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Shannon Entropy for the Feller-Pareto (FP) Family and Order Statistics of FP Subfamilies

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Abstract. In this paper, we derive the exact analytical expressions of entropy for the Feller-Pareto(FP) family and order statistics of FP subfamilies. The FP family is a very general unimodal distribution which includes a hierarchy of Pareto models, Transformed beta family , Burr families, Generalized Pareto, and Inverse Burr distributions. These distributions apply in reliability, actuarial science, economics, finance and telecommunications.We also present entropy ordering property for the sample minimum and maximum of FP subfamilies.

Keywords: Burr distribution, Entropy, Feller-Pareto distribution, Order statistics, Pareto models, Transformed beta family

1. Introduction:

The concept of entropy originated in the nineteenth century by Shannon [11]. During the last sixty years, a number of research papers and monographs discussed and extended Shannon's original work. Renyi [10], Cover and Tomas [3], Lazo and Rathie [8], Kapur [5], Kullback [7], are among the researchers in this area.Also,the Shanon information of order statistics have been studied by a few authors.Among them Wong and Chen [12], Park [9], provided some results of Shannon entropy for order statistics.The Shannon entropy for a continuous random variable X with probability density function $f_X(x)$ is defined as

$$H(X) = -\int_{-\infty}^{\infty} f_X(x) \log f_X(x) dx = -\int_0^1 \log f_X(F_X^{-1}(u)) du. \qquad (1)$$

This is a mathematical measure of information which measures the average reduction of uncertainty of X.The aim of this paper is to determine the exact form of the Shannon information for the Feller-Pareto family and order statistics of FP subfamilies.The FP family includes a variety of unimodal distributions,It includes a hierarchy of Pareto models, Transformed beta family, Generalized Pareto, and Inverse Burr distributions.These distributions arise as tractable parametric models in reliability, actuarial science, economics, finance, telecommunications.The rest of this paper is organized as follows. In Section 2 , we describe two different representations of the FP family and some general distributions which the FP family includes as the special cases.Also, we provide a technique by one-dimensional integral to calculate the entropy measure of the FP, Pareto (IV), Inverse Burr, and Generalized Pareto Distributions.In Section 3,we present Shannon entropy for jth order statistic and entropy ordering property for the sample minimum and maximum of FP subfamilies.

2. Entropy for the Feller-Pareto and related distributions:

The FP family traces its roots back to Feller [4], but the form we consider here was first defined and investigated by Arnold and Laguna [2]. Feller [4] defined the FP family as follows.

Definition 2.1: The FP distribution is defined as the distribution of $X = \mu + \theta(Y^{-1} - 1)^{\gamma}$, where $Y \sim Beta(\lambda_1, \lambda_2)$, $\theta > 0$, and $\gamma > 0$, So we write $X \sim FP(\mu, \theta, \gamma, \lambda_1, \lambda_2)$. The pdf of X is derived from the transformation of Y as

$$f_X(x) = \frac{1}{\beta(\lambda_1, \lambda_2)\theta\gamma} \left(\frac{x-\mu}{\theta}\right)^{\left(\frac{\lambda_2}{\gamma}\right)-1} \left\{1 + \left(\frac{x-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right\}^{-(\lambda_1+\lambda_2)}, \quad x \geq \mu, \tag{2}$$

where $\beta(\lambda_1, \lambda_2) = \dfrac{\Gamma(\lambda_1)\Gamma(\lambda_2)}{\Gamma(\lambda_1+\lambda_2)}$, and $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-t}dt$.

There is also another definition for the FP distribution.

Definition 2.2: Let Y_1 and Y_2 be two independent random variables having gamma distribution with scale parameter θ and shape parameters λ_1 and λ_2.Then $X = \mu + \theta\left(\frac{Y_2}{Y_1}\right)^{\gamma}$ has $FP(\mu, \theta, \gamma, \lambda_1, \lambda_2)$ distribution, where $f_X(x)$ is defined in (2).

