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Approximation of Curves by Fairness Cubic Splines

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Abstract

In this paper we present an approximation method of curves by a new type of spline functions called *fairness cubic splines* from a given Lagrangian data set under fairness constraints. An approximating problem of curves is obtained by minimizing a quadratic functional in a parametric space of cubic splines. This method is justified by a convergence result and an analysis of some numerical and graphical examples.

Keywords: Smoothness; Variational curve; Fairness spline; Cubic spline

Mathematics Subject Classifications: 65D07, 65D10, 65D17

1 Introduction

In Geology and Structural Geology the reconstruction of a curve or surface from a scattered data set is a commonly encountered problem. The theory of D^m–splines over an open bounded set has been introduced at the first time by M. Attéia [2]. By following the same idea of R. Arcangéli, we have enriched this theory and extended it to the variational spline functions [8] where the early works are therein.

Several works have used the variational approach specifically minimizing some fairness functional (see for example [5], likewise these functional also can represent the flexed energy of a thin plate [4]) on a finite element space making the most of the suitable properties of this space (see [7] and [9]) in order to simplify both characterization and computation of the solution. So we have planned to resolve in this work a variational approach problem on a finite dimensional space that is not a finite element one. This is why we focus in this paper our interest to minimize a similar fairness functional on a space of cubic spline functions that of course is not a finite element one. The resulting function is called a *fairness cubic spline*. We study some characterization of this function and we shall express it as a linear combination of the basis functions of a parametric space of cubic splines. Moreover, under adequate hypotheses we prove that such *fairness cubic spline* converges to a given function from what are proceeding the data.

We present a numerical and graphical example in order to show the effectiveness and the validity of the method of this paper. We remark that this work can be considered as an analogous case of the generic problem presented in [9]. But in this occasion–as mentioned above–we have used an other space that is not a finite element one in order to reach a great order of smoothness and to obtain a pleasing shape.

Some fields of applications of this problem can appear in Earth sciences, specially in Geology and Geophysics, as long as CAD and CAGD etc ...

The remainder of this paper is organised as follows. In Section 2, we briefly recall some preliminary notations and results. Section 3 is devoted to state the approximation problem and to present a method to solve it. In Section 4, we compute the resulting function, while convergence's Theorem is proved in Section 5. In Section 6 some numerical and graphical examples are given.

2 Notations and preliminaries

We denote by $\langle \cdot \rangle$ and $\langle \cdot , \cdot \rangle$, respectively, the Euclidean norm and inner product in $\mathbb{R}^p$, with $p \in \mathbb{N}^*$, and for any real interval $I = (a, b)$ with $a < b$ let $H^3(I; \mathbb{R}^p)$ be the usual Sobolev space of (classes of) functions u belong to $L^2(I; \mathbb{R}^p)$, together with all their derivative $d^\beta u$, in the distribution sense, of order $\beta \leq 3$. This space is equipped with the norm

$$\|u\| = \left(\sum_{\beta \leq 3} \int_I \langle d^\beta u(x) \rangle^2 \, dx \right)^{1/2},$$

the semi–norms

$$|u|_\ell = \left(\sum_{\beta = \ell} \int_I \langle d^\beta u(x) \rangle^2 \, dx \right)^{1/2}, \; 0 \leq \ell \leq 3,$$

and the corresponding inner semi–products

$$(u, v)_\ell = \sum_{\beta = \ell} \int_I \langle d^\beta u(x), d^\beta v(x) \rangle \, dx, \; 0 \leq \ell \leq 3.$$

Let $\mathbb{R}^{m,p}$ be the space of real matrices of m lines and p columns equipped with the inner product

$$\langle A, B\rangle_{m,p} = \sum_{i=1}^{m}\sum_{j=1}^{p} a_{ij}\, b_{ij}$$

with $A = (a_{ij})_{\substack{1\le i\le m\\ 1\le j\le p}}$, $B = (b_{ij})_{\substack{1\le i\le m\\ 1\le j\le p}}$ and the corresponding norm

$$\langle A\rangle_{m,p} = \left(\sum_{i=1}^{m}\sum_{j=1}^{p} (a_{ij})^2\right)^{1/2}.$$

