第33卷第2期　　中国科学技术大学学报　　Vol. 33, No. 2
2003年4月　JOURNAL OF UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA　Apr. 2003

Article ID: 0253-2778(2003)02-0134-10

Inequalities for Eigenvalues of Biharmonic Operator in Weighted Sobolev Space *

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Abstract: This paper considers the eigenvalue problem for the biharmonic equation $\Delta^2 u - \mu w(x)u = 0$ in weighted Sobolev space $W_0^{1,p}(\Omega, w(x))$, $u = 0$ on $\partial\Omega$, where $\Omega \subset R^m$ is a bounded smooth domain, $w(x) \in L^\infty(\Omega)$ is also bounded. For $m \geqslant 2$, three estimates of the eigenvalues μ_i, $i = 1, \cdots, n$ are obtained.

Key words: weighted Sobolev space; eigenvalue estimate; biharmonic

CLC number: O175.6　　**Document code**: A

AMS Subject Classifications (2000): 35J65; 35P30

1 Introduction

Let Ω be a bounded domain in R^m with smooth boundary. By [1], we know the first $n+1$ eigenvalues for the problem

$$\Delta^2 u - \mu u = 0 \quad \text{in } \Omega, \quad u = \frac{\partial u}{\partial \boldsymbol{n}} = 0 \quad \text{on } \partial\Omega \tag{1}$$

satisfy the inequality

$$\sum_{j=1}^{n} \frac{\sqrt{\mu_i}}{\mu_{n+1} - \mu_i} \geqslant \frac{m^2 n^{3/2}}{8(m+2)} \Big(\sum_{i=1}^{n} \mu_i \Big)^{-1/2}. \tag{2}$$

For the first two eigenvalues we have the stronger bound

$$\mu_2 \leqslant 7.103\mu_1 \quad (\text{in } R^2), \quad \mu_2 \leqslant 4.792 \quad \mu_1 \quad (\text{in } R^3).$$

For the case $m = 2$, Payne, Polya and Weinberger [4] obtained upper estimates, independent of the domain, for eigenvalues of Laplacian operator. G. N. Hile and R. Z. Yeh pursued the following estimates. For equation (1) and they derived the explicit bound for the $(n+1)th$ eigenvalue

$$\mu_{n+1} \leqslant \mu_n + \frac{8(m+2)}{m^2 n^{3/2}} \Big(\sum_{i=1}^{n} \mu_i \Big)^{1/2} \Big(\sum_{i=1}^{n} \sqrt{\mu_i} \Big). \tag{3}$$

* **Received date**: 2002-06-17

Foundation item: Project supported by NNSFC (10071080) and NNSFC for Young Scholars (No. 10101024)

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Next, they further obtained the estimates for some $\sigma > 0$

$$\mu_{n+1} \leqslant (1+\sigma)\mu_n + q(\sigma)\frac{M(m)}{n}\sum_{i=1}^{n}\mu_i, \tag{4}$$

where $q(\sigma) = \left[\frac{(1+\sigma)^3}{\sigma}\right]^{1/2}$, $M(m) = \frac{32}{3}\sqrt{\frac{3}{2}}m^{-1}(m+2)^{-1/2}$.

It is well known that the biharmonic equation in (1) has a strong physics background, but it is a more ideal model abstracted from the physical phenomena. Generally speaking, the weighted biharmonic equation, $\Delta^2 u - \mu w(x)u = 0$, with weighted function $w(x)$ should reflect much more physical phenomena. Therefore, in this paper, we reconsider equation system (1) in weighted Sobolev space, that is,

$$\Delta^2 u - \mu w(x)u = 0 \quad \text{in } \Omega, \quad u = \frac{\partial u}{\partial \boldsymbol{n}} = 0 \quad \text{on } \partial\Omega, \tag{5}$$

here $w(x) \in L^\infty(\Omega)$, and it is bounded by $0 < 1/a \leqslant w(x) \leqslant b$.

Obviously, when $w(x) = 1$, problem (5) is just problem (1). Hence this paper is a generalization of (1).