Arnold [1, chap.3] showed that the FP distribution is a generalization of the Pareto (IV) distribution, so the density function of the Pareto (IV) $(\mu, \theta, \gamma, \alpha)$ is given by

$$g_X(x) = \frac{\alpha}{\theta\gamma} \left(\frac{x-\mu}{\theta}\right)^{\frac{1}{\gamma}-1} \left\{1 + \left(\frac{x-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right\}^{-(\alpha+1)}, \quad x \geq \mu, \tag{3}$$

where $-\infty < \mu < +\infty$ is the location parameter, $\theta > 0$ is the scale parameter, $\gamma > 0$ is the inequality parameter and $\alpha > 0$ is the shape parameter which characterizes the tail of the distribution.Also, Klugman et al.[6] showed that the density function of the transformed beta(TB) family is given by

$$t_X(x) = \frac{\gamma}{\beta(\alpha, \tau)x} \left(\frac{x}{\theta}\right)^{\gamma\tau} \left\{1 + \left(\frac{x}{\theta}\right)^{\gamma}\right\}^{-(\alpha+\tau)}, \quad x > 0. \tag{4}$$

Moreover, we can see in the following summary, these families are still quite general and include other important distributions.

- Pareto (IV)$(\mu,\,\theta,\,\gamma,\,\alpha) = FP(\mu,\,\theta,\,\gamma,\,\alpha,1)$:

$$\begin{aligned}\text{Pareto (I)}(\,\theta,\ \alpha) &= FP(\theta,\,\theta,\,1,\,\alpha,1),\\ \text{Pareto (II)}(\mu,\,\theta,\ \alpha) &= FP(\mu,\,\theta,\,1,\,\alpha,1),\\ \text{Pareto (III)}(\mu,\,\theta,\,\gamma) &= FP(\mu,\,\theta,\,\gamma,\,1,1),\end{aligned}$$

- TB$(\theta,\,\gamma,\,\alpha,\,\tau) = FP(0,\,\theta,\,\frac{1}{\gamma},\,\alpha,\tau)$:

$$\begin{aligned}\text{Burr}(\theta,\,\gamma,\,\alpha) &= \text{TB}(\theta,\,\gamma,\,\alpha,\,1),\\ \text{loglogistic}(\theta,\,\gamma) &= \text{Burr}(\theta,\,\gamma,\,1),\\ \text{Paralogistic}(\theta,\,\alpha) &= \text{Burr}(\theta,\,\alpha,\,\alpha),\end{aligned}$$

- Generalized Pareto $(\theta,\ \alpha,\,\tau) = TB(\theta,\,1,\,\alpha,\,\tau)$:

$$\begin{aligned}\text{Scaled-F distribution}(\alpha,\tau) &= \text{Generalized Pareto}(1,\ \alpha,\tau),\\ \text{Inverse Pareto(II)}(\theta,\,\tau) &= \text{Generalized Pareto}(\theta,\ 1,\,\tau),\end{aligned}$$

- Inverse Burr $(\theta,\ \gamma,\,\tau) = TB(\theta,\,\gamma,\,1,\,\tau)$:

$$\begin{aligned}\text{loglogistic}(\theta,\,\gamma) &= \text{Inverse Burr}(\theta,\,\gamma,\,1),\\ \text{Inverse Pareto}(\theta,\,\tau) &= \text{Inverse Burr}(\theta,\,1,\,\tau),\\ \text{Inverse Paralogistic}(\theta,\,\gamma) &= \text{Inverse Burr}(\theta,\,\gamma,\,1).\end{aligned}$$

Now, Suppose X is a random variable with $FP(\mu,\,\theta,\,\gamma,\,\lambda_1,\lambda_2)$ distribution with pdf of Feller-Pareto given in (2). Formula (1) is important for our computations. The log-density of (2) is

$$\begin{aligned}\log f_X(x) &= -\log\left(\beta(\lambda_1,\lambda_2)\right) - \log(\theta\gamma) + \left(\frac{\lambda_2}{\gamma} - 1\right)\log\left(\frac{x-\mu}{\theta}\right)\\ &\quad -(\lambda_1+\lambda_2)\log\left(1 + \left(\frac{x-\mu}{\theta}\right)^{\dfrac{1}{\gamma}}\right),\end{aligned} \tag{5}$$

and the entropy is

$$\begin{aligned}H(X) &= E\left(-\log f_X(x)\right) = \log\left(\frac{\Gamma(\lambda_1+\lambda_2)}{\theta\gamma}\right) - \log(\Gamma(\lambda_1)) - \log(\Gamma(\lambda_2))\\ &\quad + \left(1 - \frac{\lambda_2}{\gamma}\right)E\left(\log\left(\frac{X-\mu}{\theta}\right)\right) + (\lambda_1+\lambda_2)E\left[\log\left(1 + \left(\frac{X-\mu}{\theta}\right)^{\dfrac{1}{\gamma}}\right)\right].\end{aligned} \tag{6}$$

