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Numerical Solution of Second-Order Matrix Differential Models Using Cubic Matrix Splines

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Abstract

This paper deals with the construction of approximate solution of second-order matrix linear differential equations using matrix cubic splines. An estimation of the approximation error, an algorithm for its implementation and some illustrative examples are included.

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1 Introduction

A great variety of phenomena in physics and engineering can be modeled in the form of matrix differential equations. Apart from the problems where the mathematical pattern is written in matrix form, they also appear when special techniques to solve scalar or vectorial problems are used. Examples of such situations are the embedding methods for the study of linear boundary value problems [5], shooting method to solve scalar or vectorial problems with boundary value conditions [6], lines method for the numerical integration of partial differential equations [7], or homotopic methods to solve nonlinear systems equations [8].

The vectorization techniques to transform a matrix problem in a set of scalar equation or vectorial independent has several drawbacks, [9]. First, the physical sense of the magnitudes is lost with vectorization techniques. Secondly, the computational cost increases. Finally, the vectorization techniques waste the advantages of those symbolic languages adapted to matrix expressions. This work we will develop a method for the numerical integration of the second-order matrix differential linear equation given by

$$\begin{array}{l} Y''(x) + A(x)Y'(x) + B(x)Y(x) = C(x) \quad , \quad a \leq x \leq b \\ Y(a) = Y_a \quad , \quad Y'(a) = Y'_a \end{array} \tag{1}$$

where Y , Y_a , $Y'_a \in \mathbf{C}^{r\times q}$ and $A, B : [a, b] \to \mathbf{C}^{r\times r}$ and $C : [a, b] \to \mathbf{C}^{r\times q}$, verify $A, B \in \mathbf{C}^1([a, b])$, it guarantees the existence of only one solution $Y(x)$ of (1), which is continuously differentiable , [10]. In the scalar case, the cubic splines were used in [11] for the resolution of ODE, obtaining approaches that, among other advantages, they were of $\mathbf{C}^1$ class in the interval $[a, b]$, easily valuables and with an approach error $O(h^4)$. In this paper , we proposed a method using cubic matrix splines for the numerical approximation to the solution of (1). The present work extends this important advantage obtained in [12] and [13] for the first-order of (1). This paper is organized as follows. In Section 2, we develop the proposed method to solve (1), and including the study of the approximation error and a constructive algorithm. Finally some example will be presented in Section 3. Along this work we will denote by $\mathbf{C}^{p\times q}$ the set of the rectangular $p \times q$ complex matrices. If $A \in \mathbf{C}^{r\times s}$, we will denote for $\|A\|$ their 2-norm, defined by

$$\|A\| = \sup_{z\neq 0} \frac{\|Az\|}{\|z\|} \ ,$$

there for a vector z in $\mathbf{C}^s$, $\|z\| = (z^t z)^{1/2}$ is the usual Euclidean norm of z. By [14],[15], it follow that

$$\max_{i,j} |a_{i,j}| \leq \|A\| \leq \sqrt{rs} \max_{i,j} |a_{i,j}| \ . \tag{2}$$

We will denote by $P_n[x]$ the set of matrix polynomial with degree n and real variable x . We will say that one matrix function $g : [a, b] \to \mathbf{C}^{r\times q}$ is of class $k \geq 0$, and we will represent it $g \in \mathbf{C}^k([a, b])$, if g it is $k-$times differentiable , and its k^{th} derivatives is continuous in $[a, b]$. Let be $[a, b] \subset \mathbf{R}$ and be

$$\Delta = \{a = x_0 < x_1 < ... < x_n = b\} \ ,$$

a partition of $[a, b]$. Given m an integer bigger or equal to zero , we define the set of matrix splines of order m as

$$M\_\mathbf{C}^{r\times r}(\Delta)^m_{m-1} = \left\{ Q : [a, b] \to \mathbf{C}^{r\times q}; \left\{ \begin{array}{l} Q|_{[x_{i-1},x_i]}(x) \in P_m[x] \quad , \ i \in \{1, ..., n\} \\ Q \in \mathbf{C}^{m-1}([a, b]) \end{array} \right. \right\},$$

If $m = 3$ the matrix splines are called *matrix cubic splines*,[16].

