np_1\xi_1^3 x_2^r + \frac{3}{2}p_1\xi_1^2 \mathrm{tr}\{\boldsymbol{\varphi}_1(x_1)\boldsymbol{\varphi}_1(x_1)^{\mathrm{T}}\} \leq N\left(p_1\xi_1^3 x_2^r + \frac{3}{2}a^2 p_1 \xi_1^{3+r}\right) \quad (13)$$

Choosing the first smooth virtual controller

$$x_2^* = -b_1\xi_1, \quad b_1 = \left(\frac{c_{1,1}}{p_1} + \frac{3}{2}a^2\right)^{\frac{1}{r}}, \quad c_{1,1} > 0 \quad (14)$$

leads to

$$\mathcal{L}V_1(\xi_1) \leq N(-c_{1,1}\xi_1^{3+r} + p_1\xi_1^3(x_2^r - x_2^{*r})) \quad (15)$$

Step $\boldsymbol{i}$ ($\boldsymbol{i = 2, \cdots, n}$). Suppose that at step $i-1$, there exist a set of virtual controllers $x_1^*, \cdots, x_i^*$ defined by

$$\begin{aligned} &x_1^* = 0, \quad \xi_1 = x_1 - x_1^* = x_1 \\ &x_k^* = -b_{k-1}\xi_{k-1}, \quad \xi_k = x_k - x_k^* = x_k + b_{k-1}\xi_{k-1} \end{aligned} \quad (16)$$

such that the $(i-1)$-th Lyapunov function $V_{i-1}(\bar{\boldsymbol{\xi}}_{i-1}) = \frac{1}{4}\sum_{j=1}^{i-1} p_j\xi_j^4$ satisfies

$$\mathcal{L}V_{i-1}(\bar{\boldsymbol{\xi}}_{i-1}) \leq N\left(-\sum_{j=1}^{i-1} c_{i-1,j}\xi_j^{3+r} + p_{i-1}\xi_{i-1}^3(x_i^r - x_i^{*r})\right) \quad (17)$$

where $k = 2, \cdots, i$, $b_1, \cdots, b_{i-1} > 0$ are designed parameters and $c_{i-1,j}$, p_j, $j = 1, \cdots, i-1$, are positive constants. We will prove that (17) still holds for Step i.

By (11) and (16), one has

$$\mathrm{d}\xi_i = N\left(x_{i+1}^r + \sum_{k=1}^{i-1} b_{i-1}\cdots b_k x_{k+1}^r\right)\mathrm{d}t + \left(\boldsymbol{\varphi}_i(\bar{\boldsymbol{x}}_i) + \sum_{k=1}^{i-1} b_{i-1}\cdots b_k \boldsymbol{\varphi}_k(\bar{\boldsymbol{x}}_k)\right)^{\mathrm{T}} \mathrm{d}\boldsymbol{\omega} \quad (18)$$

Consider the following Lyapunov function candidate

$$V_i(\bar{\boldsymbol{\xi}}_i) = V_{i-1}(\bar{\boldsymbol{\xi}}_{i-1}) + \frac{1}{4}p_i\xi_i^4 \quad (19)$$

where $p_i > 0$ is a constant. From (17) $\sim$ (19), it follows that

$$\begin{aligned}\mathcal{L}V_i(\bar{\boldsymbol{\xi}}_i) \leq N\Bigg(&-\sum_{j=1}^{i-1} c_{i-1,j}\xi_j^{3+r} + p_i\xi_i^3 x_{i+1}^r + \\ &p_{i-1}\xi_{i-1}^3(x_i^r - x_i^{*r}) + p_i\xi_i^3 \sum_{k=1}^{i-1} b_{i-1}\cdots b_k x_{k+1}^r + \\ &\frac{3}{2}p_i\xi_i^2 \left|\boldsymbol{\varphi}_i(\bar{\boldsymbol{x}}_i) + \sum_{k=1}^{i-1} b_{i-1}\cdots b_k\boldsymbol{\varphi}_k(\bar{\boldsymbol{x}}_k)\right|^2\Bigg)\end{aligned} \quad (20)$$

Using (12), (16), and Lemmas 2 $\sim$ 5, one gets

$$p_{i-1}\xi_{i-1}^3(x_i^r - x_i^{*r}) \leq (b_{i,i-1,1} + b_{i,i,1})\xi_{i-1}^{3+r} + \rho_{i,1}\xi_i^{3+r} \quad (21)$$

$$p_i\xi_i^3 \sum_{k=1}^{i-1} b_{i-1}\cdots b_k x_{k+1}^r \leq b_{i,1,2}\xi_1^{3+r} + \cdots + b_{i,i-1,2}\xi_{i-1}^{3+r} + \rho_{i,2}\xi_i^{3+r} \quad (22)$$

$$\tfrac{3}{2}p_i\xi_i^2 \left|\boldsymbol{\varphi}_i(\bar{\boldsymbol{x}}_i) + \sum_{k=1}^{i-1} b_{i-1}\cdots b_k\boldsymbol{\varphi}_k(\bar{\boldsymbol{x}}_k)\right|^2 \leq b_{i,1,3}\xi_1^{3+r} + \cdots + b_{i,i-1,3}\xi_{i-1}^{3+r} + \rho_{i,3}\xi_i^{3+r} \quad (23)$$

where $\rho_{i,1}$, $\rho_{i,2}$, $\rho_{i,3}$, $b_{i,i-1,1}, b_{i,i,1}$, $b_{i,1,2}, \cdots, b_{i,i-1,2}$, and $b_{i,1,3}, \cdots, b_{i,i-1,3}$ are positive constants. Choosing $c_{i,1} = c_{i-1,1} - b_{i,1,2} - b_{i,1,3} > 0, \cdots$ $c_{i,i-2} = c_{i-1,i-2} - b_{i,i-2,2} - b_{i,i-2,3} > 0$, $c_{i,i-1} = c_{i-1,i-1} - b_{i,i-1,1} - b_{i,i,1} - b_{i,i-1,2} - b_{i,i-1,3} > 0$, and substituting (21) $\sim$ (23) into (20), one has $\mathcal{L}V_i(\bar{\boldsymbol{\xi}}_i) \leq N(-\sum_{j=1}^{i-1} c_{i,j}\xi_j^{3+r} + p_i\xi_i^3(x_{i+1}^r - x_{i+1}^{*r}) + p_i\xi_i^3 x_{i+1}^{*r} + \rho_i\xi_i^{3+r})$, where $\rho_i = \rho_{i,1} + \rho_{i,2} + \rho_{i,3}$ is a positive real number, which together with the i-th smooth virtual controller

$$x_{i+1}^* = -b_i\xi_i, \quad b_i = \left(\frac{c_{i,i} + \rho_i}{p_i}\right)^{\frac{1}{r}}, \quad c_{i,i} > 0 \quad (24)$$

lead to

$$\mathcal{L}V_i(\bar{\boldsymbol{\xi}}_i) \leq N\left(-\sum_{j=1}^{i} c_{i,j}\xi_j^{3+r} + p_i\xi_i^3(x_{i+1}^r - x_{i+1}^{*r})\right) \quad (25)$$