Moreover, for any $n \in \mathbb{N}^*$ let $T_n = \{x_0, ..., x_n\}$ be a subset of distinct points of $[a, b]$, with $a = x_0 \le x_1 < \ldots < x_{n-1} \le x_n = b$. We denote $S_3(T_n)$ the space of cubic spline functions given by

$$S_3(T_n) = \left\{ s \in C^2([a, b]) \mid s|_{[x_{i-1},x_i]} \in \mathbb{P}_3[x_{i-1}, x_i],\ i = 1, ..., n \right\},$$

where $\mathbb{P}_3[x_{i-1}, x_i]$ is the restriction on $[x_{i-1}, x_i]$ of the vectorial space of real polynomials of degree ≤ 3. We have that $S_3(T_n)$ is a Hilbert subspace of $H^3(I) = H^3(I;\mathbb{R})$ equipped with the same norm, semi–norm and inner semi–product of $H^3(I)$.

Finally, for any $n \in \mathbb{N}^*$ we define the space of parametric cubic spline functions $V_n = (S_3(T_n))^p$ constructed from $S_3(T_n)$ which verifies

$$V_n \subset H^3(I;\mathbb{R}^p) \cap C^2([a, b];\mathbb{R}^p). \tag{2.1}$$

3 Fairness cubic spline

Let $\Upsilon_0 \subset \mathbb{R}^p$ be a curve defined by a parameterization f belonging to $H^3(I;\mathbb{R}^p)$. For each $m \in \mathbb{N}^*$ let $A^m = \{a_1, \ldots, a_m\}$ be a subset of distinct points of $[a, b]$ such that

$$\sup_{x\in I}\ \min_{i=1,\ldots,m} |x - a_i| = \mathrm{o}\left(\frac{1}{m}\right), \qquad m \to +\infty, \tag{3.1}$$

and we suppose that the set A^m contains a $\mathbb{P}_2(I;\mathbb{R}^p)$–unisolvent subset (see [3]).

Now, we consider the following problem: Find an approximating curve Υ of Υ_0 from the data points $\{f(a) \mid a \in A^m\}$ parameterized by a function σ of V_n that minimizes a certain functional in V_n.

For any $n, m \in \mathbb{N}^*$ with $m \ge 3$ and $\tau = (\tau_1, \tau_2, \tau_3) \in \mathbb{R}^3$ with τ_1, τ_2 belonging to $\mathbb{R}_+$ and $\tau_3 > 0$, let J^m_τ be the functional defined on $H^3(I;\mathbb{R}^p)$ by

$$J^m_\tau(v) = \sum_{i=1}^{m} \langle v(a_i) - f(a_i)\rangle^2 + \sum_{j=1}^{3} \tau_j\, |v|_j^2\, .$$

Remark 3.1. The first term of $J_\tau^m(v)$ indicates how well v approaches f in a least discrete squares sense. The second term can represent some different conditions as for example: Fairness conditions (see [5] and [7]), a classical smoothness measure (see [1]) etc..., while the parameter vector τ weights the importance given to each condition. □

Then, for any $n \in \mathbb{N}^*$ and any $m \geq 3$ we consider the following minimization problem: Find $\sigma_\tau^{n,m}$ such that

$$\begin{cases} \sigma_\tau^{n,m} \in V_n, \\ \forall v \in V_n, \qquad J_\tau^m(\sigma_\tau^{n,m}) \leq J_\tau^m(v). \end{cases} \tag{3.2}$$

Theorem 3.1. *The problem (3.2) has a unique solution, called fairness cubic spline in V_n relative to A^m and τ, which is also the unique solution of the following variational problem: Find $\sigma_\tau^{n,m}$ such that*

$$\begin{cases} \sigma_\tau^{n,m} \in V_n, \\ \forall v \in V_n, \qquad \sum\limits_{i=1}^m \langle \sigma_\tau^{n,m}(a_i), v(a_i)\rangle + \sum\limits_{j=1}^3 \tau_j (\sigma_\tau^{n,m}, v)_j = \sum\limits_{i=1}^m \langle f(a_i), v(a_i)\rangle \cdot \end{cases}$$