2 Construction of the approach

Consider a class of two-point boundary value problems of the form

$$\begin{aligned} &Y''(x) + A(x)Y'(x) + B(x)Y(x) = C(x), \qquad a \le x \le b \\ &Y(a) = Y_a, \qquad Y'(a) = Y'_a \end{aligned} \tag{3}$$

where Y , Y_a , $Y'_a \in \mathbf{C}^{r\times q}$ and $A, B : [a,b] \to \mathbf{C}^{r\times r}$ and $C : [a,b] \to \mathbf{C}^{r\times q}$. Let us consider the partition of the interval [a, b] given by

$$\Delta_{[a,b]} = \{a = x_0 < x_1 < ... < x_n = b\} \ , \qquad x_k = a + kh \ , \qquad k = 0, 1, ..., n, \tag{4}$$

where $h = (b-a)/n$, being n a positive integer. We will build in each subinterval $[a+kh, a+(k+1)h]$ a matrix cubic spline approximating the solution of problem (1). For the first interval $[a, a+h]$, we consider that the spline is defined by

$$S|_{[a,a+h]}(x) = Y(a) + Y'(a)(x-a) + \frac{1}{2!}Y''(a)(x-a)^2 + \frac{1}{3!}\alpha_0(x-a)^3 \tag{5}$$

where the matrix $\alpha_0 \in \mathbf{C}^{r\times q}$ is a parameter to be determined. As $S|_{[a,a+h]}(x)$ defined in (5) verifies:

$$S|_{[a,a+h]}(a) = Y(a), \tag{6}$$

$$S'|_{[a,a+h]}(a) = Y'(a), \tag{7}$$

$$S''|_{[a,a+h]}(a) = Y''(a) = C(a) - A(a)Y'(a) - B(a)Y(a), \tag{8}$$

to obtain the spline we must determine α_0 . From (5) , we have

$$S'(x) = Y'(a) + Y''(a)(x-a) + \frac{1}{2}\alpha_0(x-a)^2, \tag{9}$$

$$S''(x) = Y''(a) + \alpha_0(x-a), \tag{10}$$

We have left the problem of determining α_0 when the spline is totally determined. For that, we will impose the spline be a solution of the problem (1) in the point $x = a+h$,

$$S''|_{[a,a+h]}(a+h) = Y''(a+h) = C(a+h) - A(a+h)Y'(a+h) - B(a+h)Y(a+h).$$

From (9) and (10) , we have

$$\begin{aligned} Y''(a) + \alpha_0 h = {} & C(a+h) - A(a+h)\Big(Y'(a) + hY''(a) + \tfrac{1}{2}h^2\alpha_0\Big) \\ & - B(a+h)\Big(Y(a) + hY'(a) + \tfrac{1}{2!}h^2Y''(a) + \tfrac{1}{3!}h^3\alpha_0\Big), \end{aligned} \tag{11}$$

A simple calculation, shows that

$$\begin{aligned}
&\Big(I + \tfrac{h}{2}A(a+h) + \tfrac{h^2}{3!}B(a+h)\Big)\alpha_0 \\
&\qquad = \tfrac{1}{h}\Big[C(a+h) - A(a+h)\Big(Y'(a) + hY''(a)\Big) - \\
&\qquad\quad B(a+h)\Big(Y(a) + hY'(a) + \tfrac{h^2}{2}Y''(a)\Big)\Big] - Y''(a),
\end{aligned} \tag{12}$$

Assuming that matrix equation (12) has one solution only α_0, this technique determines the spline in the interval $[a, a+h]$. In the interval $[a+h, a+2h]$, the matrix cubic spline takes the form

$$\begin{aligned}
S|_{[a+h,a+2h]}(x) &= S|_{[a,a+h]}(a+h) + S'|_{[a,a+h]}(a+h)\Big(x-(a+h)\Big) \\
&\quad + \frac{1}{2!}S''|_{[a,a+h]}(a+h)\Big(x-(a+h)\Big)^2 + \frac{1}{3!}\alpha_1\Big(x-(a+h)\Big)^3,
\end{aligned} \tag{13}$$

so $S(x)$ defined on $[a, a+h] \cup [a+h, a+2h]$ it is of class $\mathbf{C}^2([a, a+2h])$, and all the coefficients of the spline $S|_{[a+h,a+2u]}(x)$ are determined with the exception of $\alpha_1 \in \mathbf{C}^{r\times q}$. It is easy to check that the spline (13) satisfies the differential equation (1) in $x = a+h$, for what we will determine α_1 by imposing that (13) be also a solution of (1) in $x = a+2h$

$$\begin{aligned}
S''|_{[a+h,a+2h]}(a+2h) &= C(a+2h) - A(a+2h)S'|_{[a+h,a+2h]}(a+2h) \\
&\quad - B(a+2h)S|_{[a+h,a+2h]}(a+2h).
\end{aligned} \tag{14}$$