By similar methods, we derive analogues of (3) and (4) and obtain the bound

$$\mu_{n+1} \leqslant \mu_n + \frac{8a^2b^2(m+2)}{m^2 n}\sum_{i=1}^{n}\mu_i. \tag{6}$$

For $m \geqslant 2$, $n \geqslant 1$, constant $\sigma > 0$,

$$\mu_{n+1} \leqslant (1+\sigma)\mu_n + abq(\sigma)\frac{M(m)}{n}\sum_{i=1}^{n}\mu_i. \tag{7}$$

(7) yields a sharper bound than (6) if some optimal σ is chosen.

Although we still use the Raleigh formula in variation method like the one in (1), but it is more complicated in the estimates and derivations, especially in the second part, i. e. , § 3.

2 Inequalities for μ_n

Let Ω be a bounded domain in R^m, $m \geqslant 2$, with boundary $\partial\Omega$, $w(x) \in L^\infty(\Omega)$, $1/a \leqslant w(x) \leqslant b$. Let eigenvalues of

$$\Delta^2 u - \mu w(x)u = 0 \quad \text{in } \Omega, \quad u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega$$

be designated by

$$0 < \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_n \leqslant \cdots$$

with corresponding eigenfunctions $u_1, u_2, \cdots, u_n, \cdots$, providing

$$\int_\Omega u_i u_j w(x) = \delta_{ij}, \quad i,j = 1,2,\cdots$$

In this section we have the main result as follows.

Theorem 1 For $m \geqslant 2$ and $n \geqslant 1$, for the $(n+1)th$ eigenvalue of (5) we have

$$\mu_{n+1} \leqslant \mu_n + \frac{8a^2b^2(m+2)}{m^2 n}\sum_{i=1}^{n}\mu_i. \tag{8}$$

Proof Following Payne, Polya and Weinberger[4] we consider the n trial functions

$$\phi_i = x_1 u_i - \sum_{j=1}^{n} a_{ij} u_j, \quad i = 1,2,\cdots,n,$$

where $x = (x_1, x_2, \cdots, x_m) \in R^m$, and the constant a_{ij} are defined by

$$a_{ij} = \int_\Omega x_1 u_i u_j, \quad i,j = 1,2,\cdots,n.$$

It is easy to check

$$\int_\Omega \phi_i u_j w(x) = 0, \quad \int_\Omega \phi_i^2 w(x) = \int_\Omega x_1 u_i \phi_i w(x).$$

Then each ϕ_i is orthogonal to $u_1, u_2, \cdots, u_n$. Moreover, since $\phi_i = \dfrac{\partial \phi_i}{\partial \boldsymbol{n}} = 0$ on $\partial\Omega$, by the Raleigh formula in variation method, we have

$$\mu_{n+1} \leqslant \frac{\int \phi_i \Delta^2 \phi_i}{\int \phi_i^2 w(x)}, \quad i = 1,2,\cdots,n. \tag{9}$$

Now,

$$\int \phi_i \Delta^2 \phi_i = \int \phi_i [\Delta^2 (x_1 u_i) - \sum_{j=1}^{n} a_{ij} \mu_j u_j w(x)] =$$

$$\int \phi_i [x_1 \Delta^2 u_i + 4\Delta u_{ix_1}] = \mu_i \int \phi_i^2 w(x) + 4 \int \phi_i \Delta u_{ix_1}. \tag{10}$$

After substituting (10) into (9) and summing over i we have

$$\mu_{n+1} \sum_{i=1}^{n} \int \phi_i^2 w(x) \leqslant \sum_{i=1}^{n} \mu_i \int \phi_i^2 w(x) + 4 \sum_{i=1}^{n} \int \phi_i \Delta u_{ix_1}. \tag{11}$$

Not losing generality, we make the assumption that

$$\sum_{i=1}^{n} \int u_{ix_k}^2 = \frac{1}{m} \sum_{i=1}^{n} \int |\nabla u_i|^2, \quad k = 1,2,\cdots,m. \tag{12}$$

This equality can be made to hold by rotating the coordinate system in R^m. Suppose, for example, that (12) does not hold, and that x_p and x_q denote two coordinate directions such that

$$\sum_{i=1}^{n} \int u_{ix_p}^2 < \frac{1}{m} \sum_{i=1}^{n} \int |\nabla u_i|^2 < \sum_{i=1}^{n} \int u_{ix_q}^2.$$

Then we may make a rotation of the $x_p - x_q$ plane until the equal sign in the above inequality is achieved. This operation can be repeated until (12) holds for all values of k.