To determine an expression for $H(X)$, we need to find $E\left(\log\left(\frac{X-\mu}{\theta}\right)\right)$ and $E\left[\log\left(1 + \left(\frac{X-\mu}{\theta}\right)^{\dfrac{1}{\gamma}}\right)\right]$.

Derivation of these expectations are based on the following strategy.

$$h(r) = E[(\frac{X-\mu}{\theta})^r] \quad = \int_\mu^\infty \frac{1}{\beta(\lambda_1,\lambda_2)\theta\gamma}(\frac{x-\mu}{\theta})^{r+\dfrac{\lambda_2}{\gamma}-1}\{1+(\frac{x-\mu}{\theta})^{\frac{1}{\gamma}}\}^{-(\lambda_1+\lambda_2)}dx. \tag{7}$$

By the change of variable $\left(\frac{x-\mu}{\theta}\right)^{\frac{1}{\gamma}} = \frac{u}{1-u}$, $0 < u < 1$, we obtain:

$$\begin{aligned} h(r) &= E\left[\left(\frac{X-\mu}{\theta}\right)^r\right] = \int_0^1 \frac{1}{\beta(\lambda_1,\lambda_2)} u^{r\gamma+\lambda_2-1}(1-u)^{\lambda_1-r\gamma-1}du \\ &= \frac{\Gamma(r\gamma+\lambda_2)\Gamma(\lambda_1-r\gamma)}{\Gamma(\lambda_1)\Gamma(\lambda_2)}, \qquad \lambda_1 - r\gamma \neq 0, -1, -2, ... \end{aligned} \tag{8}$$

Differentiating both side of (8) with respect to r we obtain:

$$\begin{aligned} \frac{d}{dr}h(r) &= E[(\frac{X-\mu}{\theta})^r \log(\frac{X-\mu}{\theta})] \\ &= \frac{\gamma}{\Gamma(\lambda_1)\Gamma(\lambda_2)}[\Gamma'(r\gamma+\lambda_2)\Gamma(\lambda_1-r\gamma) - \Gamma(r\gamma+\lambda_2)\Gamma'(\lambda_1-r\gamma)]. \end{aligned} \tag{9}$$

From relation (9), at $r = 0$ we obtain

$$\begin{aligned} E\left[\log\left(\frac{X-\mu}{\theta}\right)\right] &= \frac{\gamma}{\Gamma(\lambda_1)\Gamma(\lambda_2)}\left[\Gamma'(\lambda_2)\Gamma(\lambda_1) - \Gamma(\lambda_2)\Gamma'(\lambda_1)\right] \\ &= \gamma\left[\psi(\lambda_2) - \psi(\lambda_1)\right], \end{aligned} \tag{10}$$

where ψ is the digamma function defined by $\psi(\lambda) = \frac{d}{d\lambda}\log\Gamma(\lambda)$.

Now we calculate

$$\begin{aligned} k(r) &= E\left[\left(1+\left(\frac{X-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)^r\right] \\ &= \int_\mu^\infty \frac{1}{\beta(\lambda_1,\lambda_2)\theta\gamma}\left(\frac{x-\mu}{\theta}\right)^{\frac{\lambda_2}{\gamma}-1}\left\{1+\left(\frac{x-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right\}^{-(\lambda_1+\lambda_2)+r} dx. \end{aligned} \tag{11}$$