Step $\boldsymbol{n}$. Consider the Lyapunov function candidate

$$V_n(\bar{\boldsymbol{\xi}}_n) = V_{n-1}(\bar{\boldsymbol{\xi}}_{n-1}) + \frac{1}{4}p_n\xi_n^4 \quad (26)$$

where $p_n > 0$ is an appropriate constant. By (11), (12), (16), (25), and (26), one gets

$$\begin{aligned}\mathcal{L}V_n(\bar{\boldsymbol{\xi}}_n) \leq N\Bigg(&-\sum_{j=1}^{n-1} c_{n,j}\xi_j^{3+r} + p_n p_0^r \xi_n^3(u^r(x) - x_{n+1}^{*r}) + \\ &p_n p_0^r \xi_n^3 x_{n+1}^{*r} + \rho_n\xi_n^{3+r}\Bigg)\end{aligned} \quad (27)$$

where ρ_n, $c_{n,j}$, $j = 1, \cdots, n-1$, are positive constants. Choosing $p_0 = p_n^{-\frac{1}{r}}$ and defining the n-th smooth virtual control law

$$x_{n+1}^* = -b_n\xi_n, \quad b_n = (c_{n,n} + \rho_n)^{\frac{1}{r}}, \quad c_{n,n} > 0 \quad (28)$$

and by (27) and (28), one gets

$$\mathcal{L}V_n(\bar{\boldsymbol{\xi}}_n) \leq N\left(-\sum_{j=1}^{n} c_{n,j}\xi_j^{3+r} + \xi_n^3(u^r - x_{n+1}^{*r})\right) \quad (29)$$

By (14), (16), (24), and (28), one gets the smooth state-feedback control law

$$x_{n+1}^* = -b_n\xi_n = -(\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n) \quad (30)$$

where $\beta_i = \prod_{j=i}^n b_j$, $i = 1, \cdots, n$.

Remark 2. For general system (11), under the assumption that $x_1, x_2, \cdots, x_n$ are measurable, in the design procedure of controller, we can only give the existence of $\rho_{i,1}$, $\rho_{i,2}$, and $\rho_{i,3}$, $i = 2, \cdots, n$, obtained by using Lemmas 2 $\sim$ 5 rather than their explicit definitions. While for a practical example, by appropriately choosing design parameters, $\rho_{i,1}$, $\rho_{i,2}$, and $\rho_{i,3}$, $i = 2, \cdots, n$, can be concretely achieved; see Section 4 for details.

2.2 Output-feedback design

Since $(x_2, \cdots, x_n)$ of (11) are unmeasurable, in this subsection, by constructing a reduced-order observer and using the certainty equivalence principle in [15], an output-feedback controller is designed.

Introduce the unmeasurable variables

$$z_i = x_i - \ell_i \cdots \ell_2 x_1, \quad i = 2, \cdots, n \tag{31}$$

where the parameters $\ell_2, \cdots, \ell_n \geq 1$ are gain constants to be determined later. From (31) and (11), it follows that

$$\begin{aligned}
\mathrm{d}z_2 &= N(x_3^r - \ell_2 x_2^r)\mathrm{d}t + (\boldsymbol{\varphi}_2(\bar{\boldsymbol{x}}_2) - \ell_2\boldsymbol{\varphi}_1(x_1))^{\mathrm{T}}\mathrm{d}\boldsymbol{\omega}\\
&\ \ \vdots\\
\mathrm{d}z_{n-1} &= N(x_n^r - \ell_{n-1}\cdots\ell_2 x_2^r)\mathrm{d}t +\\
&\quad (\boldsymbol{\varphi}_{n-1}(\bar{\boldsymbol{x}}_{n-1}) - \ell_{n-1}\cdots\ell_2\boldsymbol{\varphi}_1(x_1))^{\mathrm{T}}\mathrm{d}\boldsymbol{\omega}\\
\mathrm{d}z_n &= N(p_0^r u^r - \ell_n\cdots\ell_2 x_2^r)\mathrm{d}t +\\
&\quad (\boldsymbol{\varphi}_n(\bar{\boldsymbol{x}}_n) - \ell_n\cdots\ell_2\boldsymbol{\varphi}_1(x_1))^{\mathrm{T}}\mathrm{d}\boldsymbol{\omega}
\end{aligned} \tag{32}$$

In view of (32), one can construct the $(n-1)$-dimensional observer as follows

$$\begin{aligned}
\dot{\hat{z}}_2 &= N(\hat{z}_3 + \ell_3\ell_2 x_1)^r - N\ell_2(\hat{z}_2 + \ell_2 x_1)^r\\
&\ \ \vdots\\
\dot{\hat{z}}_{n-1} &= N(\hat{z}_n + \ell_n\cdots\ell_2 x_1)^r - N\ell_{n-1}\cdots\ell_2(\hat{z}_2 + \ell_2 x_1)^r\\
\dot{\hat{z}}_n &= N p_0^r u^r - N\ell_n\cdots\ell_2(\hat{z}_2 + \ell_2 x_1)^r
\end{aligned} \tag{33}$$

It is obvious that the reduced-order observer (33) is implementable, and the estimate $\hat{x}_i$ of x_i can be obtained by

$$\hat{x}_i = \hat{z}_i + \ell_i\cdots\ell_2 x_1, \quad i = 2, \cdots, n \tag{34}$$