We pause to make a few technical calculations. Let us define

$$J_1 = \sum_{i=1}^{n} \int \phi_i \Delta u_{ix_1}, \quad J = \frac{m+2}{2m} \sum_{i=1}^{n} \int |\nabla u_i|^2.$$

Lemma 1 The quantities J_1 and J satisfy

(i) $J_1 = J$,

(ii) $J^2 \leqslant an[(m+2)/(2m)]^2 \sum_{i=1}^{n} \mu_i$,

(iii) $n^2(m+2)/(8ab^2) \leqslant J\sum_{i=1}^n \int \phi_i^2 w(x)$.

Proof of Lemma 1. (i) We have

$$J_1 = \sum_{i=1}^n \int x_1 u_i \Delta u_{ix_1} - \sum_{i,j=1}^n a_{ij} \int u_j \Delta u_{ix_1}.$$

The last term above vanishes since $a_{ij} = a_{ji}$, and by

$$\int u_j \Delta u_{ix_1} = -\int u_i \Delta u_{jx_1}.$$

As for the first term, we show by integration by parts that

$$\int x_1 u_i \Delta u_{ix_1} = \int \Delta(x_1 u_i) u_{ix_1} = \int (x_1 \Delta u_i + 2u_{ix_1}) u_{ix_1} =$$

$$-\int (x_1 \Delta u_i)_{x_1} u_i + 2\int u_{ix_1}^2 = -\int x_1 u_i \Delta u_{ix_1} + \int |\nabla u_i|^2 + 2\int u_{ix_1}^2.$$

Then, by (12) we have

$$J_1 = \sum_{i=1}^n \left[\frac{1}{2}\int |\nabla u_i|^2 + \int u_{ix_1}^2\right] = J.$$

(ii) By the Schwarz inequality we have

$$J^2 = \left(\frac{m+2}{2m}\right)^2 \left(\sum_{i=1}^n \int -u_i \Delta u_i\right)^2 = \left(\frac{m+2}{2m}\right)^2 \left(\sum_{i=1}^n \int u_i^2\right)\left(\sum_{i=1}^n \int (\Delta u_i)^2\right).$$

Moreover,

$$\int u_i^2 = \int u_i^2 w(x) \cdot \frac{1}{w(x)} \leqslant a\int u_i^2 w(x) = a,$$

$$\int (\Delta u_i)^2 = \int u_i \Delta^2 u_i = \mu_i \int u_i^2 w(x) = \mu_i.$$

Then,

$$J^2 \leqslant na\left(\frac{m+2}{2m}\right)\sum_{i=1}^n \mu_i.$$

(iii) Let us compute

$$\sum_{i=1}^n \int \phi_i u_{ix_1} = \sum_{i=1}^n \int x_1 u_i u_{ix_1} - \sum_{i,j=1}^n a_{ij} \int u_j u_{ix_1}. \quad (13)$$

We have, by integration by parts,

$$\int x_1 u_i u_{ix_1} = -\frac{1}{2}\int u_i^2, \quad \int u_j u_{ix_1} = -\int u_i u_{jx_1}.$$

Since $a_{ij} = a_{ji}$, the last term of (13) vanishes, and we obtain

$$\sum_{i=1}^n \int \phi_i u_{ix_1} = -\frac{1}{2}\sum_{i=1}^n \int u_i^2. \quad (14)$$

Hence, upon squaring and applying (12) we have

$$\frac{1}{4}\left(\sum_{i=1}^n \int u_i^2\right)^2 \leqslant \left(\sum_{i=1}^n \int \phi_i^2\right)\left(\sum_{i=1}^n \int u_{ix_1}{}^2\right) = \left(\sum_{i=1}^n \int \phi_i^2\right)\left(\frac{2}{m+2}J\right).$$

We still have

$$\int u_i^2 = \int u_i^2 w(x) \frac{1}{w(x)} \geqslant \frac{1}{b},$$

$$\int \phi_i^2 \leqslant a \int \phi_i^2 w(x).$$

By the above three inequalities, we finally get

$$\frac{n^2(m+2)}{8ab^2} \leqslant J \sum_{i=1}^{n} \int \phi_i^2 w(x)$$

which completes the proof of Lemma 1.