By using the change of variable$\left(\frac{x-\mu}{\theta}\right) = \left(\frac{t}{1-t}\right)^\gamma$, $0 < t < 1$, we obtain:

$$\begin{aligned} k(r) &= \frac{1}{\beta(\lambda_1,\lambda_2)}\int_0^1 t^{\lambda_2-1}(1-t)^{\lambda_1-r\gamma-1}dt \\ &= \frac{\Gamma(\lambda_1+\lambda_2)\Gamma(\lambda_1-r)}{\Gamma(\lambda_1)\Gamma(\lambda_1+\lambda_2-r)}. \end{aligned} \tag{12}$$

$$\begin{aligned} k'(r) &= E\left[\left(1+\left(\frac{X-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)^r \log\left(1+\left(\frac{X-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)\right] \\ &= \frac{\Gamma(\lambda_1+\lambda_2)\left[\Gamma(\lambda_1-r)\Gamma'(\lambda_1+\lambda_2-r) - \Gamma'(\lambda_1-r)\Gamma(\lambda_1+\lambda_2-r)\right]}{\Gamma(\lambda_1)\left[\Gamma(\lambda_1+\lambda_2-r)\right]^2}. \end{aligned} \tag{13}$$

Using relation (13), at $r = 0$ we have

$$E\left[\log\left(1+\left(\frac{X-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)\right] = \frac{\Gamma(\lambda_1+\lambda_2)\left[\Gamma(\lambda_1)\Gamma'(\lambda_1+\lambda_2)-\Gamma'(\lambda_1)\Gamma(\lambda_1+\lambda_2\right]}{\Gamma(\lambda_1)\left[\Gamma(\lambda_1+\lambda_2\right]^2}$$
$$= \psi(\lambda_1+\lambda_2)-\psi(\lambda_1). \tag{14}$$

Putting (10) and (14) in relation (6) we have:

$$\begin{aligned} H(X) &= \log\left(\frac{\Gamma(\lambda_1+\lambda_2)}{\theta\gamma}\right)-\log(\Gamma(\lambda_1))-\log(\Gamma(\lambda_2)) \\ &\quad +(\gamma-\lambda_2)\left[\psi(\lambda_2)-\psi(\lambda_1)\right] \\ &\quad +(\lambda_1+\lambda_2)\left[\psi(\lambda_1+\lambda_2)-\psi(\lambda_1)\right]. \end{aligned} \tag{15}$$

We note that Shannon entropy for other related distributions are given in Table 1.

Table 1: Table of Shannon Entropy

family name	density function	Shannon entropy
Pareto (IV)$(\mu,\,\theta,\,\gamma,\,\alpha)$	$f_X(x)\dfrac{\alpha\left(\frac{x-\mu}{\theta}\right)^{\frac{1}{\gamma}-1}}{\theta\gamma\left(1+\left(\frac{x-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)^{\alpha+1}}$ $x>\mu,\quad \mu\in(-\infty,\infty)$ $\theta>0,\ \gamma>0,\ \alpha>0$	$H(X)=\log\left(\frac{\theta\gamma}{\alpha}\right)$ $+(\gamma-1)(\psi(1)-\psi(\alpha))+1+\frac{1}{\alpha}$
Pareto (III)$(\mu,\,\theta,\,\gamma)$	$f_X(x)=\frac{1}{\theta\gamma}\left(\frac{x-\mu}{\theta}\right)^{\frac{1}{\gamma}-1}$ $\times\left\{1+\left(\frac{x-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right\}^{-2}$ $x>\mu,\mu\in R,\ \ \theta>0,\ \gamma>0$	$H(X)=\log(\theta\gamma)+2$
Pareto (II)$(\mu,\,\theta,\,\alpha)$	$f_X(x)=\dfrac{\alpha}{\theta\left(1+\left(\frac{x-\mu}{\theta}\right)\right)^{\alpha+1}}$ $x>\mu,\alpha>0,\ \ \theta>0$	$H(X)=\log\left(\frac{\theta}{\alpha}\right)+\frac{\alpha+1}{\alpha}$
Pareto (I)$(\theta,\,\alpha)$	$f_X(x)=\dfrac{\alpha}{\theta\left(1+\left(\frac{x-\theta}{\theta}\right)\right)^{\alpha+1}}$ $x>\theta,\alpha>0,\ \ \theta>0$	$H(X)=\log\left(\frac{\theta}{\alpha}\right)+\frac{\alpha+1}{\alpha}$ $=H(\text{Pareto (II)})$
Transformed Beta$(\theta,\,\gamma,\,\alpha,\,\tau)$	$f_X(x)=\frac{\Gamma(\alpha+\tau)\gamma}{\Gamma(\alpha)\Gamma(\tau)x}\left(\frac{x}{\theta}\right)^{\gamma\tau}$ $\times\dfrac{1}{\left(1+\left(\frac{x}{\theta}\right)^{\gamma}\right)^{\alpha+\tau}}$ $x>0,\ \theta>0,\alpha>0,$ $\tau>0,\ \ \gamma>0$	$H(X)=\log(\beta(\alpha,\tau))+\log\left(\frac{\theta}{\gamma}\right)$ $+\left(\frac{1}{\gamma}-\tau\right)[\psi(\tau)-\psi(\alpha)]$ $+(\alpha+\tau)[\psi(\alpha+\tau)-\psi(\alpha)]$