Let $e_i = x_i - \hat{x}_i = z_i - \hat{z}_i$, $i = 2, \cdots, n$. By (32) and (33), the error dynamics are given by

$$\begin{aligned}
\mathrm{d}e_2 &= N((x_3^r - \hat{x}_3^r) - \ell_2(x_2^r - \hat{x}_2^r))\mathrm{d}t +\\
&\quad (\boldsymbol{\varphi}_2(\bar{\boldsymbol{x}}_2) - \ell_2\boldsymbol{\varphi}_1(x_1))^{\mathrm{T}}\mathrm{d}\boldsymbol{\omega}\\
&\ \ \vdots\\
\mathrm{d}e_{n-1} &= N((x_n^r - \hat{x}_n^r) - \ell_{n-1}\cdots\ell_2(x_2^r - \hat{x}_2^r))\mathrm{d}t +\\
&\quad (\boldsymbol{\varphi}_{n-1}(\bar{\boldsymbol{x}}_{n-1}) - \ell_{n-1}\cdots\ell_2\boldsymbol{\varphi}_1(x_1))^{\mathrm{T}}\mathrm{d}\boldsymbol{\omega}\\
\mathrm{d}e_n &= -N\ell_n\cdots\ell_2(x_2^r - \hat{x}_2^r)\mathrm{d}t +\\
&\quad (\boldsymbol{\varphi}_n(\bar{\boldsymbol{x}}_n) - \ell_n\cdots\ell_2\boldsymbol{\varphi}_1(x_1))^{\mathrm{T}}\mathrm{d}\boldsymbol{\omega}
\end{aligned} \tag{35}$$

Using the certainty equivalence principle, from (30) one achieves the implementable controller

$$u = -b_n\hat{\xi}_n = -(\beta_1 x_1 + \beta_2\hat{x}_2 + \cdots + \beta_n\hat{x}_n) \tag{36}$$

To estimate $\xi_n^3(u^r - x_{n+1}^{*r})$ in (29), we first use Lemmas $3\sim5$ to develop the following estimation.

$$\begin{aligned}
|\xi_n^3(u^r - x_{n+1}^{*r})| &\leq \hat{c}|\xi_n|^3\sum_{i=2}^n\beta_i|e_i|\left(\sum_{i=2}^n e_i^{r-1} + \sum_{i=1}^n\xi_i^{r-1}\right)\leq\\
&\sum_{i=1}^n\hat{c}_{n,i}\xi_i^{3+r} + \sum_{i=2}^n\hat{\rho}_i e_i^{3+r}
\end{aligned} \tag{37}$$

where $\hat{c}$, $\hat{\rho}_i$, $i = 2, \cdots, n$, and $0 < \hat{c}_{n,i} < c_{n,i}$ are constants independent of $\ell_2, \cdots, \ell_n, N$. Substituting (37) into (29), one has

$$\mathcal{L}V_n(\bar{\boldsymbol{\xi}}_n) \leq N\left(-\sum_{i=1}^n(c_{n,i} - \hat{c}_{n,i})\xi_i^{3+r} + \sum_{i=2}^n\hat{\rho}_i e_i^{3+r}\right) \tag{38}$$

To determine the observer gains $\ell_2, \cdots, \ell_n$, one considers the change of coordinates:

$$\tilde{e}_2 = e_2, \quad \tilde{e}_3 = e_3 - \ell_3 e_2, \cdots, \tilde{e}_n = e_n - \ell_n e_{n-1} \tag{39}$$

By (39) and Lemma 4, (38) can be represented as

$$\begin{aligned}
\mathcal{L}V_n \leq\ & N\Bigg(-\sum_{i=1}^n(c_{n,i} - \hat{c}_{n,i})\xi_i^{3+r} + \hat{c}_2(\ell_3, \cdots, \ell_n)\tilde{e}_2^{3+r} +\\
&\cdots + \hat{c}_{n-1}(\ell_n)\tilde{e}_{n-1}^{3+r} + \hat{c}_n\tilde{e}_n^{3+r}\Bigg)
\end{aligned} \tag{40}$$

where $\hat{c}_2(\ell_3, \cdots, \ell_n), \cdots, \hat{c}_{n-1}(\ell_n)$ are positive real numbers independent of N, and $\hat{c}_n > 0$ is a known constant independent of N and all the ℓ_is. By (35) and (39), thus

$$\begin{aligned}
\mathrm{d}\tilde{e}_2 &= N((x_3^r - \hat{x}_3^r) - \ell_2(x_2^r - \hat{x}_2^r))\mathrm{d}t +\\
&\quad (\boldsymbol{\varphi}_2(\bar{\boldsymbol{x}}_2) - \ell_2\boldsymbol{\varphi}_1(x_1))^{\mathrm{T}}\mathrm{d}\boldsymbol{\omega}\\
&\ \ \vdots\\
\mathrm{d}\tilde{e}_{n-1} &= N((x_n^r - \hat{x}_n^r) - \ell_{n-1}(x_{n-1}^r - \hat{x}_{n-1}^r))\mathrm{d}t +\\
&\quad (\boldsymbol{\varphi}_{n-1}(\bar{\boldsymbol{x}}_{n-1}) - \ell_{n-1}\boldsymbol{\varphi}_{n-2}(\bar{\boldsymbol{x}}_{n-2}))^{\mathrm{T}}\mathrm{d}\boldsymbol{\omega}\\
\mathrm{d}\tilde{e}_n &= -N\ell_n(x_n^r - \hat{x}_n^r)\mathrm{d}t +\\
&\quad (\boldsymbol{\varphi}_n(\bar{\boldsymbol{x}}_n) - \ell_n\boldsymbol{\varphi}_{n-1}(\bar{\boldsymbol{x}}_{n-1}))^{\mathrm{T}}\mathrm{d}\boldsymbol{\omega}
\end{aligned} \tag{41}$$

Considering the following Lyapunov function

$$W_n(\tilde{\boldsymbol{e}}) = \frac{1}{4}\ell\sum_{i=2}^n\tilde{e}_i^4 \tag{42}$$

where $\ell > 0$ is a constant and $\tilde{\boldsymbol{e}} = (\tilde{e}_2, \cdots, \tilde{e}_n)$, a direct calculation leads to

$$\begin{aligned}
\mathcal{L}W_n =\ & N\Bigg(-\sum_{i=2}^n\ell\ell_i\tilde{e}_i^3((\hat{x}_i + \tilde{e}_i)^r - \hat{x}_i^r) +\\
&\sum_{i=3}^n\ell(x_i^r - \hat{x}_i^r)\tilde{e}_{i-1}^3 - \sum_{i=2}^n\ell\ell_i\tilde{e}_i^3(x_i^r - (\hat{x}_i + \tilde{e}_i)^r) +\\
&\sum_{i=2}^n\frac{3\ell}{2N}\tilde{e}_i^2|\boldsymbol{\varphi}_i(\bar{\boldsymbol{x}}_i) - \ell_i\boldsymbol{\varphi}_{i-1}(\bar{\boldsymbol{x}}_{i-1})|^2\Bigg)
\end{aligned} \tag{43}$$