Returning now to (11), if we replace μ_i by μ_n, then in view of Lemma 1 we have

$$(\mu_{n+1} - \mu_n) \sum_{i=1}^{n} \int \phi_i^2 w(x) \leqslant 4J. \tag{15}$$

Combining (ii) with (iii) in Lemma 1 yields

$$J \leqslant \frac{2a^2 b^2 (m+2)}{nm^2} \Big(\sum_{i=1}^{n} \mu_i \Big) \Big(\sum_{i=1}^{n} \int \phi_i^2 w(x) \Big). \tag{16}$$

Substituting (16) into (15) yields

$$\mu_{n+1} \leqslant \mu_n + \frac{8(m+2)a^2 b^2}{nm^2} \sum_{i=1}^{n} \mu_i.$$

The proof of Theorem 1 is now completed.

Remark. When $w(x) = 1$, the result of [3] follows from our Theorem 1.

Now we obtain a stronger result than Theorem 1 by a somewhat more lengthy argument.

Theorem 2 For the first $n+1$ eigenvalues of (5), when $m \geqslant 2$ and $n \geqslant 1$ we have the implicit bound

$$\sum_{i=1}^{n} \frac{\sqrt{\mu_i}}{\mu_{n+1} - \mu_i} \geqslant \frac{m^2 n^{3/2}}{8(m+2)(ab)^{3/2}} \Big(\sum_{i=1}^{n} \mu_i \Big)^{-1/2}, \tag{17}$$

and the explicit bound

$$\mu_{n+1} \leqslant \mu_n + \frac{8(m+2)(ab)^{3/2}}{m^2 n^{3/2}} \Big(\sum_{i=1}^{n} \mu_i \Big)^{-1/2} \Big(\sum_{i=1}^{n} \mu_i^{-1/2} \Big). \tag{18}$$

Inequality (17) is stronger than (18), and both are stronger than (8).

Proof We return again to (11) but instead of replacing each μ_i by μ_n, we introduce a new parameter α, $\alpha > \mu_n$, and write

$$\mu_{n+1} \sum_{i=1}^{n} \int \phi_i^2 w(x) \leqslant \alpha \sum_{i=1}^{n} \int \phi_i^2 w(x) + \sum_{i+1}^{n} (\mu_i - \alpha) \int \phi_i^2 w(x) + 4J. \tag{19}$$

We also apply the Cauchy inequality to (14), and set $A = \sum_{i=1}^{n} \int u_i^2$, obtaining for any $\delta > 0$,

$$\frac{A}{2} = \Big| \sum_{i=1}^{n} \int \phi_i u_{ix_1} \Big| \leqslant \frac{\delta}{2} \sum_{i=1}^{n} (\sigma - \mu_i) \int \phi_i^2 w(x) + \frac{1}{2\delta} \sum_{i=1}^{n} (\sigma - \mu_i)^{-1} \int {u_{ix_1}}^2 \cdot \frac{1}{w(x)} \leqslant$$

$$\frac{\delta}{2} \sum_{i=1}^{n} (\sigma - \mu_i) \int \phi_i^2 w(x) + \frac{a}{2\delta} \sum_{i=1}^{n} (\sigma - \mu_i)^{-1} \int {u_{ix_1}}^2. \tag{20}$$

Now we use trial functions ϕ_{ik}, based on x_k instead of x_1, for $k = 1, 2, \cdots, m$, and obtain inequalities analogous to (19) and (20) of the type

$$(\mu_{n+1}-\alpha)\sum_{i=1}^{n}\int\phi_{ik}^{2}w(x)\leqslant\sum_{i=1}^{n}(\mu_i-\alpha)\int\phi_{ik}^{2}w(x)+4J,\quad k=1,2,\cdots,m. \tag{21}$$

$$A\leqslant\delta\sum_{i=1}^{n}(\alpha-\mu_i)\int\phi_{ik}^{2}w(x)+\delta^{-1}\cdot a\sum_{i=1}^{n}(\alpha-\mu_i)^{-1}\int u_{ix_k}^{2},\quad k=1,2,\cdots,m. \tag{22}$$