family name	density function	Shannon entropy
Burr$(\theta,\ \gamma,\ \alpha)$	$f_X(x)=\frac{\alpha\gamma}{\theta}\left(\frac{x}{\theta}\right)^{\gamma-1}\left(1+\left(\frac{x}{\theta}\right)^{\gamma}\right)^{-(\alpha+1)}$ $x>0,\ \theta>0, \alpha>0,\ \ \gamma>0$	$H(X)=\log\left(\frac{\theta}{\gamma\alpha}\right)$ $+(\frac{1}{\gamma}-1)(\psi(1)-\psi(\alpha))+\frac{\alpha+1}{\alpha}$
loglogistic$(\theta,\ \gamma)$	$f_X(x)=\frac{\gamma}{\theta}\left(\frac{x}{\theta}\right)^{\gamma-1}\left(1+\left(\frac{x}{\theta}\right)^{\gamma}\right)^{-2}$ $x>0,\ \theta>0,\ \ \gamma>0$	$H(X)=\log\left(\frac{\theta}{\gamma}\right)+2$
Scaled-F distribution $(\alpha,\ \tau)$	$f_X(x)=\frac{1}{\beta(\alpha,\tau)}x^{\tau-1}\left(1+x\right)^{-(\alpha+\tau)}$ $x>0,\ ,\ \alpha>0,\ \ \tau>0$	$H(X)=\log(\beta(\alpha,\tau))$ $+\left(1-\tau\right)\left[\psi(\tau)-\psi(\alpha)\right]$ $+(\alpha+\tau)\left[\psi(\alpha+\tau)-\psi(\alpha)\right]$
Paralogistic$(\theta,\ \alpha)$	$f_X(x)=\frac{\alpha^2}{\theta}\left(\frac{x}{\theta}\right)^{\alpha-1}\left(1+\left(\frac{x}{\theta}\right)^{\alpha}\right)^{-(\alpha+1)}$ $x>0,\ \theta>0, \alpha>0$	$H(X)=\log\left(\frac{\theta}{\alpha^2}\right)$ $+(\frac{1-\alpha}{\alpha})(\psi(1)-\psi(\alpha))$ $+\frac{\alpha+1}{\alpha}$
Generalized Pareto $(\theta,\ \alpha,\ \tau)$	$f_X(x)=\frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)x}\left(\frac{x}{\theta}\right)^{\tau}\frac{1}{\left(1+\frac{x}{\theta}\right)^{\alpha+\tau}}$ $x>0,\ \theta>0,\ \ \alpha>0,\ \ \tau>0$	$H(X)=\log(\beta(\alpha,\tau))+\log\left(\theta\right)$ $+\left(1-\tau\right)\left[\psi(\tau)-\psi(\alpha)\right]$ $+(\alpha+\tau)\left[\psi(\alpha+\tau)-\psi(\alpha)\right]$
Inverse Pareto $(\theta,\ \ \tau)$	$f_X(x)=\frac{\tau}{x}\left(\frac{x}{\theta}\right)^{\tau}\left(\left(1+\frac{x}{\theta}\right)^{-(1+\tau)}\right)$ $x>0,\ \theta>0,\ \ \ \tau>0$	$H(X)=\log\left(\theta\tau\right)$ $+\left(1-\tau\right)\left[\psi(\tau)-\psi(1)\right]$ $+(1+\tau)\left[\psi(1+\tau)-\psi(1)\right]$
Inverse Burr $(\theta,\ \gamma,\ \tau)$	$f_X(x)=\frac{\gamma\tau}{x}\left(\frac{x}{\theta}\right)^{\tau\gamma}\left(\left(1+\left(\frac{x}{\theta}\right)^{\gamma}\right)^{-(1+\tau)}\right)$ $x>0,\ \theta>0,\ \ \ \tau>0,\ \ \gamma>0$	$H(X)=\log\left(\frac{\theta\tau}{\gamma}\right)$ $+\left(\frac{1}{\gamma}-\tau\right)\left[\psi(\tau)-\psi(1)\right]$ $+(1+\tau)\left[\psi(1+\tau)-\psi(1)\right]$
Inverse Paralogistic $(\theta,\ \gamma)$	$f_X(x)=\frac{\gamma}{x}\left(\frac{x}{\theta}\right)^{\gamma}\left(\left(1+\left(\frac{x}{\theta}\right)^{\gamma}\right)^{-2}\right)$ $x>0,\ \theta>0,\ \ \ \gamma>0$	$H(X)=\log\left(\frac{\theta}{\gamma}\right)$ $+2\left[\psi(2)-\psi(1)\right]$