We concentrate on each term on the right-hand side of (43). By Lemma 6, one has

$$-\ell\ell_i\tilde{e}_i^3\left((\hat{x}_i + \tilde{e}_i)^r - \hat{x}_i^r\right) \leq -\frac{\ell\ell_i}{2^{r-1}}\tilde{e}_i^{3+r}, i = 2, \cdots, n \tag{44}$$

Using (16) and (39), Lemmas $2\sim4$, and $e_i = x_i - \hat{x}_i$, for $i = 3, \cdots, n$, one obtains

$$\begin{aligned}
&\left|\ell\,(x_i^r - \hat{x}_i^r)\tilde{e}_{i-1}^3\right| \leq\\
&\quad \ell|\tilde{e}_{i-1}|^3(2^{r-1}(2^{r-1}+1)(|\xi_i|^r + |b_{i-1}\xi_{i-1}|^r) +\\
&\quad (i-1)^r 2^{r-1}(\ell_i^r\cdots\ell_3^r|\tilde{e}_2|^r + \cdots + \ell_i^r|\tilde{e}_{i-1}|^r + |\tilde{e}_i|^r)) \leq\\
&\quad \varrho_{1,i}(\xi_{i-1}^{3+r} + \xi_i^{3+r}) + \tilde{c}_{i,2}(\ell_3, \cdots, \ell_i)\tilde{e}_2^{3+r} + \cdots +\\
&\quad \tilde{c}_{i,i-2}(\ell_{i-1}, \ell_i)\tilde{e}_{i-2}^{3+r} + \tilde{c}_{i,i-1}(\varrho_{1,i}, \ell_i)\tilde{e}_{i-1}^{3+r} + \tilde{c}_{i,i}\tilde{e}_i^{3+r}
\end{aligned} \tag{45}$$

where $\tilde{c}_{i,2}(\ell_3, \cdots, \ell_i)$, $\cdots$, $\tilde{c}_{i,i-2}(\ell_{i-1}, \ell_i)$, $\tilde{c}_{i,i-1}(\varrho_{1,i}, \ell_i)$, $\tilde{c}_{i,i}$ are positive real numbers independent of N, and $\varrho_{1,i} > 0$ is a constant independent of N and ℓ_is. Using (16), Lemmas $2\sim5$, and $x_i - \hat{x}_i - \tilde{e}_i = e_i - \tilde{e}_i = \ell_i\cdots\ell_3\tilde{e}_2 + \cdots + \ell_i\tilde{e}_{i-1}$

for $i = 2, \cdots, n$, one has

$$
\begin{aligned}
&|-\ell\ell_i\tilde{e}_i^3(x_i^r-(\hat{x}_i+\tilde{e}_i)^r)| \le \\
&r\ell\ell_i|\tilde{e}_i|^3(\ell_i\cdots\ell_3|\tilde{e}_2|+\cdots+\ell_i|\tilde{e}_{i-1}|)|(\xi_i-b_{i-1}\xi_{i-1})^{r-1}+ \\
&(\xi_i-b_{i-1}\xi_{i-1}-\ell_i\cdots\ell_3\tilde{e}_2-\cdots-\ell_i\tilde{e}_{i-1})^{r-1}| \le \\
&\tilde{d}_i(\ell_i^2\cdots\ell_3|\tilde{e}_2|+\cdots+\ell_i^2|\tilde{e}_{i-1}|)(|\xi_i|^{2+r}+|b_{i-1}\xi_{i-1}|^{2+r}+ \\
&(\ell_i\cdots\ell_3)^{2+r}|\tilde{e}_2|^{2+r}+\cdots+\ell_i^{2+r}|\tilde{e}_{i-1}|^{2+r}+|\tilde{e}_i|^{2+r}) \le \\
&\varrho_{2,i}(\xi_{i-1}^{3+r}+\xi_i^{3+r})+\hat{c}_{i,2}(\varrho_{2,i},\ell_3,\cdots,\ell_i)\tilde{e}_2^{3+r}+\cdots+ \\
&\hat{c}_{i,i-1}(\varrho_{2,i},\ell_i)\tilde{e}_{i-1}^{3+r}+\hat{c}_{i,i}\tilde{e}_i^{3+r}
\end{aligned} \tag{46}
$$

where $\varrho_{2,i} > 0$, $\tilde{d}_i > 0$ are constants independent of N and ℓ_is, and $\hat{c}_{i,2}(\varrho_{2,i}, \ell_3, \cdots, \ell_i)$, $\cdots$, $\hat{c}_{i,i-1}(\varrho_{2,i}, \ell_i)$, $\hat{c}_{i,i}$ are positive real numbers independent of N. Using (12), (16), Lemmas $2\sim 4$, and $\ell_i \ge 1$ for $i = 2, \cdots, n$, one gets

$$
\begin{aligned}
&\frac{3\ell}{2N}\,\tilde{e}_i^2|\boldsymbol{\varphi}_i-\ell_i\boldsymbol{\varphi}_{i-1}|^2 \le \\
&\quad\frac{3ia^2\ell}{N^{\frac{1}{r^{i-1}}}}\tilde{e}_i^2\bigg((1+\ell_i^2)|x_1|^{1+r}+\cdots+ \\
&\quad(1+\ell_i^2)|x_{i-1}|^{1+r}+|x_i|^{1+r}\bigg) \le \\
&\quad\varrho_{3,i}(\xi_1^{3+r}+\xi_2^{3+r}+\cdots+\xi_i^{3+r})+\frac{\hat{d}_i(\varrho_{3,i})\ell_i^{3+r}}{N^{\frac{3+r}{2r^{i-1}}}}\tilde{e}_i^{3+r}
\end{aligned} \tag{47}
$$

where $\varrho_{3,i} > 0$ and $\hat{d}_i(\varrho_{3,i}) > 0$ are positive constants independent of N and ℓ_is. Substituting (44) $\sim$ (47) into (43), a tedious but straightforward calculation leads to

$$
\begin{aligned}
\mathcal{L}W_n \le\; & N\Bigg(\sum_{i=1}^n\varrho_{4,i}\xi_i^{3+r}-\sum_{i=2}^n\frac{\ell\ell_i}{2^{r-1}}\tilde{e}_i^{3+r}+ \\
&\bigg(\tilde{c}_2(\varrho_{1,3},\varrho_{2,3},\cdots,\varrho_{2,n},\ell_3,\cdots,\ell_n)+ \\
&\frac{\hat{d}_2(\varrho_{3,2})\ell_2^{3+r}}{N^{\frac{3+r}{2r}}}\bigg)\tilde{e}_2^{3+r}+\cdots+ \\
&\bigg(\tilde{c}_{n-1}(\varrho_{1,n},\varrho_{2,n},\ell_n)+\frac{\hat{d}_{n-1}(\varrho_{3,n-1})\ell_{n-1}^{3+r}}{N^{\frac{3+r}{2r^{n-2}}}}\bigg)\tilde{e}_{n-1}^{3+r}+ \\
&\bigg(\tilde{c}_n+\frac{\hat{d}_n(\varrho_{3,n})\ell_n^{3+r}}{N^{\frac{3+r}{2r^{n-1}}}}\bigg)\tilde{e}_n^{3+r}\Bigg)
\end{aligned} \tag{48}
$$