(Because of (12) and its consequence (i) of Lemma 1, the quantity J is the same for each value of k.) We sum each of (21) and (22) over k, denoting

$$S=\sum_{k=1}^{m}\sum_{i=1}^{n}\int\phi_{ik}^{2}w(x),\quad T=\sum_{k=1}^{m}\sum_{i=1}^{n}(\alpha-\mu_i)\int\phi_{ik}^{2}w(x)$$

to obtain the inequalities

$$(\mu_{n+1}-\alpha)S+T\leqslant 4mJ, \tag{23}$$

$$mA\leqslant\delta T+\delta^{-1}\sum_{i=1}^{n}(\alpha-\mu_i)^{-1}\int|\nabla u_i|^2\cdot a. \tag{24}$$

Noticing

$$\int|\nabla u_i|^2=\int-u_i(\Delta u_i)\leqslant\left(\int u_i^2\right)^{1/2}\left(\int(\Delta u_i)^2\right)^{1/2}$$

we set $A_i=\int u_i^2$, then $A=\sum_{i=1}^{n}A_i$, and the above estimate could be written as

$$\int|\nabla u_i|^2\leqslant A_i^{1/2}\sqrt{\mu_i},$$

which, when applied to (24), yields

$$mA\leqslant\delta T+\delta^{-1}\sum_{i=1}^{n}\frac{\sqrt{A_i\mu_i}}{\alpha-\mu_i}\cdot a. \tag{25}$$

The right side of this inequality is minimized by choosing

$$\delta=T^{-1/2}\left(\sum_{i=1}^{n}\frac{\sqrt{A_i\mu_i}}{\alpha-\mu_i}\right)^{1/2}\cdot a^{1/2}. \tag{26}$$

From (ii) of Lemma 1 we also have

$$J\leqslant\sqrt{A}\,\frac{m+2}{2m}\left(\sum_{i=1}^{n}\mu_i\right)^{1/2}. \tag{27}$$

Thus upon substituting (26) and (27) into (23) we arrive at inequality

$$(\mu_{n+1}-\alpha)S\leqslant 2\sqrt{A}(m+2)\left(\sum_{i=1}^{n}\mu_i\right)^{1/2}-\frac{m^2A^2}{4}\left(\sum_{i=1}^{n}\frac{\sqrt{A_i\mu_i}}{\alpha-\mu_i}\right)^{-1}\cdot a^{-1}. \tag{28}$$

Recall that α is restricted so that $\alpha>\mu_n$, we choose α as α_0 so that the right side of (28) is zero. Hence the left side must be nonpositive, i. e., $\alpha_0\geqslant\mu_{n+1}$. Therefore,

$$a\sum_{i=1}^{n}\frac{\sqrt{A_i\mu_i}}{\mu_{n+1}-\mu_i}\geqslant\frac{m^2A^{3/2}}{8(m+2)}\left(\sum_{i=1}^{n}\mu_i\right)^{-1/2}. \tag{29}$$

Since the right side of (28) is a monotone decreasing function of α on (μ_n,∞) from a positive number to $-\infty$, such a number α_0 exists and, in fact, it is unique. By the obvious estimates

$$\frac{1}{b}\leqslant A_i\leqslant a,\quad\frac{n}{b}\leqslant A\leqslant an,$$

we obtain the inequality from (29)

$$\sum_{i=1}^{n}\frac{\sqrt{\mu_i}}{\mu_{n+1}-\mu_i}\geqslant\frac{m^2n^{3/2}}{8(m+2)(ab)^{3/2}}\Big(\sum_{i=1}^{n}\mu_i\Big)^{-1/2}.$$

It yields (17).