3. The Shannon Entropy for Order statistics of FP Subfamilies:

Let $X_1,...,X_n$ be a random sample from a distribution $F_X(x)$ with density $f_X(x)>0$. The order statistics of this sample is defined by the arrangement of $X_1,...,X_n$ from the smallest to the largest, by $Y_1<Y_2<...<Y_n$. The density of Y_j , $j=1,...,n$, is

$$f_{Y_j}(y)=\frac{n!}{(j-1)!(n-j)!}f_X(y)\left[F_X(y)\right]^{j-1}\left[1-F_X(y)\right]^{n-j}. \tag{16}$$

Now, let $U_1,U_2,...,U_n$ be a random sample from $U(0,1)$ with the order statistics $W_1<W_2<...<W_n$. The density of W_j , $j=1,...,n$, is

$$f_{W_j}(w)=\frac{1}{B(j,n-j+1)}w^{j-1}\left[1-w\right]^{n-j},\quad 0<w<1, \tag{17}$$

where $B(j, n-j+1) = \dfrac{\Gamma(j)\Gamma(n-j+1)}{\Gamma(n+1)} = \dfrac{(j-1)!(n-j)!}{n!}$.

The entropy of the beta distribution is

$$H_n(W_j) = -(j-1)\left[\psi(j) - \psi(n+1)\right] - (n-j)\left[\psi(n+1-j) - \psi(n+1)\right] + \log B(j, n-j+1),$$

where $\psi(t) = \dfrac{d \log \Gamma(t)}{dt}$, $\psi(n+1) = \psi(n) + \dfrac{1}{n}$.

Using the fact that $W_j = F_X(Y_j)$ and $Y_j = F_X^{-1}(W_j)$, $j = 1, 2, ..., n$, are one to one transformations, the entropies of order statistics can be computed by

$$\begin{aligned} H(Y_j) &= H_n(W_j) - E_{g_j}\left[\log f_X(F_X^{-1}(W_j))\right] \\ &= H_n(W_j) - \int f_j(y) \log f_X(y) dy \end{aligned} \tag{18}$$

where $H_n(W_j)$ is the entropy and E_{g_j} is an expectation of beta distribution. Now, we can have an application of (18) for the pareto (IV) distribution. Let X be a random variable having Pareto type (IV) (μ, θ, γ, α) with $F_X(x) = 1 - \left\{1 + \left(\dfrac{x-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right\}^{-\alpha}$. For computing $H(Y_j)$, we find $F_X^{-1}(W_j) = \dfrac{1}{\theta}\left[\left((1-W_j)^{-\frac{1}{\alpha}} - 1\right)^{\gamma} + \mu\right]$ and the expectation term in (18) as follows:

$$\begin{aligned} E_{g_j}\left[\log f_X\left(F_X^{-1}(W_j)\right)\right] &= \log\left(\frac{\alpha}{\theta\gamma}\right) + (1-\gamma).\frac{n!}{(n-j)!} \\ &\quad \times \sum_{k=0}^{j-1}(-1)^k \frac{\psi(1) - \psi(\alpha(n-j+k+1))}{k!(j-k-1)!(n-j+k+1)} \\ &\quad + \frac{\alpha+1}{\alpha}\left(\psi(n-j+1) - \psi(n+1)\right) \\ &= \log\left(\frac{\alpha}{\theta\gamma}\right) + (1-\gamma)t_{\alpha,j}(n) \\ &\quad + \frac{\alpha+1}{\alpha}\left(\psi(n-j+1) - \psi(n+1)\right), \end{aligned} \tag{19}$$

where $t_{\alpha,j}(n) = \dfrac{n!}{(n-j)!}\displaystyle\sum_{k=0}^{j-1}(-1)^k \dfrac{\psi(1) - \psi(\alpha(n-j+k+1))}{k!(j-k-1)!(n-j+k+1)}$.

Now, by (18) and (19) the entropy of order statistics is

$$H(Y_j) = H_n(W_j) - \log\left(\frac{\alpha}{\theta\gamma}\right) + (-1+\gamma)t_{\alpha,j}(n) + \frac{\alpha+1}{\alpha}\left(\psi(n) + \frac{1}{n} - \psi(n-j+1)\right). \tag{20}$$

In particular cases, the entropy for the sample minimum and maximum are

$$H(Y_1) = -\log\left(\frac{n\alpha}{\theta\gamma}\right) + (-1+\gamma)\left(\psi(1) - \psi(n\alpha)\right) + \frac{n\alpha+1}{n\alpha}, \tag{21}$$

and

$$H(Y_n) = -\log\left(\frac{n\alpha}{\theta\gamma}\right) + (-1+\gamma)\sum_{k=0}^{n-1}(-1)^k\binom{n}{k+1}\left(\psi(1)-\psi(\alpha(k+1))\right) + \frac{n-1}{n} + \frac{\alpha+1}{\alpha}\left(\psi(n+1)-\psi(1)\right). \quad (22)$$

If $n = 2m+1$ and $0 < \gamma \leq 1$, then the difference between $H(Y_{2m+1})$ and $H(Y_1)$ is

$$(1-\gamma)\sum_{k=0}^{2m-1}(-1)^k\binom{2m+1}{k+1}\left[\psi(\alpha(k+1))-\psi(1)\right] + \frac{\alpha+1}{\alpha}\left(\psi(2m+1)-\psi(1)\right) \geq 0. \quad (23)$$

Thus, when n is odd ,the entropy about the maximum is more than the entropy about the minimum in Pareto (IV) samples. Finally, we present Shannon entropy of order statistic and entropy ordering property for the sample minimum and maximum of several FP subfamilies in Table 2.