Proof. Taking into account (2.1) and that the following norm

$$v \longmapsto [[v]] = \left(\sum_{i=1}^m \langle v(a_i)\rangle^2 + \sum_{j=1}^3 \tau_j \, |v|_j^2 \right)^{1/2}$$

is equivalent in V_n to the norm $\|\cdot\|$, one easily checks that the symmetric bilinear $\widetilde{a} : V_n \times V_n \longrightarrow \mathbb{R}$ given by

$$\widetilde{a}(u, v) = \sum_{i=1}^m \langle u(a_i), v(a_i)\rangle + \sum_{j=1}^3 \tau_j (u, v)_j$$

is a continuous and V_n–elliptic. Likewise, the linear form

$$\varphi : \; v \in V_n \longmapsto \varphi(v) = \sum_{i=1}^m \langle f(a_i), v(a_i)\rangle$$

is continuous. The result is then a consequence of the Lax–Milgram Lemma (see [3]). □

4 Computation

Now well, we are going to show how to obtain in practice any *fairness cubic spline*. The $\sigma_\tau^{n,m}([a, b])$ set for $n \in \mathbb{N}^*$ with $m \geq 3$ and a given value of the parameter vector τ provides a solution for our problem.

For any $n \in \mathbb{N}^*$ it is known that dim $S_3(T_n) = n + 3$. We consider $\{w_1, \ldots, w_{n+3}\}$ a basis of the space $S_3(T_n)$ and $\{e_1, \ldots, e_p\}$ the canonical one of $\mathbb{R}^p$. Then, the family $\{v_1, \ldots, v_Z\}$ is a basis of V_n with $Z = p\ (n+3)$ and

$$\forall i = 1, \ldots, n+3,\ \forall \ell = 1, \ldots, p,\ j = p(i-1) + \ell,\ v_j = w_i e_\ell.$$

Thus, $\sigma_\tau^{n,m}$ can be written as $\sigma_\tau^{n,m} = \sum_{i=1}^{Z} \alpha_i v_i$, with $\alpha_i \in \mathbb{R}$, for $i = 1, \ldots, Z$. Applying Theorem 3.1 we obtain a linear system of order Z as follows

$$C\ \alpha = b \tag{4.1}$$

with

$$\begin{aligned} C &= (c_{ij})_{1 \le i,j \le Z}, \\ \alpha &= (\alpha_i)_{1 \le i \le Z}, \\ b &= (b_1, \ldots, b_Z)^T, \end{aligned}$$

where for $i, j = 1, \ldots, Z$ we have

$$\begin{cases} c_{ij} = \sum_{k=1}^{m} \langle v_i(a_k), v_j(a_k) \rangle + \sum_{r=1}^{3} \tau_r \left(v_i, v_j\right)_r, \\ b_j = \sum_{k=1}^{m} \langle f(a_k), v_j(a_k) \rangle. \end{cases}$$

Let L be the Lagrangian operator defined from $H^3(I, \mathbb{R}^p)$ into $\mathbb{R}^{m,p}$ by

$$Lv = \sum_{i=1}^{m} v(a_i).$$

In this case the coefficients of the matrix C are given by

$$c_{ij} = \langle Lv_i, Lv_j \rangle_{m,p} + \sum_{r=1}^{3} \tau_r \left(v_i, v_j\right)_r,\ \forall i, j = 1, \ldots, Z.$$

Finally, we point out the following result.

Proposition 4.1. *The matrix C is symmetric, positive definite and of band type.*

Proof. Obviously the matrix C is symmetric.