Inequality (18) is derived from (17) by replacing each μ_i with μ_n in the denominators of the left-hand side, and then solving for μ_{n+1}. Inequality (8) is obtained from (18) by noting that

$$\sum_{i=1}^{n}\sqrt{\mu_i}\leqslant\sqrt{n}\Big(\sum_{i=1}^{n}\mu_i\Big)^{1/2}.$$

Thus the proof of Theorem 2 is complete.

Remark. When $w(x)=1$ the corresponding results to (17) and (18) in [3] are obtained.

3 Stronger inequalities for low eigenvalues

For the eigenvalues of (5), when $m\geqslant 2$, $n\geqslant 1$, and any constant $\sigma>0$ we have

$$\mu_{n+1}\leqslant(1+\sigma)\mu_n+q(\sigma)\cdot\frac{M(m)}{n}\cdot ab\sum_{i=1}^{n}\mu_i\leqslant[1+\sigma+q(\sigma)M(m)ab]\mu_n,\qquad(30)$$

where

$$q(\sigma)=\left[\frac{(1+\sigma)^3}{\sigma}\right]^{1/2},\qquad M(m)=\frac{32}{3}\sqrt{\frac{2}{3}}m^{-1}(m+2)^{-1/2}.\qquad(31)$$

Proof. We begin again with inequality (11), apply (i) of Lemma 1, and introduce a real parameter β to obtain

$$\mu_{n+1}\sum_{i=1}^{n}\int\phi_i^2w(x)\leqslant\sum_{i=1}^{n}\mu_i\int\phi_i^2w(x)+4(1+\beta)J_1-4\beta J.\qquad(32)$$

We also introduce parameters $\alpha>0$, $\tau_i>0$, $i=1,2,\cdots,n$, and apply Cauchy's inequality to J_1 to obtain

$$\begin{aligned}4(1+\beta)J_1&=4(1+\beta)\sum_{i=1}^{n}\int-\nabla\phi_i\cdot\nabla u_{ix_1}\leqslant\\&\sum_{i=1}^{n}2\tau_i\int|\nabla\phi_i|^2+2(1+\beta)^2\sum_{i=1}^{n}\tau_i^{-1}\int|\nabla u_{ix_1}|^2=\\&\sum_{i=1}^{n}2\tau_i\int-\phi_i\Delta\phi_i+2(1+\beta)^2\sum_{i=1}^{n}\tau_i^{-1}\int u_{ix_1x_1}\Delta u_i.\end{aligned}\qquad(33)$$

Again by Cauchy's inequality and (10) we have

$$\begin{aligned}\sum_{i=1}^{n}2\tau_i\int-\phi_i\Delta\phi_i&\leqslant\sum_{i=1}^{n}\sigma^{-1}\tau_i^2\int\phi_i^2+\sum_{i=1}^{n}\sigma\int(\Delta\phi_i)^2\leqslant\\&\sum_{i=1}^{n}(\sigma^{-1}\tau_i^2a+\sigma\mu_i)\int\phi_i^2w(x)+4\sigma J.\end{aligned}$$

Substitute this expression into (33), then the resulting inequality goes into (32), and we get

$$\mu_{n+1}\sum_{i=1}^{n}\int\phi_i^2w(x)\leqslant\sum_{i=1}^{n}(\mu_i+\sigma\mu_i+\sigma^{-1}\tau_i^2a)\int\phi_i^2w(x)+$$

$$2(1+\beta)^2\sum_{i=1}^{n}\tau_i^{-1}\int u_{ix_1x_1}\Delta u_i+4(\sigma-\beta)J. \tag{34}$$