where $\tilde{c}_2(\varrho_{1,3}, \varrho_{2,3}, \cdots \varrho_{2,n}, \ell_3, \cdots, \ell_n)$, $\cdots$, $\tilde{c}_{n-1}(\varrho_{1,n}, \varrho_{2,n}, \ell_n)$ are positive real numbers independent of N, $0 < \varrho_{4,i} < c_{n,i} - \hat{c}_{n,i}$, and $\tilde{c}_n > 0$ are known constants independent of N and all the ℓ_is. Considering the following Lyapunov function

$$
U_n(\bar{\boldsymbol{\xi}}_n, \tilde{\boldsymbol{e}}) = V_n(\bar{\boldsymbol{\xi}}_n) + W_n(\tilde{\boldsymbol{e}}) \tag{49}
$$

by (40) and (48), one has

$$
\begin{aligned}
\mathcal{L}U_n \le\; & -N\Bigg(\sum_{i=1}^n\varrho_i\xi_i^{3+r}+\bigg(\frac{\ell\ell_2}{2^{r-1}}-\bar{d}_2(\varrho_{1,3},\varrho_{2,3},\cdots,\varrho_{2,n}, \\
&\ell_3,\cdots,\ell_n)-\frac{\hat{d}_2(\varrho_{3,2})\ell_2^{3+r}}{N^{\frac{3+r}{2r}}}\bigg)\tilde{e}_2^{3+r}+\cdots+\bigg(\frac{\ell\ell_{n-1}}{2^{r-1}}- \\
&\bar{d}_{n-1}(\varrho_{1,n},\varrho_{2,n},\ell_n)-\frac{\hat{d}_{n-1}(\varrho_{3,n-1})\ell_{n-1}^{3+r}}{N^{\frac{3+r}{2r^{n-2}}}}\bigg)\tilde{e}_{n-1}^{3+r}+ \\
&\bigg(\frac{\ell\ell_n}{2^{r-1}}-\bar{d}_n-\frac{\hat{d}_n(\varrho_{3,n})\ell_n^{3+r}}{N^{\frac{3+r}{2r^{n-1}}}}\bigg)\tilde{e}_n^{3+r}\Bigg)
\end{aligned} \tag{50}
$$

where $\bar{d}_2(\varrho_{1,3}, \varrho_{2,3}, \cdots \varrho_{2,n}, \ell_3, \cdots, \ell_n)$, $\cdots$, $\bar{d}_{n-1}(\varrho_{1,n}, \varrho_{2,n}, \ell_n)$ are positive real numbers independent of N, and $\bar{d}_n > 0$, $\varrho_i = c_{n,i} - \hat{c}_{n,i} - \varrho_{4,i} > 0$ are known constants independent of N and all the ℓ_is. If the gain parameters ℓ_is and N are assigned one by one in the following manner: $\ell_n \ge \max\{1, \frac{2^{r-1}}{\ell}(\alpha_n + 1 + \bar{d}_n)\}$, $\ell_{n-1} \ge \max\{1, \frac{2^{r-1}}{\ell}(\alpha_{n-1} + 1 + \bar{d}_{n-1}(\varrho_{1,n}, \varrho_{2,n}, \ell_n))\}$, $\cdots$, $\ell_2 \ge \max\{1, \frac{2^{r-1}}{\ell}(\alpha_2 + 1 + \bar{d}_2(\varrho_{1,3}, \varrho_{2,3}, \cdots, \varrho_{2,n}, \ell_3, \cdots, \ell_n))\}$, $N \ge \max\{1, \hat{d}_2^{\frac{2r}{3+r}}(\varrho_{3,2})\ell_2^{2r}, \hat{d}_3^{\frac{2r^2}{3+r}}(\varrho_{3,3})\ell_3^{2r^2}, \cdots, \hat{d}_n^{\frac{2r^{n-1}}{3+r}}(\varrho_{3,n})\ell_n^{2r^{n-1}}\}$, then (50) obviously becomes

$$
\mathcal{L}U_n \le -N\left(\sum_{i=1}^n\varrho_i\xi_i^{3+r}+\sum_{i=2}^n\alpha_i\tilde{e}_i^{3+r}\right) \tag{51}
$$

where $\alpha_i > 0$ $(i = 2, \cdots, n)$ are some constants.

Remark 3. The idea of the rescaling transformation (10) and the observer (33) originates from [16], in which a simple deterministic high-order nonlinear system was considered from the viewpoint of reducing the control effort.

Remark 4. Indeed, since this paper is based on [11], these two papers have some similarities, such as system form, most of inequalities, Assumption 1, and conclusion. However, some differences need to be emphasized: 1) This paper studies stochastic high-order nonlinear system with a ratio of odd integers power, which is much more general than that in [11]; 2) All inequalities in [11] are only suitable for the case of r being positive odd integer, while for r being any positive real number, these inequalities need to be reproved; 3) Compared with [11], in the design procedure of controller, the operations of most of inequalities, whose powers involve more operations between fraction and integer, are much more complicated.

3 Controller analysis

We state the main results in this paper.

Theorem 1. If Assumption 1 holds for stochastic high-order nonlinear system (1), under the output-feedback controller (10), (33), (34), and (36), then

1) The closed-loop system (1), (10), (33), (34), and (36) has an almost surely unique solution in $[0, \infty)$ for any initial value η_0;

2) The equilibrium at the origin of the closed-loop system is GAS in probability;

3) The output $y(t)$ can be regulated to the origin almost surely;

4) The control law

$$
u^*(\hat{\boldsymbol{x}}) = -\hat{\xi}_n\left(\frac{3+r}{6}\beta b_n^r\right)^{\frac{1}{r}}, \quad \beta \ge 2 \tag{52}
$$

guarantees that the equilibrium at the origin of the closed-loop system is GAS in probability and also minimizes the cost functional

$$
\begin{aligned}
J(u)=\mathrm{E}\bigg\{\int_0^\infty\bigg[&l(\hat{\boldsymbol{x}},\tilde{\boldsymbol{e}})+Np_np_0^rb_n^{-3}\beta^2\frac{r}{3+r}\left(\frac{3+r}{3}\right)^{-\frac{3}{r}}\times \\
&\left(\frac{2}{\beta}\right)^{\frac{3+r}{r}}u^{3+r}\bigg]\mathrm{d}\tau\bigg\}
\end{aligned} \tag{53}
$$

where $l(\hat{\boldsymbol{x}}, \tilde{\boldsymbol{e}})$ is defined by Lemma 7, $\hat{\boldsymbol{x}} = (x_1, \hat{x}_2, \cdots, \hat{x}_n)^{\mathrm{T}}$.