Let now $x = (x_1, \ldots, x_Z)^T \in \mathbb{R}^Z$, we have

$$\begin{aligned}
x^T\, C\, x &= \sum_{i,j=1}^{Z} x_i c_{ij} x_j \\
&= \sum_{i,j=1}^{Z} x_i \left(\langle L v_i, L v_j \rangle_{m,p} + \sum_{r=1}^{3} \tau_r\, (v_i, v_j)_r \right) x_j \\
&= \left\langle L \left(\sum_{i=1}^{Z} x_i v_i \right), L \left(\sum_{j=1}^{Z} x_j v_j \right) \right\rangle_{m,p} + \sum_{r=1}^{3} \tau_r \left(\sum_{i=1}^{Z} x_i v_i, \sum_{j=1}^{Z} x_j v_j \right)_r \\
&= \left\langle L \left(\sum_{i=1}^{Z} x_i v_i \right) \right\rangle_{m,p}^2 + \sum_{r=1}^{3} \tau_r \left| \sum_{i=1}^{Z} x_i v_i \right|_r^2 \geq 0.
\end{aligned}$$

Let $w = \sum_{i=1}^{Z} x_i v_i$ and we suppose that $x^T\, C\, x^T = 0$ then one has

$$\langle L w \rangle_{m,p}^2 + \sum_{i=1}^{3} \tau_i\, |w|_i^2 = 0.$$

Hence, $[[\ w\ ]]^2 = 0$ that implies $w = 0$ (where $[[\ \cdot\ ]]$ designs the norm defined in the proof of Theorem 3.1).

Moreover, for the independent linearity of the family $\{v_i\}_{1 \leq i \leq Z}$. Consequently $x = 0$ and C is positive definite.

Finally, the matrix C is of band type because for each $i = 1, \ldots, Z$ the function v_i has a local support. □

To compute in practice the coefficients of the linear system (4.1), we simplify such linear system using the notations

$$\begin{aligned}
A &= \left(\sum_{k=1}^{m} v_i(a_k) \right)_{1 \leq i \leq Z}, \\
B &= (B_s)_{1 \leq s \leq 3} = \Big(((v_i, v_j)_s)_{1 \leq i,j \leq Z} \Big)_{1 \leq s \leq 3}, \\
\hat{f} &= \sum_{k=1}^{m} f(a_k),
\end{aligned}$$

hence the system (4.1) is equivalent to

$$\left(A\, A^T + \tau^T\, B \right)\, \alpha = A\, \hat{f}.$$

5 Convergence

Under adequate conditions, we are going to show that the *fairness cubic spline* in V_n relative to A^m and τ converges to f when m tends to $+\infty$.

To do this, let f be a function of $H^3(I;\mathbb{R}^p)$ and $\sigma_\tau^{n,m}$ be the *fairness cubic spline* in V_n corresponding to the data given from f.

Theorem 5.1. *Suppose that the hypotheses (2.1), (3.1) hold and that*

$$\forall i = 1,2,\ \tau_i = \mathrm{o}(\tau_3),\ m \to +\infty, \tag{5.1}$$

and

$$\frac{m}{\tau_3\ n^3} = \mathrm{o}(1),\ m \to +\infty. \tag{5.2}$$

Then, one has

$$\lim_{m\to+\infty} \|f - \sigma_\tau^{n,m}\|_3 = 0.$$

Proof. Step 1. First, let s be a parametric cubic spline function of V_n interpolating f on the knots $x_0, \ldots, x_n$ such that $s''(x_0) = f''(x_0)$ and $s''(x_n) = f''(x_n)$, then we have $J_\tau^m(\sigma_\tau^{n,m}) \le J_\tau^m(s)$ which implies that

$$|\sigma_\tau^{n,m}|_3^2 \ \le\ \frac{1}{\tau_3}\sum_{i=1}^{m} \langle s(a_i) - f(a_i)\rangle^2 + \frac{\tau_1}{\tau_3}|s|_1^2 + \frac{\tau_2}{\tau_3}|s|_2^2 + |s|_3^2 .$$

From the properties of the function s it follows that $|s|_2^2 \ \le\ |f|_2^2$ (see for example [6]) and

$$\begin{aligned} |s|_1^2 &\le |s-f|_1^2 + |f|_1^2 = \int_a^b (s'-f')^2 dx + |f|_1^2 \\ &\le \frac{(b-a)^2}{n}\ |f|_2^2 + |f|_1^2 . \end{aligned}$$