In order to simplify (34) we choose each τ_i so that

$$(1+\sigma)\mu_i+\sigma^{-1}\tau_i^2 a=\tau,$$

where τ is a new parameter. The conditions τ_i, $\sigma>0$ require that $\tau>\mu_n$. In fact, we have

$$\tau>(1+\sigma)\mu_n>\mu_n,$$

$$\tau_i=[\tau-(1+\sigma)\mu_i]^{1/2}\sigma^{1/2}\cdot a^{-1/2},\quad i=1,2,\cdots,n,$$

$$\tau_1\geqslant\tau_2\geqslant\cdots\geqslant\tau_n>0.$$

If we use trial functions ϕ_{ik} based on x_k instead of x_1, inequality (34) has its counterpart

$$\mu_{n+1}\sum_{i=1}^{n}\int\phi_{ik}^2 w(x)\leqslant\tau\sum_{i=1}^{n}\int\phi_{ik}^2 w(x)+$$

$$2(1+\beta)^2\sum_{i=1}^{n}\tau_i^{-1}\int u_{ix_kx_k}\Delta u_i+4(\sigma-\beta)J,\quad k=1,2,\cdots,m. \tag{35}$$

We sum these inequalities over k, using

$$\sum_{k=1}^{m}\int u_{ix_kx_k}\Delta u_i=\int(\Delta u_i)^2=\mu_i,$$

and obtain

$$(\mu_{n+1}-\tau)S\leqslant 2(1+\beta)^2\sum_{i=1}^{n}\frac{\mu_i}{\tau_i}+4m(\sigma-\beta)J. \tag{36}$$

The counterpart of (iii) of Lemma 1 for x_k is

$$\frac{m+2}{8}\cdot A^2\leqslant J\sum_{i=1}^{n}\phi_{ik}^2,\quad k=1,2,\cdots,m.$$

Summing over k leads to

$$\frac{m(m+2)}{8}A^2\leqslant J\sum_{k=1}^{m}\sum_{i=1}^{n}\phi_{ik}^2\leqslant J\Big(\sum_{k=1}^{m}\sum_{i=1}^{n}\phi_{ik}^2 w(x)\Big)\cdot a=aJS,$$

and using $A\geqslant\frac{n}{b}$, we have

$$\frac{m(m+2)}{8}\cdot\frac{n^2}{b^2}\leqslant aJS. \tag{37}$$

By restricting $\sigma-\beta<0$ we can use (37) to eliminate J in (36), and then multiply by S to obtain

$$(\mu_{n+1}-\tau)S^2-2(1+\beta)^2\sum_{i=1}^{n}\frac{\mu_i}{\tau_i}S+\frac{m^2}{2}(m+2)n^2\frac{\beta-\sigma}{ab^2}\leqslant 0.$$

Therefore, we have a quadratic inequality in S of the form

$$\theta S^2-2\xi S+\eta\leqslant 0.$$

We can assert that

$$\theta\leqslant\xi^2\eta^{-1},$$

because $\theta\leqslant 0$ and the quadratic equation $\theta x^2-2\xi x+\eta=0$ must have a real root. Thus

$$\mu_{n+1}-\tau\leqslant\frac{2(1+\beta)^4 ab^2}{m^2(m+2)n^2(\beta-\sigma)}\Big(\sum_{i=1}^{n}\frac{\mu_i}{\tau_i}\Big)^2.$$

Since $f(\beta) = (1+\beta)^4/(\beta-\sigma)$ is minimized by taking $\beta = (4\sigma+1)/3$, we substitute this value in the preceding inequality to obtain

$$\mu_{n+1} \leqslant \tau + \frac{2\cdot 4^4(1+\sigma)^3 ab^2}{3^3 m^2(m+2)n^2}\left(\sum_{i=1}^{n}\frac{\mu_i}{\tau_i}\right)^2. \tag{38}$$

Since $\tau_1 \geqslant \tau_2 \geqslant \cdots \geqslant \tau_n$, replacing each τ_i by τ_n in (38) and further eliminating τ_n in favor of τ gives

$$\mu_{n+1} \leqslant \tau + \frac{2\cdot 4^4(1+\sigma)^3 ab^2}{3^3 m^2(m+2)n^2}\left(\sum_{i=1}^{n}\mu_i\right)^2[\tau-(1+\sigma)\mu_n]^{-1}. \tag{39}$$

The right side of inequality (39) has the form

$$\tau + C(\tau - B)^{-1},$$

which is minimized by letting $\tau = \sqrt{C} + B > (1+\sigma)\mu_n$. Substitution of this value of τ into (39) gives the desired result (31). The weakened version of (31) is obtained by replacing each μ_i with μ_n, thereby completing the proof of Theorem 3.