Table 2: Entropy of order statistics and entropy ordering for FP subfamilies

FP subfamilies and Distribution function	Entropy for order statistics	Entropy ordering
Pareto (III)(μ,θ,γ) $F_X(x)=1-\left(1+\left(\frac{x-\mu}{\theta}\right)^{\frac{1}{\gamma}}\right)^{-1}$ $x>\mu,\ \theta>0,\ 0<\gamma\le 1$	$H(Y_j)=H_n(W_j)+\log(\theta\gamma)$ $+(\gamma-1)t_{1,j}(n)$ $+2\left(\psi(n+1)-\psi(n-j+1)\right)$	$H(Y_1)\le H(Y_n)$ $for\ \ n=2m+1$
Pareto (II)(μ,θ,α) $F_X(x)=1-\left(1+\left(\frac{x-\mu}{\theta}\right)\right)^{-\alpha}$ $x>\mu,\alpha>0,\ \theta>0$	$H(Y_j)=H_n(W_j)+\log\left(\frac{\theta}{\alpha}\right)$ $+\frac{\alpha+1}{\alpha}\left(\psi(n+1)-\psi(n-j+1)\right)$	$H(Y_1)\le H(Y_n)$ $for\ \ all\ \ n$
Pareto (I)(θ,α) $F_X(x)=1-\left(1+\left(\frac{x-\theta}{\theta}\right)\right)^{-\alpha}$ $x>\theta,\alpha>0,\ \theta>0$	$H(Y_j)=H_n(W_j)+\log\left(\frac{\theta}{\alpha}\right)$ $+\frac{\alpha+1}{\alpha}\left(\psi(n+1)-\psi(n-j+1)\right)$	$H(Y_1)\le H(Y_n)$ $for\ \ all\ \ n$
Transformed Beta$(\theta,\gamma,\alpha,1)$ $F_X(x)=1-\left(1+\left(\frac{x}{\theta}\right)^{\gamma}\right)^{-\alpha}$ $x>0,\ \theta>0,\alpha>0,\quad 0<\gamma\le 1$	$H(Y_j)=H_n(W_j)-\log\left(\frac{\alpha\gamma}{\theta}\right)$ $+\frac{1-\gamma}{\gamma}t_{\alpha,j}(n)$ $+\frac{\alpha+1}{\alpha}\left(\psi(n+1)-\psi(n-j+1)\right)$	$H(Y_1)\le H(Y_n)$ $for\ \ n=2m+1$
Loglogistic(θ,γ) $F_X(x)=1-\left(1+\left(\frac{x}{\theta}\right)^{\gamma}\right)^{-1}$ $x>0,\ \theta>0,\quad 0<\gamma\le 1$	$H(Y_j)=H_n(W_j)+\log(\theta)$ $+\frac{1-\gamma}{\gamma}t_{1,j}(n)$ $+2\left(\psi(n+1)-\psi(n-j+1)\right)$	$H(Y_1)\le H(Y_n)$ $for\ \ n=2m+1$
Paralogistic(θ,α) $F_X(x)=1-\left(1+\left(\frac{x}{\theta}\right)^{\alpha}\right)^{-\alpha}$ $x>0,\ \theta>0,\quad 0<\alpha\le 1$	$H(Y_j)=H_n(W_j)-2\log(\alpha)$ $+\log(\theta)+\frac{1-\alpha}{\alpha}t_{\alpha,j}(n)$ $+\frac{\alpha+1}{\alpha}\left(\psi(n+1)-\psi(n-j+1)\right)$	$H(Y_1)\le H(Y_n)$ $for\ \ n=2m+1$
Generalized Pareto$(\theta,\ \alpha,1)$ $F_X(x)=1-\left(1+\frac{x}{\theta}\right)^{-\alpha}$ $x>0,\ \theta>0,\quad \alpha>0$	$H(Y_j)=H_n(W_j)+\log(\frac{\theta}{\alpha})$ $+\frac{\alpha+1}{\alpha}\left(\psi(n+1)-\psi(n-j+1)\right)$	$H(Y_1)\le H(Y_n)$ $for\ \ all\ \ n$
Inverse Pareto$(\theta,\ 1)$ $F_X(x)=1-\left(1+\frac{x}{\theta}\right)^{-1}$ $x>0,\ \theta>0$	$H(Y_j)=H_n(W_j)+\log(\theta)$ $+2\left(\psi(n+1)-\psi(n-j+1)\right)$	$H(Y_1)\le H(Y_n)$ $for\ \ all\ \ n$
Inverse Burr$(\frac{1}{\theta},\ 1)$ $F_X(x)=1-(1+\theta x)^{-1}$ $x>0,\ \theta>0$	$H(Y_j)=H_n(W_j)-\log(\theta)$ $+2\left(\psi(n+1)-\psi(n-j+1)\right)$	$H(Y_1)\le H(Y_n)$ $for\ \ all\ \ n$

Conclusion

We have derived the exact form of Shannon entropy for the Feller-Pareto(FP) family and order statistics of FP Subfamilies.These families cover a wide spectrum of areas ranging from actuarial science,economics, financial to medicine and telecommunications, for distributions of important variables such as sizes of insurance claims, incomes in a population of people, stock price fluctuations, and length of telephone calls. So we believe that the results that we have presented in this paper will be important as a reference for scientists and engineers from many areas.

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