Furthermore, as the function $s'''(x)$ is a constant C_i in each sub-interval $[x_{i-1}, x_i]$ for $i = 1, \ldots, n$, then we have

$$\int_a^b f'''(x)^2 dx = \int_a^b s'''(x)^2 dx + \int_a^b E'''(x)^2 dx + 2\int_a^b E'''(x)\, s'''(x) dx \tag{5.3}$$

where $E(x) = f(x) - s(x)$, but we have that

$$\int_a^b E'''(x)\, s'''(x) dx = \sum_{i=0}^{n-1} C_i \left(E''(x_{i+1}) - E''(x_i)\right).$$

It follows that

$$\begin{aligned} \min_{0\le i\le n-1}\ C_i\ \sum_{i=0}^{n-1} \left(E''(x_{i+1}) - E''(x_i)\right) &\le \int_a^b E'''(x)\, s'''(x) dx \\ &\le \max_{0\le i\le n-1}\ C_i\ \sum_{i=0}^{n-1} \left(E''(x_{i+1}) - E''(x_i)\right). \end{aligned}$$

As $\sum_{i=0}^{n-1}(E''(x_{i+1})-E''(x_i))=E''(x_n)-E''(x_0)=0$ then we deduce that

$$\int_a^b E'''(x)\, s'''(x)dx=0.$$

Since, from (5.3) one has

$$|s|_3^2 \;\le\; |f|_3^2. \tag{5.4}$$

Likewise, for one dimension it is known, for all $j=1,\dots,p$, that

$$|(p_j\circ s)(a_i)-(p_j\circ f)(a_i)| \;\le\; \left(\frac{1}{n}\right)^{3/2}\left(\int_a^b((p_j\circ f'')(t))^2dt\right)^{1/2}$$

where p_j, for $j=1,\dots,p$, is the j-th canonical projection from $\mathbb{R}^p$ into $\mathbb{R}$. So, we obtain that

$$\sum_{i=1}^{m}\langle s(a_i)-f(a_i)\rangle^2 \;\le\; \frac{m\ p}{n^3}\,|f|_2^2.$$

Then, we have

$$|\sigma_\tau^{n,m}|_3^2 \;\le\; \frac{m\ p}{\tau_3\ n^3}|f|_2^2+\frac{\tau_1}{\tau_3}\left(\frac{(b-a)^2}{n}\,|f|_2^2+|f|_1^2\right)+\frac{\tau_2}{\tau_3}\,|f|_2^2+|f|_3^2. \tag{5.5}$$

Hence, from (5.1), (5.2) and (5.5) we deduced that there exist a constant $C>0$ and $M\in\mathbb{N}$ such that

$$|\sigma_\tau^{n,m}| \;\le\; C, \qquad \forall m\ge M.$$

We conclude that the family $(\sigma_\tau^{n,m})_{m\in\mathbb{N}^*}$ is bounded in V_n. It follows that there exists a sub-sequence $\left(\sigma_{\tau_l}^{n_l,m_l}\right)_{l\in\mathbb{N}}$, with $\tau_l=\tau(m_l)$, $\lim\limits_{l\to+\infty} m_l=+\infty$, and an element $f^*\in H^3(I;\mathbb{R}^p)$ such that

$$\sigma_{\tau_l}^{n,m} \text{ converges weakly to } f^* \text{ in } H^3(I;\mathbb{R}^p). \tag{5.6}$$

Step 2. Let us now prove that $f^*=f$.

We suppose that $f^*\ne f$. From the continuous injection of $H^3(I;\mathbb{R}^p)$ into $C^2([a,b];\mathbb{R}^p)$ it follows that there exists $\theta>0$ and an open sub-interval J of I such that

$$\forall x\in J,\ \langle f^*(x)-f(x)\rangle \;>\; \theta.$$

As such injection is also compact then from (5.6) we obtain

$$\exists l_0\in\mathbb{N},\ \forall l\ge l_0,\ \forall x\in J,\ \left\langle\sigma_{\tau_l}^{n_l,m_l}(x)-f^*(x)\right\rangle \;\le\; \frac{\theta}{2}.$$