From (13) we obtain

$$\begin{aligned}
S|_{[a+h,a+2h]}(a+2h) &= S|_{[a,a+h]}(a+h) + hS'|_{[a,a+h]}(a+h) \\
&\quad + \tfrac{1}{2!}h^2 S''|_{[a,a+h]}(a+h) + \tfrac{1}{3!}h^3\alpha_1,
\end{aligned} \tag{15}$$

$$\begin{aligned}
S'|_{[a+h,a+2h]}(a+2h) &= S'|_{[a,a+h]}(a+h) + hS''|_{[a,a+h]}(a+h) \\
&\quad + \tfrac{1}{2}h^2\alpha_1,
\end{aligned} \tag{16}$$

$$S''|_{[a+h,a+2h]}(a+2h) = S''|_{[a,a+h]}(a+h) + h\alpha_1, \tag{17}$$

replace (15),(16) and (17) in (14) , we obtain the matrix equation with an only matrix unknown α_1

$$\begin{aligned}
&\Big(I + \tfrac{h}{2}A(a+2h) + \tfrac{h^2}{3!}B(a+2h)\Big)\alpha_1 \\
&\quad = \tfrac{1}{h}\Big[C(a+2h) - A(a+2h)\Big(S'|_{[a,a+h]}(a+h) + hS''|_{[a,a+h]}(a+h)\Big) \\
&\quad - B(a+2h)\Big(S|_{[a,a+h]}(a+h) + hS'|_{[a,a+h]}(a+h) + \tfrac{h^2}{2}S''|_{[a,a+h]}(a+h)\Big)\Big] \\
&\quad - S''|_{[a,a+h]}(a+h),
\end{aligned} \tag{18}$$

Assuming that matrix equation (18) has one solution only α_1, in this way the spline is totally determined in the interval $[a+h, a+2h]$.Iterating this process, let us consider the matrix cubic spline constructed until the subinterval $[a+(k-1)h, a+kh]$ and we define it in the next subinterval $[a+kh, a+(k+1)h]$ as

$$S|_{[a+kh,a+(k+1)h]}(x) = \beta_k(x) + \frac{1}{3!}\alpha_k(x-(a+kh))^3, \tag{19}$$

where

$$\beta_k(x) = \sum_{k=0}^{2} \frac{1}{k!} S^{(k)}|_{[a+(k-1)h,a+kh]}(a+kh)(x-(a+kh))^k, \tag{20}$$

Defined so, the matrix cubic spline $S(x) \in \mathbf{C}^2(\cup_{j=0}^{k}[a+jh, a+(j+1)h])$, and it is easy to cheek that it verifies the differential equation (1) in the point $x = a+kh$. We determine α_k by imposing that differential equation (1) is satisfied at the point $x = a+(k+1)h$

$$\begin{aligned} S''|_{[a+kh,a+(k+1)h]}(a+(k+1)h) &= C(a+(k+1)h) \\ &-A(a+(k+1)h)S'|_{[a+kh,a+(k+1)h]}(a+(k+1)h) \\ &-B(a+(k+1)h)S|_{[a+kh,a+(k+1)h]}(a+(k+1)h), \end{aligned}$$

that developing takes us to the matrix equation

$$\begin{aligned} &\Big(I + \tfrac{h}{2}A(a+(k+1)h) + \tfrac{h^2}{3!}B(a+(k+1)h)\Big)\alpha_k = \\ &\tfrac{1}{h}\Big[A(a+(k+1)h)\beta_k(a+(k+1)h]) + hS''|_{[a,a+h]}(a+h)\Big) - \\ &B(a+2h)\Big(S|_{[a,a+h]}(a+h) + hS'|_{[a,a+h]}(a+h) + \tfrac{h^2}{2}S''|_{[a,a+h]}(a+h)\Big)\Big], \end{aligned} \tag{21}$$