Now we show Theorem 3 is sharper than Theorem 1.

For domains in R^2 we have

$$\mu_2 \leqslant (1+8a^2b^2)\mu_1, \quad \text{(by Theorem 1)}$$

$$\mu_2 \leqslant (1+8a^{3/2}b^{3/2})\mu_1, \quad \text{(by Theorem 2)}$$

$$\mu_2 \leqslant \left[1+\sigma+q(\sigma)\frac{8}{3}\sqrt{\frac{2}{3}}ab\right]\mu_1. \quad \text{(by Theorem 3)}$$

We choose $\sigma = 0.5$, then

$$8(ab)^{3/2} - \left[\sigma + q(\sigma)\frac{8}{3}\sqrt{\frac{2}{3}}\right] = 8(ab)^{1/2}\left[ab - \frac{\sqrt{6}}{9}q(\sigma)\right] - \sigma \geqslant$$

$$8\left[1 - \frac{\sqrt{6}}{9}q(\sigma)\right] - \sigma = 8\left[1 - \frac{\sqrt{6}}{9}\cdot\frac{3\sqrt{3}}{2} - \frac{1}{16}\right] > 0.$$

General comparisons between Theorem 3 and Theorems 1, 2 are difficult to make for general m and n. We will compare Theorems 3 and 1 for some simple case theoretically. We compare the two inequalities

$$\mu_{n+1} \leqslant \left[1 + \frac{8(m+2)a^2b^2}{m^2}\right]\mu_n, \tag{40}$$

$$\mu_{n+1} \leqslant [1+\sigma+abq(\sigma)M(m)]\mu_n. \tag{41}$$

Inequality (41) holds for all $\sigma > 0$. Thus the best bound is obtained by choosing σ so that the right-hand side is minimized. In general a closed form expression for optimal σ is difficult to attain, since one is required to solve a cubic equation. We will show nevertheless that this optimal σ always yields a better bound in (41) than the bound (40).

Theorem 4 For all $m \geqslant 2$ inequality (41) is strictly sharper than (40) provided that the optimal value of σ is chosen.

Proof. Let us denote

$$B_m(\sigma) = \sigma^{-1}[8(m+2)m^{-2}a^2b^2 - M(m)q(\sigma)ab].$$

It must be shown that for some σ we have $B_m(\sigma) > 1$.

First, because $ab \geqslant 1$, we have

$$B_m(\sigma) \geqslant \sigma^{-1} ab[8(m+2)m^{-2} - M(m)q(\sigma)] = ab\tilde{B}_m(\sigma),$$

here

$$\tilde{B}_m(\sigma) = \sigma^{-1}[8(m+2)m^{-2} - M(m)q(\sigma)].$$

One can show that after some lengthy computations that the maximum of $\tilde{B}_m(\sigma)$ occurs at

$$\sigma = \tilde{\sigma}_m = \left[\frac{3}{8}(m+2)^3 m^{-2} - 1\right]^{-1}.$$

where obviously $\tilde{\sigma}_m > 0$, and the corresponding maximum of $\tilde{B}_m(\sigma)$ in

$$\tilde{B}_m(\tilde{\sigma}_m) = m^{-4}[m^4 + 8m^2 + 32m + 16], \tag{42}$$

then

$$B_m(\tilde{\sigma}_m) \geqslant ab\tilde{B}_m(\tilde{\sigma}_m) > ab \geqslant 1. \tag{43}$$

Thereby we complete the proof of Theorem 4.

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一类加权 Sobolev 空间中重调和算子的特征值估计

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摘要:论文讨论了加权 Sobolev 空间 $W_0^{1,p}(\Omega,w(x))$ 中重调和方程 $\Delta^2 u - \mu w(x)u = 0, u|\partial\Omega = 0$ 的特征值估计,其中 $\Omega \subset R^m$ 是边界光滑的有界区域,$w(x) \in L^\infty$ 有界. 在 $m \geqslant 2$ 的情况,对 μ_i, $i = 1,\cdots,n$ 做出了逐步加细的三个估计.

关键词:加权的 Sobolev 空间;特征值估计;双调和算子