Proof. By $V_1(\xi_1) = \frac{1}{4}p_1\xi_1^4$, (19), (26), (42), (49), (51), and Lemma 1, 1) and 2) hold, and $P\{\lim_{t\to\infty}\left(\sum_{i=1}^n|\xi_i(t)| + \sum_{i=2}^n|\tilde{e}_i(t)|\right) = 0\} = 1$, from which and $y(t) = \xi_1(t)$, 3) also holds.

$$\begin{bmatrix} \mathrm{d}\tilde{e}_2 \\ \vdots \\ \mathrm{d}\tilde{e}_n \\ \mathrm{d}y \\ \vdots \\ \mathrm{d}\hat{x}_{n-1} \\ \mathrm{d}\hat{x}_n \end{bmatrix} = \begin{bmatrix} N(((\tilde{e}_3+\ell_3\tilde{e}_2+\hat{x}_3)^r-\hat{x}_3^r)-\ell_2((\tilde{e}_2+\hat{x}_2)^r-\hat{x}_2^r)) \\ \vdots \\ -N\ell_n((\tilde{e}_n+\ell_n\tilde{e}_{n-1}+\cdots+\ell_n\cdots\ell_3\tilde{e}_2+\hat{x}_n)^r-\hat{x}_n^r) \\ N(\hat{x}_2+\tilde{e}_2)^r \\ \vdots \\ N(\hat{x}_n^r+\ell_{n-1}\cdots\ell_2((\hat{x}_2+\tilde{e}_2)^r-\hat{x}_2^r)) \\ N\ell_n\cdots\ell_2((\hat{x}_2+\tilde{e}_2)^r-\hat{x}_2^r) \end{bmatrix}\mathrm{d}t + \begin{bmatrix} (\boldsymbol{\varphi}_2^*(\bar{\hat{\boldsymbol{x}}}_2,\tilde{e}_2)-\ell_2\boldsymbol{\varphi}_1(y))^{\mathrm{T}} \\ \vdots \\ (\boldsymbol{\varphi}_n^*(\bar{\hat{\boldsymbol{x}}}_n,\bar{\tilde{\boldsymbol{e}}}_n)-\ell_n\boldsymbol{\varphi}_{n-1}^*(\bar{\hat{\boldsymbol{x}}}_{n-1},\bar{\tilde{\boldsymbol{e}}}_{n-1})^{\mathrm{T}} \\ \boldsymbol{\varphi}_1(y)^{\mathrm{T}} \\ \vdots \\ \ell_{n-1}\cdots\ell_2\boldsymbol{\varphi}_1(y)^{\mathrm{T}} \\ \ell_n\cdots\ell_2\boldsymbol{\varphi}_1(y)^{\mathrm{T}} \end{bmatrix}\mathrm{d}\boldsymbol{\omega} +$$

$$\begin{bmatrix} 0,\cdots,0,Np_0^r \end{bmatrix}^{\mathrm{T}} u^r\mathrm{d}t = \hat{\boldsymbol{f}}(\hat{\boldsymbol{x}},\bar{\tilde{\boldsymbol{e}}}_n)\mathrm{d}t + \hat{g}_1(\hat{\boldsymbol{x}},\bar{\tilde{\boldsymbol{e}}}_n)\mathrm{d}\boldsymbol{\omega} + \hat{\boldsymbol{g}}_2u^r\mathrm{d}t \tag{54}$$

Now, we prove conclusion 4). By (11), (35), (39), (41), $e_i = x_i - \hat{x}_i$, and $x_i = \hat{x}_i + \tilde{e}_i + \ell_i\cdots\ell_3\tilde{e}_2 + \cdots + \ell_i\tilde{e}_{i-1}$ for $i = 2,\cdots,n$, one gets (54), where $\bar{\tilde{\boldsymbol{e}}}_i = (\tilde{e}_2,\cdots,\tilde{e}_i)^{\mathrm{T}}$, $\bar{\hat{\boldsymbol{x}}}_i = (x_1,\hat{x}_2,\cdots,\hat{x}_i)^{\mathrm{T}}$, $\boldsymbol{\varphi}_i^*(\bar{\hat{\boldsymbol{x}}}_i,\bar{\tilde{\boldsymbol{e}}}_i)$ is obtained by replacing $\bar{\boldsymbol{x}}_i$ in $\boldsymbol{\varphi}_i(\bar{\boldsymbol{x}}_i)$ with $\bar{\hat{\boldsymbol{x}}}_i$ and $\bar{\tilde{\boldsymbol{e}}}_i$, $i = 2,\cdots,n$. By $\hat{\xi}_i = \hat{x}_i - \hat{x}_i^*$, $\hat{\boldsymbol{g}}_2 = [0,\cdots,0,Np_0^r]^{\mathrm{T}}$, $U_n(\bar{\hat{\boldsymbol{x}}}_n,\bar{\tilde{\boldsymbol{e}}}_n) = \frac{1}{4}\sum_{i=1}^n p_i\hat{\xi}_i^4 + \frac{1}{4}\ell\sum_{i=2}^n \tilde{e}_i^4$, one obtains

$$L_{\hat{\boldsymbol{g}}_2}U_n = \frac{\partial U_n}{\partial\hat{\boldsymbol{x}}}\hat{\boldsymbol{g}}_2 = \frac{\partial U_n}{\partial\hat{x}_n}Np_0^r = Np_0^rp_n\hat{\xi}_n^3 \tag{55}$$

Thus, (7) becomes

$$u = -\left(N^{-1}p_n^{-1}p_0^{-r}\hat{\xi}_n^{-3}\ell_\gamma(|Np_0^rp_n\hat{\xi}_n^3R(\hat{\boldsymbol{x}})^{-\frac{1}{2}}|)\right)^{\frac{1}{r}} \tag{56}$$

where $R(\hat{\boldsymbol{x}}) > 0$ is a scalar-valued function. Choosing

$$\gamma(s) = \frac{r}{3+r}s^{\frac{3+r}{r}} \tag{57}$$

one gets $(\dot{\gamma})^{-1}(s) = s^{\frac{r}{3}}$, which one substitutes into the definition of $\ell_\gamma(s)$ to obtain

$$\ell_\gamma(s) = ss^{\frac{r}{3}} - \frac{r}{3+r}s^{\frac{3+r}{3}} = \frac{3}{3+r}s^{\frac{3+r}{3}} \tag{58}$$