Note that solubility of equation (21) is guaranteed showing that the matrix $\Big(I + \frac{h}{2}A(a+(k+1)h) + \frac{h^2}{3!}B(a+(k+1)h)\Big)$ is invertible, for $k = 0, 1, ..., n-1$. Let us denote

$$M = \max_{a \le x \le b} \Big\{||Ax||, ||Bx||\Big\}, \tag{22}$$

and then

$$\begin{aligned} &\Big\|I - \Big(I + \frac{h}{2}A(a+(k+1)h) + \frac{h^2}{3!}B(a+(k+1)h)\Big)\Big\| \\ &\qquad = \frac{h}{2}\Big\|A(a+(k+1)h) + \frac{h}{3}B(a+(k+1)h)\Big\| \\ &\qquad \le \frac{h}{2}\Big\{\|A(a+(k+1)h)\| + \frac{h}{3}\|B(a+(k+1)h)\|\Big\} \\ &\qquad \le \frac{h}{2}\Big\{M + \frac{h}{3}M\Big\} = \Big(\frac{h}{2} + \frac{h^2}{6}\Big)M. \end{aligned}$$

thus taking $h \le \sqrt{1+\frac{4}{M}} - 1$, one has

$$\Big\|I - \Big(I + \frac{h}{2}A(a+(k+1)h) + \frac{h^2}{3!}B(a+(k+1)h)\Big)\Big\| \le 1,$$

which guarantees, for the perturbation lemma, [20, p. 58],that matrix $\Big(I + \frac{h}{2}A(a+(k+1)h) + \frac{h^2}{3!}B(a+(k+1)h)\Big)$ is invertible, and therefore, the equation (21) has unique solution α_k, for $k = 0, 1, ..., n-1$. Taking into account [17, Theorem 5] and (2), the following result has been established.

Algorithm.

- Step 1. Determine the constants M and $Y''(a)$ given by (22) and (8), respectively. Take $h \leq \sqrt{1+\frac{4}{M}} - 1$ and $h = (b-a)/n$ and consider the partition $\Delta_{[a,b]}$ given by (4).

- Step 2. For $k = 0$, solve the matrix equation (12) and compute $S|_{[a,a+h]}(x)$ defined by (5).

- Step 3. For $k = 1, ..., n-l$, solve the matrix equation (21) and compute $S|_{[a+kh,a+(k+l)h]}(x)$ defined by (19).

3 Numerical illustrations

These are the main results of the paper. In this section, we test the algorithm in a situation where the exact solution is known.

Example 1. Let us consider the problem

$$\begin{aligned} &Y''(x) + A(x)Y'(x) + B(x)Y = C(x), && 0 \leq x \leq 1, \\ &Y(0) = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \quad Y'(0) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, && Y(x) \in \mathbf{C}^2, \end{aligned}$$

where

$$\begin{aligned} A(x) &= \begin{pmatrix} 1 & 2 \\ 0 & x \end{pmatrix}, \\ B(x) &= \begin{pmatrix} x & e^x \\ 1 & 0 \end{pmatrix}, \\ C(x) &= \begin{pmatrix} (x+1)e^x + x^3 + 2x + 4 & (x^2+3x+2)e^x + 2 \\ x^2 + x & xe^x + x \end{pmatrix}, \end{aligned}$$

This problem has the exact solution $Y(x) = \begin{pmatrix} x^2 & xe^x \\ x+1 & x-1 \end{pmatrix}$,so we will be able to calculate the approximation error. Taking derivatives it follows that

$$Y''(x) = C(x) - A(x)Y'(x) - B(x)Y(x)$$

so $Y''(0) = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$. As $\max_{x\in[0,1]} \{\|A(x)\|, \|B(x)\|\} \leq 3$, we take $M = 3$. We have $h \leq \sqrt{1+\frac{4}{M}} - 1 = 0.5275$, so we take $h = 0.1$.

The approximation solution , are shown in Table 1. In Table 1. the values in the error column are the maximum error in each subinterval.