Hence, for all $l \geq l_0$ and $x \in J$ we have

$$\left\langle \sigma_{\tau_l}^{n_l,m_l}(x) - f(x) \right\rangle \;\geq\; \langle f^*(x) - f(x) \rangle - \left\langle \sigma_{\tau_l}^{n_l,m_l}(x) - f^*(x) \right\rangle \;>\; \frac{\theta}{2} \cdot \qquad (5.7)$$

Now well, for l sufficiently great and using (2.1) we deduce that there exists a point $a_{m_l} \in A^m \cap J$ such that

$$\left\langle \sigma_{\tau_l}^{n_l,m_l}(a_{m_l}) - f(a_{m_l}) \right\rangle = \mathrm{o}(1),\; l \to +\infty,$$

which is a contradiction with (5.7). Consequently $f^* = f$.

Step 3. As $H^3(I;\mathbb{R}^p)$ is compactly injected in $H^2(I;\mathbb{R}^p)$, using (5.6) and taking into account that $f^* = f$ we have

$$f = \lim_{l \to +\infty} \sigma_{\tau_l}^{n_l,m_l} \text{ in } H^2(I;\mathbb{R}^p).$$

Then, one has

$$\lim_{l \to +\infty} ((\sigma_{\tau_l}^{n_l,m_l}, f))_2 = \|f\|_2^2 \,. \qquad (5.8)$$

Using again (5.6) and that $f^* = f$ we obtain

$$\lim_{l \to +\infty} (\sigma_{\tau_l}^{n_l,m_l}, f)_3 = \lim_{l \to +\infty} \left(((\sigma_{\tau_l}^{n_l,m_l}, f))_3 - \lim_{l \to +\infty} ((\sigma_{\tau_l}^{n_l,m_l}, f))_2 \right) = |f|_3^2 \,. \quad (5.9)$$

Moreover, for all $l \in \mathbb{N}$ we have

$$\left|\sigma_{\tau_l}^{n_l,m_l} - f\right|_3^2 = \left|\sigma_{\tau_l}^{n_l,m_l}\right|_3^2 + |f|_3^2 - 2(\sigma_{\tau_l}^{n_l,m_l}, f)_3$$

we deduce from (5.9) together with (5.4) that

$$\lim_{l \to +\infty} \left|\sigma_{\tau_l}^{n_l,m_l} - f\right|_3 = 0$$

which implies with (5.8) that

$$\lim_{l \to +\infty} \left\|\sigma_{\tau_l}^{n_l,m_l} - f\right\|_3 = 0.$$

Step 4. Finally, by reasoning with reduction to absurd we prove that the result is true. To do this, we suppose that it is false, so there exists a real number $\gamma > 0$ and the following sequences $(n_{l'})_{l' \in \mathbb{N}}$, $(m_{l'})_{l' \in \mathbb{N}}$ and $(\tau_{l'})_{l' \in \mathbb{N}}$ with $\lim\limits_{l' \to +\infty} m_{l'} = +\infty$ such that

$$\left\|\sigma_{\tau_{l'}}^{n_{l'},m_{l'}} - f\right\|_3 \;\geq\; \gamma, \qquad \forall l' \in \mathbb{N}. \qquad (5.10)$$

Now well, the sequence $\left(\sigma_{\tau_{l'}}^{n_{l'},m_{l'}}\right)_{l' \in \mathbb{N}}$ is bounded in V_n. Hence, by following the same way of the Steps 1), 2) and 3) we deduce that from such sequence we can extract a sub-sequence that converges towards f, which produces a contradiction with (5.10). □

Remark 5.1. We denote that when m tends to $+\infty$ then from (5.2) it follows that n also tends to $+\infty$. This is the reason that we have'nt put for the convergence result that both n and m tend to $+\infty$. □

6 Numerical and graphical example

In order to test the smoothness method we consider the following example:

Let Υ_0 be a curve parameterized by the function $f : (0,1) \to \mathbb{R}^2$ defined by

$$f(x) = \Big(\text{Exp}\{\frac{\pi x}{2}\}\ \text{Cos}(5\pi x), \text{Exp}\{\frac{\pi x}{2}\}\ \text{Sin}(5\pi x)\Big).$$

The graph of this function appears in Figure 1.