Choosing

$$R(\hat{\boldsymbol{x}}) = \left(\frac{3+r}{3}(Np_n)^{-\frac{r}{3}}p_0^{-\frac{r^2}{3}}b_n^r\right)^{-\frac{6}{3+r}} \tag{59}$$

by (56) and (58), one has

$$u(\hat{\boldsymbol{x}}) = -\left(N^{-1}p_n^{-1}p_0^{-r}\hat{\xi}_n^{-3}\frac{3}{3+r}(Np_0^rp_n\hat{\xi}_n^3)^{\frac{3+r}{3}} \times \frac{3+r}{3}(Np_n)^{-\frac{r}{3}}p_0^{-\frac{r^2}{3}}b_n^r\right)^{\frac{1}{r}} = -b_n\hat{\xi}_n \tag{60}$$

which has exactly the same form as (36). Since (60) achieves GAS in probability, by (8), (55), (57), and (59), one can get the inverse optimal controller (52). From (9), (57) and (59), one gets (53). □

4 A simulation example

Consider the following system

$$\begin{aligned} \mathrm{d}\eta_1 &= \eta_2^{\frac{5}{3}}\mathrm{d}t + \frac{1}{10}\eta_1^{\frac{4}{3}}\mathrm{d}\omega \\ \mathrm{d}\eta_2 &= \upsilon^{\frac{5}{3}}\mathrm{d}t + \frac{1}{10}\left(\eta_1^{\frac{4}{3}}+\eta_2\right)\sin\eta_2\mathrm{d}\omega \\ y &= \eta_1 \end{aligned} \tag{61}$$

where $\psi_1(\eta_1) = \frac{1}{10}\eta_1^{\frac{4}{3}}$ and $\psi_2(\eta_1,\eta_2) = \frac{1}{10}(\eta_1^{\frac{4}{3}}+\eta_2)\sin\eta_2$. Next, we need to prove the following inequality

$$\left|\frac{1}{10}\eta_2\sin\eta_2\right| \le \frac{1}{10}\eta_2^{\frac{4}{3}} \tag{62}$$

When $|\eta_2| = 0$, one has $|\frac{1}{10}\eta_2\sin\eta_2| = \frac{1}{10}\eta_2^{\frac{4}{3}}$; when $0 < |\eta_2| < 1$, one has $\left|\frac{\frac{1}{10}\sin\eta_2}{\eta_2}\right| \le \frac{1}{10} \le \frac{1}{10}|\eta_2|^{-\frac{2}{3}}$, so $|\frac{1}{10}\eta_2\sin\eta_2| \le \frac{1}{10}\eta_2^{\frac{4}{3}}$; when $|\eta_2| \ge 1$, one has $|\frac{1}{10}\eta_2\sin\eta_2| \le \frac{1}{10}|\eta_2| \le \frac{1}{10}|\eta_2|^{\frac{4}{3}}$.

From (62) and the definition of $\psi_1(\eta_1)$ and $\psi_2(\eta_1,\eta_2)$, we get $a = 1/10$ in Assumption 1. We apply the above design procedure to (61). Introduce

$$\eta_1 = x_1, \quad \eta_2 = N^{\frac{3}{5}}x_2, \quad \upsilon = p_0N^{\frac{3}{5}+\frac{9}{25}}u \tag{63}$$

define $\xi_1 = x_1$, $\xi_2 = x_2 - x_2^* = x_2 + b_1\xi_1$, and choose $V_2(\xi_1,\xi_2) = \frac{1}{4}p_1\xi_1^4 + \frac{1}{4}p_2\xi_2^4$. A direct calculation leads to

$$\begin{aligned} \mathcal{L}V_2 \le\ & N(-(c_1 - b_{2,1,1} - b_{2,2,1} - b_{2,1,2} - b_{2,1,3})\xi_1^{\frac{14}{3}} + \\ & p_2p_0^{\frac{5}{3}}\xi_2^3u^{\frac{5}{3}} + (\rho_{2,1}+\rho_{2,2}+\rho_{2,3})\xi_2^{\frac{14}{3}}) \end{aligned} \tag{64}$$

where $N \ge 1$, $p_0 > 0$ are rescaling factors, and $p_1 > 0$, $p_2 > 0$, $x_2^* = -b_1\xi_1 = -(\frac{c_1}{p_1}+\frac{3}{200})^{\frac{3}{5}}x_1$, $\rho_{2,1} = \frac{5}{14}(\frac{9}{14d_1})^{\frac{9}{5}}(\frac{5p_1}{3})^{\frac{14}{5}} + \frac{3}{14}(\frac{11}{14d_2})^{\frac{11}{3}}(\frac{10p_1}{3}b_1^{\frac{2}{3}})^{\frac{14}{3}}$, $\rho_{2,2} = \frac{9}{14}(\frac{5}{14d_3})^{\frac{5}{9}}(2^{\frac{2}{3}}p_2b_1^{\frac{8}{3}})^{\frac{14}{9}} + 2^{\frac{2}{3}}b_1p_2$, $\rho_{2,3} = \frac{3}{7}(\frac{4}{7d_4})^{\frac{4}{3}}((\frac{3\cdot2^{\frac{5}{3}}}{100}p_2b_1^{\frac{8}{3}} + \frac{3}{100}p_2(1+b_1^2))^{\frac{7}{3}} + \frac{3\cdot2^{\frac{5}{3}}}{100}p_2)$. In simulation, we choose $c_1 = 2$, $c_2 = 1.75$, $b_{2,1,1} = 0.25$, $b_{2,2,1} = 1$, $b_{2,1,3} = 0.2$, $b_{2,1,4} = 0.05$, $p_1 = 0.1$, $p_2 = 0.001$, $p_0 = 1000^{\frac{3}{5}}$, and (64) becomes

$$\mathcal{L}V_2 \le N\left(-0.5\xi_1^{\frac{14}{3}} - 1.75\xi_2^{\frac{14}{3}} + \xi_2^3(u^{\frac{5}{3}} - x_3^{*\frac{5}{3}})\right) \tag{65}$$

where $x_3^* = -b_2\xi_2 = -2^{\frac{3}{5}}(x_2+b_1x_1)$ and $b_2 = 2^{\frac{3}{5}}$.