Example 2. Consider the matrix problem

$$Y''(x) + A(x)Y'(x) + B(x)Y(x) = C(x),$$
$$Y(0) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad Y'(0) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 2 \end{pmatrix}, \quad x \in [0,1],$$

with

$$A(x) = \begin{pmatrix} 1 & 1 & 2 \\ x & 2 & x+1 \\ x^2+1 & 3 & x \end{pmatrix}, \quad B(x) = \begin{pmatrix} 1 & 3 & x \\ e^x & 2 & 1 \\ 0 & 1 & x \end{pmatrix},$$
$$C(x) = \begin{pmatrix} x^3+5x+1 & 4e^x + xe^x + x^2 + 5 \\ 3x^2+3x+xe^x & e^{2x} + 2e^x + 2xe^x + 2x + 3 \\ x^3+3x^2+3 & x^2e^x + 2xe^x + 2e^x + x^2 + 2x + 1 \end{pmatrix},$$

with an exact solution given by $Y(x) = \begin{pmatrix} x & e^x \\ 0 & 1 \\ x^2 & x+e^x \end{pmatrix}$ so we will be able to calculate the approximation error.As $max_{x\in[0,1]}\|A(x)\|, \|B(x)\| \leq 6$, we take $M = 6$. Taking derivatives it follows that

$$Y''(x) = C(x) - A(x)Y'(x) - B(x)Y(x),$$

so $Y''(0) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 2 & 1 \end{pmatrix}$. We have $h \leq \sqrt{1+\frac{4}{M}} - 1 = 0.2910$, so we take $h = 0.1$. The approximation solution , are shown in Table 1. In Table 1. the values in the error column are the maximum error in each subinterval.

Table 1. The matrix cubic spline and max error of example 1.

$[x_i, x_{i+1}]$	Approximation	Error $\times 10^3$
$[0, 0.1]$	$\begin{pmatrix} 0.038461x + 0.500000x^2 + 0.158932x^3 & x + x^2 + 0.534202x^3 \\ 1 + x - 0.157809x^3 & -1 + x - 0.000028x^3 \end{pmatrix}$	$\begin{pmatrix} 0.994914 & -0.017109 \\ 0.157809 & 0.0000283 \end{pmatrix}$
$[0.1, 0.2]$	$\begin{pmatrix} -0.001543 + 0.084775x + 0.036861x^2 + 1.70273x^3 & -0.000076 + 1.00227x + 0.977324x^2 + 0.609789x^3 \\ 0.999666 + 1.01003x - 0.100322x^2 + 0.176598x^3 & -1 + 0.999998x + 0.000017x^2 - 0.000084x^3 \end{pmatrix}$	$\begin{pmatrix} 0.949224 & -0.068648 \\ 0.928066 & 0.000282 \end{pmatrix}$
$[0.2, 0.3]$	$\begin{pmatrix} 0.031024 - 0.121459x + 1.06803x^2 - 0.015893x^3 & -0.000760 + 1.01254x + 0.925985x^2 + 0.695354x^3 \\ 1.00286 + 0.990732x - 0.003820x^2 + 0.015761x^3 & -1 + 0.999992x + 0.000051x^2 - 0.000139x^3 \end{pmatrix}$	$\begin{pmatrix} -0.932722 & -0.068648 \\ -0.981517 & 0.000282 \end{pmatrix}$
$[0.3, 0.4]$	$\begin{pmatrix} 0.031364 - 0.124856x + 1.07936x^2 - 0.028475x^3 & -0.003374 + 1.03867x + 0.838858x^2 + 0.792161x^3 \\ 1.00377 + 0.981652x + 0.026445x^2 - 0.017866x^3 & -0.999998 + 0.999978x + 0.000097x^2 - 0.000192x^3 \end{pmatrix}$	$\begin{pmatrix} 0.769059 & -0.184402 \\ 0.447901 & 0.003767 \end{pmatrix}$
$[0.4, 0.5]$	$\begin{pmatrix} 0.044719 - 0.136132x + 1.07424x^2 - 0.024210x^3 & -0.010379 + 1.09122x + 0.707495x^2 + 0.901630x^3 \\ 1.00188 + 0.995800x - 0.008924x^2 + 0.011607x^3 & -0.999995 + 0.999954x + 0.000156x^2 - 0.000241x^3 \end{pmatrix}$	$\begin{pmatrix} 0.781272 & -0.446339 \\ 0.996639 & 0.009084 \end{pmatrix}$
$[0.5, 0.6]$	$\begin{pmatrix} 0.047843 - 0.100694x + 1.00514x^2 + 0.021856x^3 & -0.025846 + 1.18401x + 0.521907x^2 + 1.02536x^3 \\ 1.00332 + 0.987185x + 0.008306x^2 + 0.000121x^3 & -0.999989 + 0.999921x + 0.000222x^2 - 0.000285x^3 \end{pmatrix}$	$\begin{pmatrix} -0.151314 & -0.446339 \\ 0966391 & 0.009084 \end{pmatrix}$
$[0.6, 0.7]$	$\begin{pmatrix} 0.157528 - 0.451164x + 1.35241x^2 - 0.047714x^3 & -0.056036 + 1.33497x + 0.270318x^2 + 1.16513x^3 \\ 1.00573 + 0.980674x + 0.019158x^2 - 0.005908x^3 & -0.999981 + 0.999880x + 0.000291x^2 - 0.000323x^3 \end{pmatrix}$	$\begin{pmatrix} -0.390527 & -0.654124 \\ 0.243266 & 0.018502 \end{pmatrix}$
$[0.7, 0.8]$	$\begin{pmatrix} 0.042773 - 1.60056x + 2.99441x^2 - 0.829618x^3 & -0.111171 + 1.56697x - 0.061117x^2 + 1.32295x^3 \\ 1.00951 + 0.964487x + 0.042282x^2 - 0.016919x^3 & -0.999970 + 0.999832x + 0.000359x^2 - 0.000356x^3 \end{pmatrix}$	$\begin{pmatrix} 0.163976 & -0.210524 \\ 0.504071 & 0.056843 \end{pmatrix}$
$[0.8, 0.9]$	$\begin{pmatrix} 0.005285 - 0.014647x + 1.01201x^2 - 0.003620x^3 & -0.203025 + 1.91143x - 0.491685x^2 + 1.50236x^3 \\ 1.00253 + 0.990641x + 0.009590x^2 - 0.003298x^3 & -1.00077 + 1.00284x - 0.003401x^2 + 0.001211x^3 \end{pmatrix}$	$\begin{pmatrix} 0.805669 & -0.568024 \\ 0.525341 & 0.088235 \end{pmatrix}$
$[0.9, 1]$	$\begin{pmatrix} 0.003468 - 0.008590x + 1.00528x^2 - 0.001128x^3 & -0.348388 + 2.39597x - 1.03007x^2 + 1.70176x^3 \\ 1.00063 + 0.996989x + 0.002536x^2 - 0.0006852x^3 & -0.999471 + 0.998504x + 0.001417x^2 - 0.000573x^3 \end{pmatrix}$	$\begin{pmatrix} 0.966375 & -0.989543 \\ 0.530279 & 0.124039 \end{pmatrix}$