We have computed an approximating curve Υ of Υ_0 parameterized by a *fairness cubic spline* from a set of m scattered points of $[0,1]$. Likewise, we have taken $n = 30$ so $\dim S_3(T_n) = 33$ and $\dim V_n = 66$ which is the order of the system linear given in (4.1).

Graphically, for $n = 30$ and $\tau = (10^{-4}, 10^{-7}, 10^{-9})$ Figure 2 shows the graphs of $m = 75$ arbitrary points of the curve Υ_0 and an approximating curve Υ of these points parameterized by the *fairness cubic spline* $\sigma_\tau^{n,m}$.

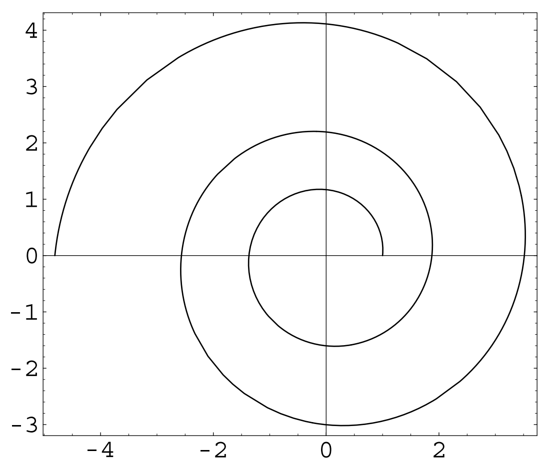

Figure 1: The graph of Υ_0.

We conclude graphically from Figure 2 that the approximating curve Υ is similar to the original one Υ_0.

For $n, m \in \mathbb{N}$ and $\tau \in \mathbb{R}^3$, with $\tau_1 \geq 0$, $\tau_2 \geq 0$ and $\tau_3 > 0$, we have computed an estimation of the relative error by

$$E_r = \frac{\sum_{i=1}^{1000} \langle f(a_i) - \sigma_\tau^{n,m}(a_i) \rangle}{1000}$$

where a_i are arbitrary points of the interval I. So, for the data given for V_n, $m = 75$ and for $\tau = (10^{-4}, 10^{-4}, 10^{-7})$, we have computed **Table 1** which shows the effect of the number of data points in the estimation of E_r.

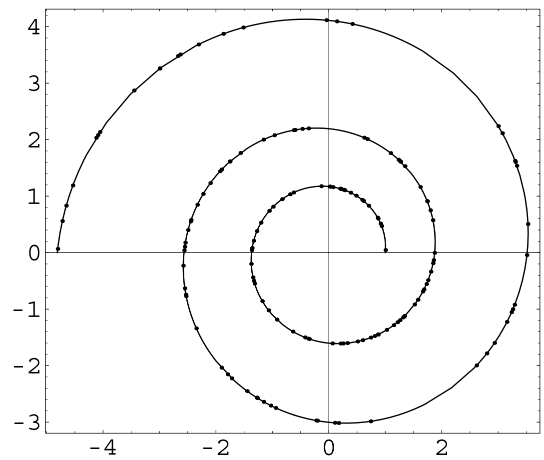

Figure 2: The graph of an approximating curve Υ of Υ_0 from $m = 75$ arbitrary points of Υ_0 parameterized by the *fairness cubic spline* for $n = 30$ and $\tau = (10^{-4}, 10^{-7}, 10^{-9})$.

m	E_r
15	2.20263
25	1.60455×10^{-2}
50	1.41512×10^{-2}
75	2.78664×10^{-3}
150	2.26171×10^{-4}

Table 1: Influence on E_r of the points(m).

Let us give here some numerical interpretation of each terms of fairness. To this end, for $n = 30$, $m = 150$ and fixed value $\tau_3 = 10^{-9}$ (it means that we treat to preserve the same degree of approximation E_r), we have computed **Table 2** which shows the effect of the variation of τ_1 produced in the value of the estimation of the length of the approximating curve. To do this, we have fixed $\tau_2 = 0$ and we have taken some distinct values of τ_1 which is associated to the minimization of the semi–norm $|\cdot|_{1,(a,b),\mathbb{R}^2}$. Hence, we observe that the estimation of the curve length, denoted by LC, is a decrease function of τ_1, which justifies the interpretation given in Remark 3.1.