The reduced-order observer and the certainty equivalence controller are, respectively, chosen as

$$\begin{aligned} \dot{\hat{z}}_2 &= Np_0^{\frac{5}{3}}u^{\frac{5}{3}} - N\ell_2\hat{x}_2^{\frac{5}{3}} \\ u &= -b_2(\hat{x}_2+b_1x_1) = -2^{\frac{3}{5}}(\hat{x}_2+b_1x_1) \end{aligned} \tag{66}$$

With $W_2 = \frac{1}{4}\ell\tilde{e}_2^4$, $\ell = 60$, $N = 25$, $\ell_2 = 10$, by (65), (66), and the definition of x_3^*, a tedious but straightforward calculation leads to $\mathcal{L}U_2 \le -25(0.04\xi_1^{\frac{14}{3}} + 0.04\xi_2^{\frac{14}{3}} + 7\tilde{e}_2^{\frac{14}{3}})$, where $U_2(\xi_1,\xi_2,\tilde{e}_2) = V_2(\xi_1,\xi_2) + W_2(\tilde{e}_2)$.

With the initial values $\eta_1(0) = 0.01$, $\eta_2(0) = -0.5$, and $\hat{z}_2(0) = -0.2$, Fig.1 gives the response of the closed-loop system (61), (63), and (66), which demonstrates the effectiveness of the output-feedback controller.

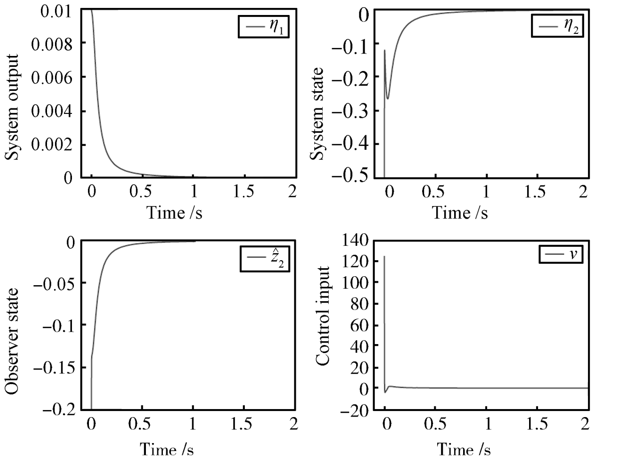


Fig. 1 The responses of closed-loop system (61), (63), and (66)

5 Concluding remarks

This paper deals with the output-feedback stabilization problem for a class of stochastic high-order nonlinear systems with a ratio of odd integers power for the first time. The designed smooth output-feedback controller ensures that the equilibrium at the origin of the closed-loop system is GAS in probability, the output can be regulated to the origin. Furthermore, the problem of inverse optimal stabilization in probability is also solved.

A remaining problem is that one needs to continually choose $c_{i,j}$, which will lead to more limitations in practical applications. Our future work is to construct an effective searching algorithm to choose these coefficients by computer.

References

1 Deng H, Krstić M. Output-feedback stochastic nonlinear stabilization. *IEEE Transactions on Automatic Control*, 1999, **44**(2): 328−333

2 Liu S J, Zhang J F, Jiang Z P. Decentralized adaptive output-feedback stabilization for large-scale stochastic nonlinear systems. *Automatica*, 2007, **43**(2): 238−251

3 Liu Y G, Zhang J F. Practical output-feedback risk-sensitive control for stochastic nonlinear systems with stable zero-dynamics. *SIAM Journal on Control and Optimization*, 2006, **45**(3): 885−926

4 Wu Z J, Xie X J, Zhang S Y. Adaptive backstepping controller design using stochastic small-gain theorem. *Automatica*, 2007, **43**(4): 608−620

5 Professor Wei Lin [Online], available: http://nonlinear.cwru.edu/~linwei/, March 4, 2009

6 Xie X J, Tian J. State-feedback stabilization for high-order stochastic nonlinear systems with stochastic inverse dynamics. *International Journal of Robust and Nonlinear Control*, 2007, **17**(14): 1343−1362

7 Li W Q, Xie X J. Inverse optimal stabilization for stochastic nonlinear systems whose linearizations are not stabilizable. *Automatica*, 2009, **45**(2): 498−503

8 Tian J, Xie X J. Adaptive state-feedback stabilization for high-order stochastic nonlinear systems with uncertain control coefficients. *International Journal of Control*, 2007, **80**(9): 1503−1516

9 Tian Jie, Xie Xue-Jun. Adaptive state-feedback stabilization for high-order stochastic nonlinear systems with time-varying control coefficients. *Acta Automatica Sinica*, 2008, **34**(9): 1188−1191

10 Xie X J, Tian J. Adaptive state-feedback stabilization of high-order stochastic systems with nonlinear parameterization. *Automatica*, 2009, **45**(1): 126−133

11 Xie X J, Li W Q. Output-feedback control of a class of high-order stochastic nonlinear systems. *International Journal of Control*, 2009, **82**(9): 1692−1705

12 Krstic M, Deng H. *Stabilization of Nonlinear Uncertain Nonlinear Systems*. New York: Springer, 1998

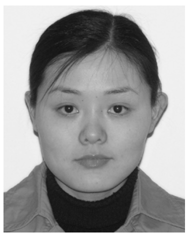
13 Polendo J, Qian C J. A generalized framework for global output feedback stabilization of nonlinear systems. In: Proceedings of the 44th IEEE Conference on Decision and Control. Sevilla, Spain: IEEE, 2005. 2646−2651

14 Qian C J, Lin W. Non-Lipschitz continuous stabilizers for nonlinear systems with uncontrollable unstable linearization. *Systems and Control Letters*, 2001, **42**(3): 185−200

15 Ioannou P A, Sun J. *Robust Adaptive Control*. New Jersey: Prentice-Hall, 1996

16 Duan Na, Xie Xue-Jun, Liu Hai-Kuan. An improved control algorithm of high-order nonlinear systems. *Acta Automatica Sinica*, 2008, **34**(10): 1262−1267

LIU Liang Master student at the Institute of Automation, Qufu Normal University. His research interest covers stochastic nonlinear control.
E-mail: smithll@163.com



DUAN Na Ph. D. candidate at Qufu Normal University and a lecturer at Xuzhou Normal University. Her research interest covers nonlinear adaptive control and stochastic nonlinear control.
E-mail: duanna08@163.com

XIE Xue-Jun Professor at Xuzhou Normal University. His research interest covers stochastic nonlinear control system and adaptive control. Corresponding author of this paper.
E-mail: xuejunxie@126.com