Table 2. The matrix cubic spline and max error of example 2.

$[x_i, x_{i+1}]$	Approximation	Error $\times 10^3$
$[0, 0.1]$	$\begin{pmatrix} 1.000000x & 1 + x + 0.142464x^3 \\ 0.000000 & 1 + 0.006339x^3 \\ 1.000000x^2 & 1 + 2x + 0.5x^2 + 0.509093x^3 \end{pmatrix}$	$\begin{pmatrix} 0 & 0.523698 \\ 0.000000 & -0.000000 \\ 0 & -0.301597 \end{pmatrix}$
$[0.1, 0.2]$	$\begin{pmatrix} 1.000000x & 1.010700 + 0.998632x + 0.013675x^2 + 0.096879x^3 \\ -0.526189 \times 10^{-16}x^3 & 0.999991 + 0.255199 \times 10^{-3}x - 0.002552x^2 + 0.014846x^3 \\ 1.000000x^2 & 0.999979 + 2.00063x + 0.493659x^2 + 0.530231x^3 \end{pmatrix}$	$\begin{pmatrix} 0 & 0.976258 \\ 0.000000 & -0.125893 \\ 0 & -0.272569 \end{pmatrix}$
$[0.2, 0.3]$	$\begin{pmatrix} 1.000000x & 1.02663 + 1.00916x + 0.044307x^2 + 0.055078x^3 \\ +0.754607 \times 10^{-16}x^3 & 1.00013 - 0.001765x + 0.007548x^2 - 0.001986x^3 \\ 1.00000x^2 + 0.711762 \times 10^{-15}x^3 & 0.996180 + 2.00433x + 0.475188x^2 + 0.561016x^3 \end{pmatrix}$	$\begin{pmatrix} 0 & 0.501056 \\ 0.000000 & -0.222387 \\ 0.000000 & -0.553388 \end{pmatrix}$
$[0.3, 0.4]$	$\begin{pmatrix} 1.00000x - 0.277280 \times 10^{-15}x^2 + 0.219315 \times 10^{-15}x^3 & 1.08716 + 0.736346x + 0.626170x^2 - 0.079499x^3 \\ (0.113218x - 0.408477x^2 + 0.434823x^3) \times 10^{-15} & 1.00083 - 0.008762x + 0.030872x^2 - 0.027903x^3 \\ -0.216604 \times 10^{-15}x + 1.000000x^2 - 0.802238 \times 10^{-15}x^3 & 0.986958 + 2.00773x + 0.463850x^2 + 0.573613x^3 \end{pmatrix}$	$\begin{pmatrix} 0.000000 & 0.891012 \\ 0.000000 & -0.344674 \\ -0.000000 & -0.201107 \end{pmatrix}$
$[0.4, 0.5]$	$\begin{pmatrix} 1.00000x + 0.378279 \times 10^{-16}x^2 - 0.432749 \times 10^{-16}x^3 & 1.58821 - 1.95262x + 5.05939x^2 - 1.86619x^3 \\ (-0.297555x + 0.618455x^2 - 0.420953x^3) \times 10^{-15} & 1.01063 - 0.082315x + 0.214755x^2 - 0.181139x^3 \\ 0.550626 \times 10^{-15}x + 1.000000x^2 + 0.114714 \times 10^{-14}x^3 & 0.990948 + 1.90461x + 0.721644x^2 + 0.358785x^3 \end{pmatrix}$	$\begin{pmatrix} 0.000000 & 0.525171 \\ 0.000000 & -0.521823 \\ 0.000000 & -0.979197 \end{pmatrix}$