Table 3 shows the effect of the variation of τ_2 produced in the value of the estimation of the curvature of the approximating curve. To do this, we have fixed $\tau_1 = 0$ and we have taken some distinct values of τ_2 which is associated to the minimization of the semi–norm $|\cdot|_{2,(a,b),\mathbb{R}^2}$. Hence, we observe that the

τ_1	LC	E_r
8×10^{-2}	31.5475	3.28436×10^{-1}
1×10^{-1}	30.4715	3.9954×10^{-1}
2×10^{-1}	26.2314	6.43437×10^{-1}
3×10^{-1}	23.1503	8.44617×10^{-1}

Table 2: Influence on LC of τ_1.

estimation of the curvature, denoted by CU, is a decrease function of τ_2, which justifies the interpretation given in Remark 3.1.

τ_2	CU	E_r
3×10^{-2}	6.30333	2.01639
5×10^{-2}	2.41021	2.0819
6×10^{-2}	0.853592	2.18999

Table 3: Influence on CU of τ_2.

Now, in order to analyze our smoothing method, we compare it with the least square one. To this end, for $n = 30$ and $m = 150$ points we present the following results:

1. The problem of only least square, it means that $\tau = (0, 0, 0)$, we obtain the following estimations $E_r = 2.33781 \times 10^{-4}$, LC $= 38.29$ and CU $= 0.501939$ of the approximating curve of Υ_0 which graph appears in Figure 3 (left side).

2. Our problem, for example $\tau = (10^{-7}, 10^{-7}, 10^{-9})$, we obtain the following estimations $E_r = 1.87622 \times 10^{-4}$, LC $= 38.2944$ and CU $= 0.501836$ of the approximation curve parameterized by a *fairness cubic spline* which graph appears in Figure 3 (right side).

Conclusion: Obviously, when we impose more conditions to any problem of minimization then its approximating method can lose its concept. But, if we observe the results given in **Tables** 2 and 3 (imposing the fairness constraint), we observe that the estimations of the values LC and CU of the approximating curve are minimized, that test the validity between the theory (Remark 3.1) and practice. Moreover, from the comparison of our method with the classical least square one, specially the values estimated and both graphs of Figure 3 are very similar, from what we conclude that our approximation method with *fairness constraint* presented in this paper is well as an other one.

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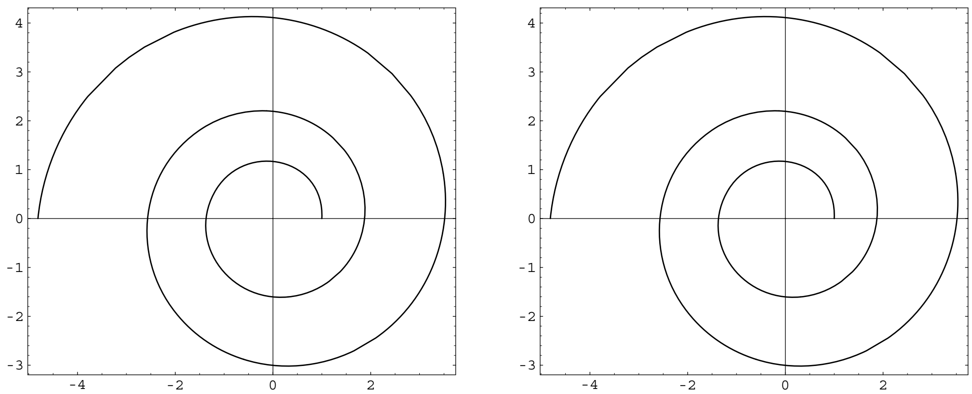

Figure 3: An approximating curve of Υ_0 parameterized by the *fairness cubic spline* for $\tau = (0,0,0)$, with only a least square, (left side). An other one parameterized by the *fairness cubic spline* for $\tau = (10^{-7}, 10^{-9}, 10^{-9})$, i.e. with fairness constraint, (right side).

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