$[0.5, 0.6]$	$\begin{pmatrix} 1.000000x - 0.288507 \times 10^{-16}x^2 & 2.44082 - 7.06828x + 15.2907x^2 - 8.68706x^3 \\ 0.864365 \times 10^{-15}x^2 - 0.584893 \times 10^{-15}x^3 & 1.07411 - 0.463168x + 0.976461x^2 - 0.688942x^3 \\ -0.473948 \times 10^{-15}x + 1.000000x^2 - 0.631930 \times 10^{-15}x^3 & 0.765008 + 2.50823x + 0.772895x^2 - 0.514372x^3 \end{pmatrix}$	$\begin{pmatrix} 0.000000 & 0.525171 \\ 0.000000 & -0.521823 \\ 0.000000 & 0.671654 \end{pmatrix}$
$[0.6, 0.7]$	$\begin{pmatrix} 1.000000x - 0.681856 \times 10^{-15}x^2 + 0.363958 \times 10^{-15}x^3 & 0.245403 + 3.89878x - 2.98773x^2 + 1.46762x^3 \\ (0.393841x - 0.492886x^2 + 0.169136x^3) \times 10^{-15} & 0.674019 + 1.56728x - 2.40762x^2 + 1.19110x^3 \\ (-0.143292 + 0.716460x + 0.663389x^3) \times 10^{-15} + 1.00000x^2 & -0.577844 + 9.14454x - 10.2876x^2 + 5.63036x^3 \end{pmatrix}$	$\begin{pmatrix} 0.000000 & -0.205384 \\ 0.000000 & 0.070564 \\ -0.000000 & 0.142011 \end{pmatrix}$
$[0.7, 0.8]$	$\begin{pmatrix} 1.00000x + (-0.107133 - 0.820830x^2 + 0.430136x^3) \times 10^{-15} & 0.770447 + 1.64859x + 0.226825x^2 - 0.063117x^3 \\ (-0.179852 + 0.743079x - 0.991798x^2 + 0.406712x^3) \times 10^{-15} & 1.09450 - 0.234789x + 0.166765x^2 - 0.034795x^3 \\ (0.031936 - 0.136868x - 0.093107x^3) \times 10^{-15} + 1.00000x^2 & 1.06385 + 2.10872x - 0.236448x^2 + 0.844086x^3 \end{pmatrix}$	$\begin{pmatrix} 0.000000 & -0.205384 \\ 0.000000 & 0.0705643 \\ 0.000000 & 0.1420113 \end{pmatrix}$
$[0.8, 0.9]$	$\begin{pmatrix} 1.00000x + (0.214044 + 0.580337x^2 - 0.153683x^3) \times 10^{-15} & 0.817421 + 1.55904x + 0.338771x^2 - 0.109762x^3 \\ (-0.063063 + 0.243350x - 0.310483x^2 + 0.128714x^3) \times 10^{-14} & 1.07853 - 0.174890x + 0.091892x^2 - 0.003598x^3 \\ (-0.063553 + 0.238324x + 0.124127x^3) \times 10^{-14} + 1.00000x^2 & 1.00288 + 2.33735x - 0.522238x^2 + 0.963165x^3 \end{pmatrix}$	$\begin{pmatrix} 0.000000 & 0.273847 \\ 0.000000 & 0.415084 \\ -0.000000 & -0.612482 \end{pmatrix}$
$[0.9, 1]$	$\begin{pmatrix} 1.00000x + (+0.329089 + 0.888065x^2 - 0.267656x^3) \times 10^{-15} & 1.01536 + 1.29374x + 0.633550x^2 - 0.218939x^3 \\ (0.199800 - 0.632860x + 0.663084x^2 - 0.231866x^3) \times 10^{-14} & 1.10271 - 0.255493x + 0.181450x^2 - 0.036768x^3 \\ (0.102033 - 0.340110x - 0.139963x^3) \times 10^{-14} + 1.00000x^2 & 0.823301 + 2.73063x - 0.959217x^2 + 1.12501x^3 \end{pmatrix}$	$\begin{pmatrix} 0.000000 & -0.542728 \\ 0.000000 & 0.810109 \\ 0.000000 & -0.144241 \end{pmatrix}$

4 Conclusions

This article develops a new method for the numerical solution of second-order matrix differential equations of the non-linear type $Y''(x) + A(x)Y'(x) + B(x)Y(x) = C(x)$, using matrix-cubic splines. Our method is well-suited for implementation on numerical and/or symbolical computer systems (Mathematica, Matlab, etc.) as we have shown in Section 2 giving the explicit algorithm.

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References

[1] E.A. Al-Said, M.A. Noor, Cubic splines method for a system of third-order boundary value problems, *Appl. Math. Comput,* **142** (2003),195 - 204.

[2] L.D. Faddeyev, The inverse problem in the quantum theory of scattering,*J. Math. Physics,* **4** (1963), 72-104.

[3] S. Barnett, Matrices in Control Theory,*Van Nostrand, Reinhold,* 1971.

[4] W.T. Reid, Riccati Differential Equations, Academic Press, 1972.

[5] M. Scott, Invariant imbedding and its Applications to Ordinary Differential Equations, Addison-Wesley, 1973.

[6] P. Marzulli, Global error estimates for the standard parallel shooting method,*J. Computat. Appl. Maths.* **34** (1991) 233-241 .

[7] K. Raktorys,The Method of Discretization in Time and Partial Differential Equations, *D. Reidel Pub. Co.,Dordrecht* , 1982.

[8] P.T. Boggs, The solution of nonlinear systems of equations by A-stable integration techniques ,*SIAM J.Numer. Anal.* **8 (4)** (1971), 767-785.

[9] A. Graham, Kvanecker Products and Matrix Calculus with Applications, John Wiley, New York, 1981.

[10] T.M. Flett, Differential Analysis, Cambridge University Press, 1980.

[11] A.S.V.Ravi Kanth and Y.N.Reddy, Cubice spline for a class of singular two-point boundary value problems, *Appl.Math.Comput.* Article in press, (2005).

[12] F.R Loscalzo and T.D. Talbot, Spline function approximations for solutions of ordinary differential equations, *SIAM J. Numer. Anal.* **4** (1967) 433-445.

[13] E. Defez, L. Soler, A. Hervas, C. Santamaria, Numerical solutions of matrix differential models using cubic matrix splines, *Comput.Math. Appl.* **50** (2005), 693-699.

[14] G.H. Golub and C.F. VanLoan, Matrix Computations, *Third Edition,* Johns Hopkins University Press, 1996.

[15] Carl.D.Meyer, Matrix analysis and applied linear algebra, *SIAM,* 2000.

[16] E. Defez, A. Hervas, A. Law, J. Villanueva-Oller and R. Villanueva, Matrix cubic splines for progressive transmission of images, *Journal of Mathemahcal Imaging and Vision,* **17 (1)**(2002), 41-53.

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